For problems 1-5, let
\[
\begin{bmatrix}
a \\
b
\end{bmatrix},
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix},
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}.
\]
and show the following:

1) for \( f(x) = a^T x \), \( \frac{df}{dx} = a \)

2) for \( f(x) = x^T a \), \( \frac{df}{dx} = a \)

3) for \( f(x) = Ax \), \( \frac{df}{dx} = A^T \)

4) for \( f(x) = A^T x \), \( \frac{df}{dx} = A \)

5) for \( f(x) = x^T A x \), \( \frac{df}{dx} = (A + A^T)x \)

6) The error vector \( e \) between observation vector \( d \) and estimate of the input \( \dot{x} \) is \( e = d - A\dot{x} \). We want to weight the errors by a matrix \( R \), where \( R \) is symmetric \( (R = R^T) \). Find \( \hat{x} \) to minimize \( e^T R e \). (This is a weighted least squares.)

7) Show that any matrix \( A \) can be written as the sum of a symmetric matrix and a skew symmetric matrix. That is,
\[
A = R + Q,
\]
\[
R = R^T,
\]
\[
Q = -Q^T
\]
Determine \( R \) and \( Q \).
8) Assume we expect a process to follow the following equation
\[ y(t) = \frac{1}{ct + d\sqrt{t}} \]
Assume we measure the \( y(t) \) at various times \( t \):

<table>
<thead>
<tr>
<th>( t )</th>
<th>( y(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>0.30</td>
</tr>
<tr>
<td>2.0</td>
<td>0.21</td>
</tr>
<tr>
<td>3.0</td>
<td>0.14</td>
</tr>
<tr>
<td>4.0</td>
<td>0.12</td>
</tr>
<tr>
<td>5.0</td>
<td>0.11</td>
</tr>
<tr>
<td>6.0</td>
<td>0.09</td>
</tr>
</tbody>
</table>

a) Determine a least squares estimate of the parameters \( c \) and \( d \).
b) Estimate the value of \( y(t) \) at \( t = 2.5 \).
c) Suppose we believe all measurements made before time \( t = 3.5 \) are twice as reliable as those made later. Determine a reasonable weighted least squares estimated of \( c \) and \( d \).

9) Assume we expect a process to follow the following equation
\[ \gamma(x) = \epsilon e^{\beta x} \]
Assume we measure the \( \gamma(x) \) at various locations \( x \):

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \gamma(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>2.45</td>
</tr>
<tr>
<td>0.1</td>
<td>2.38</td>
</tr>
<tr>
<td>0.4</td>
<td>2.30</td>
</tr>
<tr>
<td>2.0</td>
<td>1.40</td>
</tr>
<tr>
<td>4.0</td>
<td>0.70</td>
</tr>
</tbody>
</table>

a) Determine a least squares estimate of the parameters \( \epsilon \) and \( \beta \). (Hint: Try logarythms...)
b) Estimate the value of \( \gamma(x) \) at \( x = 3.0 \).
10) Assume we have an experimental process we are modeling and, based on sound physical principles, we assume a relationship between $x$ and $y$ to be

$$y(x) = \left(\frac{\alpha}{x}\right)^\beta$$

and we have the following measurements

<table>
<thead>
<tr>
<th>$y$</th>
<th>$x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>0.3</td>
<td>3</td>
</tr>
<tr>
<td>0.1</td>
<td>4</td>
</tr>
</tbody>
</table>

write out and describe how to use a least squares technique to estimate the parameters $\alpha$ and $\beta$. Assume we have the general form

$$a = Bc$$

What are in the $a$ and $c$ vectors? What is in the $B$ matrix? What are your estimates for $\alpha$ and $\beta$?

*Hint: you cannot solve for $\alpha$ directly. Let $w = \beta \log \alpha$ and solve for $w$ and $\beta$, then infer $\alpha$.*

11) Linearize the following two systems about the origin, and write the results in state variable form

$$\dot{x}_1 = x_1 x_2 + 3 x_2 + u_1^2 + u_2$$
$$\dot{x}_2 = 4 x_1 + x_2 + x_1 u_2^2 + 2 u_1$$

$$\dot{x}_1 = x_1^2 - \sin(3 x_2) + u_1^3 - u_2$$
$$\dot{x}_2 = x_2 - u_1 + x_1 e^{-x_2}$$
\[ \dot{x} = (A^T R A)^{-1} A^T R \dot{d} \]
\[ \dot{y}(t) = (0.8104 t + 2.4707 \sqrt{t})^{-1} \]
\[ \dot{y}(x) = 2.519 e^{-0.3142 x} \]
\[ \dot{y}(x) = \left( \frac{1.974}{x} \right)^{3.1183} \]
\[ \delta \ddot{x} = \begin{bmatrix} 0 & 3 \\ 1 & 1 \end{bmatrix} \delta \dot{x} + \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \delta u \]
\[ \delta \ddot{x} = \begin{bmatrix} 0 & -3 \\ 1 & 1 \end{bmatrix} \delta \dot{x} + \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \delta u \]