Instructions

- Answer all the questions directly on the test.
- Show all the necessary work and write your answers out neatly in English sentences. Use mathematical notation to express your answers, not Maple or Matlab notation.
- It is not necessary to use your computer to answer all of the questions but you can use it for various calculations, and to check your work. If you use Maple or Matlab be sure to say how you use it in a sentence, e.g., “Using Matlab, the solution to these equations is:
- You may not use any prepared Maple or Matlab worksheets or scripts.

<table>
<thead>
<tr>
<th>Question</th>
<th>Possible Points</th>
<th>Points Obtained</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25</td>
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<td>2</td>
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<td>3</td>
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<td>4</td>
<td>20</td>
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<td>Total</td>
<td>100</td>
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</tbody>
</table>
1. Orthogonality and Fundamental Spaces

Consider the matrix $A$ and its reduced echelon form $R$

$$A = \begin{bmatrix} A_1 & A_2 & A_3 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 3 & 5 \\ 3 & 2 & 5 \\ 4 & 1 & 5 \end{bmatrix}$$

$$R = \begin{bmatrix} R_1 & R_2 & R_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

1.a Consider these four spaces $\mathcal{R}(A), \mathcal{N}(A), \mathcal{R}(A^\top), \mathcal{N}(A^\top)$. Fill in the table below. Because of time considerations just fill in two spots in the last column.

<table>
<thead>
<tr>
<th>$V$</th>
<th>$V^\perp$</th>
<th>dim $V$</th>
<th>dim $V^\perp$</th>
<th>basis of $V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{R}(A)$</td>
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<tr>
<td>$\mathcal{N}(A)$</td>
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<tr>
<td>$\mathcal{R}(A^\top)$</td>
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<tr>
<td>$\mathcal{N}(A^\top)$</td>
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</table>
2. Least Squares Projections

2.a Let $A$ and $Y$ be the matrix and vector

\[ A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 5 \\ 1 \\ 0 \\ 1 \end{bmatrix} \]

Write out, in matrix form, the normal equations for finding the least squares solution to $Y = AX + E$, then find $X$. 

2.b Now suppose that
\[
A = \begin{bmatrix}
\frac{3}{5} & -\frac{2}{5} & 0 \\
0 & 0 & -\frac{2}{5} \\
\frac{2}{5} & \frac{3}{5} & 0 \\
0 & 0 & \frac{3}{5}
\end{bmatrix}, \quad Y = \begin{bmatrix}
5 \\
1 \\
0 \\
1
\end{bmatrix}
\]

Verify that the columns of \( A \) are orthogonal (suggestion: what is \( A^\top A \)).

2.c When the columns of \( A \) are orthogonal, such as above, how are the normal equations simplified. Now calculate \( X \) in this case.
3. Gram Schmidt & QR

3.a Perform Gram-Schmidt on the columns of $A$ following

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

3.b Find a $QR$ decomposition of $A$, in which diagonal elements of $R$ are all positive.

3.c In this problem the inner product is $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$. Consider the following set of polynomials $p_1 = 1$, $p_2 = \frac{1-2t}{\sqrt{3}}$ and $p_3 = t^2$. Observe that $\{p_1, p_2\}$ is an orthonormal set. Find a decomposition

$$p_3 = q_1 + q_2$$

where $q_1$ the projection of $p_3$ onto the span of $p_1$ and $p_2$ and $q_2$ is orthogonal to the span of $p_1$ and $p_2$. 
4. Similarity

Recall that matrices $A$ and $B$ are similar if $B = MAM^{-1}$ for some matrix $M$.

4.a Let $A$ be an arbitrary $2 \times 2$ matrix and let $M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Show directly that $\det B = \det A$.

4.b Suppose that $x_0$ is a fixed point of $A$. Show by calculation that $y_0 = Mx_0$ is a fixed point of $B$, i.e., that $By_0 = y_0$. 