1 Complex Numbers

Introduction

Even though most engineering students have seen complex numbers and probably have had some experience with the arithmetic and algebra of complex numbers by the sophomore year, it is usually the case that the exposure has been random. We would like to present the structure and rules of complex numbers in an organized way and also give you a chance to do a few routine exercises so that complex numbers do not appear as mysterious as they once did.

In the same way that we think of real numbers as being points on the real line, we think of complex numbers as being points in the plane. The horizontal axis will be the usual real numbers and the vertical axis will be the purely imaginary numbers. For instance, numbers like 3, 5, \(-6\), and 17 will be on the real axis, and 5\(i\), 7\(i\), \(-13i\), and \(-46i\) will be on the imaginary axis. Any number in the plane can then be written as a number from the real axis plus a number from the imaginary axis. For example, the number two units to the right of the origin and 3 units up would be \(2 + 3i\). An arbitrary complex number would be written as \(z = x + iy\) (it seems that if actual numbers are used, the \(i\) is written on the right, but if symbols are being used, the \(i\) is written on the left). This form of the complex number, \(z = x + iy\), is called the rectangular form, and the number \(x\) is called the real part of \(z\) and \(y\) is called the imaginary part of \(z\). Standard notation for the real and imaginary parts of a complex number are \(x = \text{Re}(z)\) and \(y = \text{Im}(z)\). The symbol \(i\) is usually used in mathematics, but in some engineering disciplines, \(j\) is used since \(i\) has other meanings (current, for example).

Thinking of complex numbers as vectors can often be useful. Addition and subtraction of complex numbers is the same as it is for vectors. When adding or subtracting complex numbers, we collect terms with respect to \(i\).

\[
(2 + 3i) + (5 + 7i) = (2 + 5) + (3 + 7)i = 7 + 10i \\
(2 + 3i) - (5 + 7i) = (2 - 5) + (3 - 7)i = -3 - 4i
\]

It's the multiplication and division of complex numbers that makes complex numbers different from vectors. Since \(i = \sqrt{-1}\), it follows that \(i^2 = -1\), so when multiplying complex numbers, we simplify using \(i^2 = -1\) and then collect terms with respect to \(i\). For example

\[
(2 + 3i) \cdot (5 + 7i) = 10 + 14i + 15i + 21i^2 = 10 + 14i + 15i - 21 = -11 + 29i
\]

Division, on the other hand, can be a bit more challenging. Here we make use of the fact that the product of a complex number with its conjugate \((\overline{z} = x - iy)\) is \(x^2 + y^2\), a real number. In other words,

\[
z \cdot \overline{z} = (x + iy) \cdot (x - iy) = x^2 - ixy + ixy - iy^2 = x^2 + y^2
\]

When dividing a complex number \(z\) by another complex number \(w\), we multiply the numerator and denominator by the conjugate of \(w\), producing a real number.
in the denominator and a complex multiplication in the numerator. For example, to divide \( z = 2 + 3i \) by \( w = 5 + 2i \),

\[
\frac{2 + 3i}{5 + 2i} = \frac{(2 + 3i) \cdot (5 - 2i)}{(5 + 2i) \cdot (5 - 2i)} = \frac{(2 + 3i) \cdot 10 - 4i + 15i - 6i^2}{25 + 4} = \frac{10 - 4i + 15i - 6i^2}{25 + 4} = \frac{10 - 4i + 15i - 6(-1)}{29} = \frac{10 + 11i}{29} = \frac{10}{29} + \frac{11}{29}i
\]

The magnitude (length) of a complex number \( z = x + iy \) is \( r = \sqrt{x^2 + y^2} \).

If \( \theta \) is the angle between \( z \) and the positive \( x \) axis, then the real part of \( z \) is \( \text{Re}(z) = x = r \cos(\theta) \) and the imaginary part is \( \text{Im}(z) = y = r \sin(\theta) \). Using \( r \) and \( \theta \), the complex number \( z \) can be expressed in the form \( z = re^{i\theta} \), known as the polar form of the complex number. We now have a number of ways of representing complex numbers.

\[
z = x + iy = r \cos(\theta) + ir \sin(\theta) = r (\cos(\theta) + i \sin(\theta)) = re^{i\theta}
\]

where we have use Euler’s Formula

\[
e^{i\theta} = \cos(\theta) + i \sin(\theta)
\]

We will justify Euler’s Formula later. Notice that multiplication of two complex numbers in polar form is easy. If \( z = 5e^{3i} \) and \( w = 7e^{4i} \), then \( z \cdot w = 5e^{3i} \cdot 7e^{4i} = (5 \cdot 7) (e^{3i} \cdot e^{4i}) = 35e^{3i+4i} = 35e^{7i} \); the \( r \)'s are multiplied and the \( \theta \)'s are added.

