Circular convolution  Let $f$ and $g$ be two signals with $N$ samples each.

The circular convolution of $f$ and $g$ is the $n$ point sample given by:

$$ f \ast g(k) = \sum_{r=0}^{N-1} f(k-r)g(r) = \sum_{r=0}^{N-1} f(r)g(k-r) \pmod{N} $$

The mod $N$ means that we use clock arithmetic in computing $k - r$, or that we assume that $f$ and $g$ are defined for all integers, but are periodic with period $N$, i.e., $f(k + N) = f(k)$, $g(k + N) = g(k)$. For example, suppose that $N = 4$ and that $f$ and $g$ are defined by vectors $[x_0, x_1, x_2, x_3]$ and $[y_0, y_1, y_2, y_3]$:

$$ f(r) = x_{r \mod 4}, \ g(r) = y_{r \mod 4}. $$

Then

$$ f \ast g(0) = \sum_{r=0}^{3} f(-r)g(r) = f(0)g(0) + f(-1)g(1) + f(-2)g(2) + f(-3)g(3) $$
$$ = f(0)g(0) + f(3)g(1) + f(2)g(2) + f(1)g(3) $$
$$ = x_0y_0 + x_3y_1 + x_2y_2 + x_1y_3 $$

$$ f \ast g(1) = x_1y_0 + x_0y_1 + x_3y_2 + x_2y_3 $$

$$ f \ast g(2) = x_2y_0 + x_1y_1 + x_0y_2 + x_3y_3 $$

$$ f \ast g(3) = x_3y_0 + x_2y_1 + x_1y_2 + x_0y_3 $$

1. Let $N = 8$, $f(0) = \frac{1}{2}$, $f(1) = \frac{1}{2}$, $f(r) = 0$ otherwise. Let $g$ be defined by an arbitrary vector $[y_0, y_1, \ldots, y_7]$. Find the formulas (for eight samples) for $f \ast g$. 

2. Find a matrix $H_f$ such that

$$
\begin{bmatrix}
  f \ast g(0) \\
  f \ast g(1) \\
  \vdots \\
  f \ast g(7)
\end{bmatrix}
= H_f Y = H_f
\begin{bmatrix}
  y_0 \\
  y_1 \\
  \vdots \\
  y_7
\end{bmatrix}.
$$

3. Now let $N = 8$, $f(7) = x_7, f(0) = x_0, f(1) = x_1, f(r) = 0$ otherwise. Now write out the matrix for $H_f$.

4. Based on the examples in 3 and 4 write out the matrix for $H_f$ for a general $f$. 
**Convolution theorem**  For a signal \( f(n), n \in \mathbb{Z} \) which is zero for sufficiently large negative \( n \) define the \( z \)-transform by

\[
F(z) = \sum_{n=-\infty}^{\infty} \frac{f(n)}{z^n} = \sum_{n=-\infty}^{\infty} f(n)z^{-n}.
\]

or if \( f \in \mathbb{C}^N \) is a finite signal then

\[
F(z) = \sum_{n=0}^{N-1} \frac{f(n)}{z^n} = \sum_{n=0}^{N-1} f(n)z^{-n}.
\]

5. Demonstrate that

\[
\hat{f}(k) = F(e^{2\pi i \frac{k}{N}}).
\]

6. Let \( f \) and \( g \) be arbitrary 4 point functions defined by the vectors \[ \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix} \]
and \[ \begin{bmatrix} b_0 & b_1 & b_2 & b_3 \end{bmatrix} \], respectively. Compute \( h = f \ast g \).
7. Compute $F(z)$, $G(z)$, $H(z)$, and $F(z)G(z)$, Reduce the exponents of $F(z)G(z)$ mod 4 and compare to $H(z)$. What do you observe?

8. Based on previous questions give a short argument about why

$$\hat{f} \ast \hat{g} = \hat{fg}$$

as functions.