Instructions

- Answer all the questions directly on the test.
- Show all the necessary work and write your answers out neatly in English sentences. Use mathematical notation to express your answers, not Maple or Matlab notation.
- It is not necessary to use your computer to answer all of the questions but you can use it for various calculations, and to check your work. If you use Maple or Matlab be sure to say how you use it in a sentence, e.g., “Using Matlab, the solution to these equations is:
- You may not use any prepared Maple or Matlab worksheets or scripts.
- All questions have equal value.

<table>
<thead>
<tr>
<th>Question</th>
<th>Possible Points</th>
<th>Points Obtained</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
1. Linear Independence, Span, Basis

1.a In the table below are listed several vector spaces over \( \mathbb{R} \) and set of vectors in these spaces. Complete the table by writing yes or in each of the empty boxes.

<table>
<thead>
<tr>
<th>Space</th>
<th>Set of vectors</th>
<th>Linearly Indep</th>
<th>Spanning Set</th>
<th>Basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{R}^2 )</td>
<td>( \begin{bmatrix} 1 &amp; 2 &amp; 5 \ 2 &amp; 1 &amp; 4 \end{bmatrix} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mathbb{R}^3 )</td>
<td>( \begin{bmatrix} 1 &amp; 2 &amp; 3 \ 2 &amp; 3 &amp; 1 \ 3 &amp; 2 &amp; -1 \end{bmatrix} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mathbb{P}_4 )</td>
<td>( {x^3 - 1, x - 1, x + 1} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \mathbb{R}^3 )</td>
<td>( \begin{bmatrix} 1 &amp; 2 &amp; 0 \ 2 &amp; 1 &amp; 0 \ 3 &amp; 2 &amp; 1 \end{bmatrix} )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1.b Consider the set \( B = \{e_1, e_2, e_3\} \) and the vector \( v \)

\[
e_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}.
\]

Find the coordinate vector \([v]_B\) by converting the equation \( v = a_1e_1 + a_2e_2 + a_3e_3 \) into a matrix equation and solving.
2. Range and Null Space

Let $A$ be the matrix of a projection from $\mathbb{R}^4$ to $\mathbb{R}^3$

\[
A = \begin{bmatrix}
1 & 2 & 3 & 4 \\
0 & 1 & 1 & 2 \\
1 & 0 & 1 & 0
\end{bmatrix} = [A_1 \ A_2 \ A_3 \ A_4].
\]

The reduced echelon form of $A$ is

\[
R = \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

2.a Find a basis for the nullspace $A$, showing the steps.

2.b Find a basis for the column space, selected from the columns $A_1, A_2, A_3, A_4$.
Write the other columns as linear combinations of the basis columns.

2.c Find $\text{nullity}(A)$ and $\text{rk}(A$). Verify that the rank theorem holds in this case.
3. Change of Basis

Consider these three bases of $\mathbb{R}^3$, $B = \{e_1, e_2, e_3\}$, $B' = \{e'_1, e'_2, e'_3\}$, $B'' = \{e''_1, e''_2, e''_3\}$:

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

$$e'_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e'_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad e'_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix},$$

$$e''_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e''_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad e''_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

Observe that the change of basis matrix from $B$ to $B'$ is given by:

$$[e_1, e_2, e_3] = [e'_1, e'_2, e'_3] A = [e''_1, e''_2, e''_3].$$

3.a If $[v]_B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, find $[v]_{B'}$.

3.b Find the change of basis matrix from $B'$ to $B''$, call it $B$.

3.c Let $C$ be the change of basis matrix from $B$ to $B''$, and $D$ be the change of basis matrix from $B'$ to $B$. Express $C$ and $D$ in terms of $A$ and $B$, and then compute them.
4. Matrix of linear transformation

Let $B = \{p_1, p_2, p_3\} = \{1, x, x^2\}$ be the standard basis of $P_3$ and let $L : P_3 \to P_3$ be the differential operator

$$L(p(x)) = (x + 1)p'(x) - 2p(x)$$

4.a Find the matrix of $L$ with respect to $B$ as a basis for both the domain and range, call it $A$.

4.b Find a $p(x)$ such that $L(p(x)) = 0$.

4.c Find a $p(x)$ such that $L(p(x)) = x$, if possible.