**Exercise 1** If \( z = 2 + 3i \) and \( w = 4 - 2i \), calculate the following:

\[
a) \ 2z \quad b) \ iw \quad c) \ z/i \\
d) \ 3z - 5w \quad e) \ \text{Re}(z) \quad f) \ \text{Im}(w) \\
g) \ \overline{z} \quad h) \ |w| \\
j) \ w/\overline{w} \quad k) \ zw \quad l) \ z/\overline{w}
\]

The conjugation operation can be used in a variety of ways. For instance if \( z \) and \( \overline{z} \) are added, the result is \( 2 \text{Re}(z) \); that is, \( z + \overline{z} = (x + iy) + (x - iy) = 2x \). In the same manner, you can show that the difference of \( z \) and \( \overline{z} \) is \( 2i \text{Im}(z) \). Furthermore, it’s easy to see that the conjugate of a sum is what it should be (i.e. the sum of the conjugates), and it’s remarkable that the conjugate of a product is also what it should be (i.e. the product of the conjugates). The next exercise asks you to verify these results by doing some examples.

**Exercise 2** If \( z = 2 + 4i \) and \( w = 5 + 2i \)

\[
a) \ \text{show} \ (z + w) = \overline{z} + \overline{w} \\
b) \ \text{show} \ \overline{z - w} = \overline{z} \cdot \overline{w}
\]
In addition to adding, subtracting, multiplying, and dividing complex numbers, we can also evaluate functions of complex numbers. As long as the function is algebraic, we can use our basic rules.

**Exercise 3** Let \( f(z) = z^2 \). Calculate \( f(1+3i) \). Convert both \( 1+3i \) and \( f(1+3i) \) to polar form. How are the magnitudes related? How are the \( \theta \)'s related?

As another example, let \( f(z) = z^2 + 3z \). Evaluating \( f \) requires multiplying and adding. For example, \( f(2 + 3i) = (2 + 3i)^2 + 3(2 + 3i) = 1 + 21i \). Even though this function is more complicated, it is still algebraic and the evaluation can be done fairly easily. The problem comes when the function is not a simple algebraic function. For instance, what is \( \ln(z) \), \( \sin(z) \), or even \( \sqrt{z} \)? Some of these functions are difficult to analyze, and in fact, a detailed study of these types of functions is usually done in an upper level mathematics course devoted to complex variables. Since the non-algebraic function that is most often used in engineering and science is the exponential function, \( e^z \), we will concentrate on its properties (and you will see these properties being applied to our study of second order differential equations).

The algebraic properties for \( e^z \) are the same as for \( e^x \). That is

\[
\begin{align*}
e^{z+w} &= e^z e^w \\
e^0 &= 1 \\
e^{-z} &= \frac{1}{e^z}
\end{align*}
\]

Euler’s formula \((e^{i\theta} = \cos(\theta) + i\sin(\theta))\) allows us to write \( e^z \) in rectangular form. For example,

\[
e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos(y) + i\sin(y)) = e^x \cos(y) + ie^x \sin(y)
\]

The term \( e^z \) is the magnitude and \( y \) determines the angle. Notice that since \( \cos(y) \) and \( \sin(y) \) are periodic, \( e^z \) will also be periodic in the variable \( y \). For instance,

\[
e^{3+(2+2\pi)i} = e^{3+2i+2\pi i} = e^{3+2i} e^{2\pi i} = e^{3+2i}
\]

Also note that adding \( \pi \) to the imaginary part will simply change the direction of \( e^z \).

\[
e^{x+i(y+\pi)} = e^{x+iy+i\pi} = e^{x+iy} e^{i\pi} = e^{x+iy}(-1) = -e^{x+iy}
\]

The next two exercises ask you to evaluate and plot a variety of complex numbers of the form \( e^z \). Note that the number \( e^{i\theta} \) will always be on the unit circle and that the conjugate of \( e^{i\theta} \) is \( e^{-i\theta} \). That is,

\[
e^{i\theta} = \cos(\theta) + i\sin(\theta) = \cos(-\theta) - i\sin(-\theta) = e^{-i\theta}
\]

Note that we used two facts from trigonometry: \( \cos(-\theta) = \cos(\theta) \) and \( \sin(-\theta) = -\sin(\theta) \). A similar argument shows that the conjugate of \( e^z \) is \( e^{-z} \).
Exercise 4 Evaluate and plot (on one set of axes) the following complex numbers.

\[ e^{0 \cdot i} = e^{2 \pi \cdot i} = e^{\pi \cdot i} = e^{2n \pi \cdot i} = e^{\pi \cdot i} = e^{3 \pi \cdot i} = e^{\frac{\pi}{2} \cdot i} = e^{\frac{3 \pi}{2} \cdot i} = e^{\frac{5 \pi}{4} \cdot i} = e^{\frac{7 \pi}{4} \cdot i} = e^{\frac{9 \pi}{4} \cdot i} = \]

An arbitrary complex number of the form \( e^z = e^{x+iy} \) will not be on the unit circle unless \( x = 0 \). For example,

\[ e^{2 + \frac{\pi}{4}i} = e^2 \cdot e^{\frac{\pi}{4}i} = e^2 \left( \cos \left( \frac{\pi}{4} \right) + \sin \left( \frac{\pi}{4} \right)i \right) = e^2 \left( \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right) = 5.22 + 5.22i \]

So when determining the placement of \( e^{x+iy} \) in the complex plane, use \( x \) to determine how far the complex number is from the origin, and use \( y \) to determine the angle the complex number makes with the positive real axis. The real and imaginary parts of \( e^{x+iy} \) are also easy to determine: \( \text{Re}(e^{x+iy}) = e^x \cos(y) \) and \( \text{Im}(e^{x+iy}) = e^x \sin(y) \).

Exercise 5 Evaluate and plot (on one set of axes) the following complex numbers.

\[ e^{1+2\pi i} = e^{0.5+3\pi i} = e^{-2+3\pi i} = e^{1+\pi i} = e^{1-\frac{\pi}{2} i} = e^{-1+\frac{\pi}{2} i} = e^{2+\frac{\pi}{2} i} = e^{-0.01+\frac{\pi}{2} i} = \]

4
Exercise 6  

a) Find a $z$ for which $e^z = e^{i}$.  
b) Find a $z$ for which $e^z = 4i$.  
c) Find 2 $z$’s for which $e^z = 4$.  
d) Find a $z$ for which $e^z = 1 + 2i$. 

Now that we’ve had some practice working with Euler’s formula, let’s look at why it’s true. One way to prove this relationship between the exponential, sine, and cosine functions is to make use of Taylor series, which were introduced in calculus. If we assume (and we will) that the Taylor series for $e^z$ has the same form for complex numbers as for real numbers, then

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{\theta^3}{3!}i + \frac{\theta^4}{4!} + \frac{\theta^5}{5!}i - \frac{\theta^6}{6!} - \frac{\theta^7}{7!}i + \frac{\theta^8}{8!} + \cdots$$

$$= (1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots) + i(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots)$$

$$= \cos(\theta) + i\sin(\theta)$$

The proof isn’t long or difficult; it just requires some knowledge of the Taylor series expansion for the exponential, sine, and cosine functions.
Application to Differential Equations

This section contains an example related to solutions of linear, second order differential equations. As seen in class, when we arrive at the solution to a linear, second order differential equation which has complex numbers, the graph of the solution appears to be a sine or cosine curve (which is possibly decaying). Also when using \texttt{dsolve} in Maple, complex numbers do not appear.

Where did all the complex numbers go?

We give an example showing how to convert the complex form of solutions to a form which has only real numbers. Even though the following example is a bit messy and technical, it does illustrate how Euler’s formula and some complex arithmetic are used to eliminate the complex numbers which initially appear in solutions.

Example

When solving \( \frac{d^2x(t)}{dt^2} + 4x(t) = 0 \), \( x(0) = 0 \), \( x'(0) = 1 \), the characteristic equation is \( r^2 + 4 = 0 \) which has solutions \( r = \pm 2i \). Therefore, the general solution is

\[ x(t) = e^{2it} + e^{-2it} \]

The initial conditions give the equations

\[ c_1 + c_2 = 0 \]
\[ 2ic_1 - 2ic_2 = 1 \]

whose solution is \( \{ c_1 = -\frac{1}{4}i, c_2 = \frac{1}{4}i \} \). Please notice in this example (and in the others) that \( c_1 \) and \( c_2 \) are conjugates. Therefore, the complex form of the solution is

\[ x(t) = -\frac{1}{4}ie^{2it} + \frac{1}{4}ie^{-2it} \]

If we were to plot \( x(t) \) in Maple, we would see a sin curve. In other words, even though there are complex numbers in the expression, the complex numbers seem to disappear when plotting. Euler’s formula comes to the rescue.

\[
x(t) = -\frac{1}{4}ie^{2it} + \frac{1}{4}ie^{-2it} \\
= -\frac{1}{4}i (\cos(2t) + i \sin(2t)) + \frac{1}{4}i (\cos(-2t) + i \sin(-2t)) \\
= -\frac{1}{4}i (\cos(2t) + i \sin(2t)) + \frac{1}{4}i (\cos(2t) - i \sin(2t)) \\
= -\frac{1}{4}i \cos(2t) + \frac{1}{4}i \sin(2t) + \frac{1}{4}i \cos(2t) + \frac{1}{4}i \sin(2t) \\
= \frac{1}{2} \sin(2t)
\]