Consider a slab of material experiencing transient heat conduction:

\[ \begin{align*}
L & \quad s = \text{const} \\
R & \quad \alpha = \text{const} \\
\end{align*} \]

The new addition to this problem is temporal (time) dependence and as such the solution \( T \) becomes a function of both \( t \) and \( x \). For the following analyses, the unknown solution \( T \) will be called \( Q \) for sake of generality.

\( \mathcal{L}(Q) = \frac{\partial^2 Q(x,t)}{\partial x^2} - \alpha \frac{\partial^2 Q(x,t)}{\partial t^2} - s = 0 \quad \Omega \in (x_L, x_R) \cup t \) (1)

with boundary and initial conditions of

\( Q(x_L,t) = Q_L \quad \partial \Omega_1 \in x_L \\
Q(x_R,t) = Q_R \quad \partial \Omega_2 \in x_R \\
Q(x,t_0) = g(x) \quad \Omega \in (x_L, x_R) \) (2)

Thus, equation (1) has two Dirichlet boundary conditions and an initial condition and the problem statement is therefore well-posed. We begin, as usual, by assuming a continuum approximate form for the unknown solution of temperature \( Q \). Having invested significant theoretical development into \( \Psi_a(x) \), we shall opt to put the time variable into the expansion coefficient \( Q_a \), hence

\( Q(x,t) \approx Q^\Psi(x,t) = \sum_{\alpha=1}^{N} \Psi_a(x) Q(t)_{\alpha} \) (3)

Pushing the approximate form of the solution through the GWS recipe results in

\( GWS^h = S \left( \int_{\Omega} \int_{\Omega} \left[ N_k \right]^T \frac{d}{dt} Q_{\alpha} \frac{d}{dt} + \alpha \int_{\Omega} \frac{d}{dx} \left[ N_k \right]^T \frac{d}{dx} Q_{\alpha} \frac{d}{dx} \frac{d}{dx} \right) \) (4)

which, upon assembly, can be expressed as the syntactical matrix statement

\( [\text{MASS}] \frac{dQ}{dt} + [\text{DIFF}] [Q] - [\text{SRC}] = \{0\} \) (5)

where we now have a first order ODE in time along with our usual spatially discrete form. Our solution plan now becomes

1) Solve the spatial discretization at the current time level

2) “Somehow” propagate the solution forward some time increment \( \Delta t \)

Let’s now figure out how to propagate a scalar ODE equation forward in time and then apply our results to (5).
Taylor series

Suppose we have a scalar function \( f(t) \). Plotting it below:

\[
f(t + \Delta t) \approx f(t) + f'(t) \Delta t + \frac{f''(t)}{2!} (\Delta t)^2 + \frac{f'''(t)}{3!} (\Delta t)^3 + \ldots
\]

We can use the Taylor Series (TS) to estimate values of the function at specific points via the derivatives.

**Forward TS:**

\[
f(t_{n+1}) = f(t_n) + \frac{\Delta t}{1!} f'(t_n) + \frac{\Delta t^2}{2!} f''(t_n) + \frac{\Delta t^3}{3!} f'''(t_n) + \ldots
\]  

(6a)

**Backward TS:**

\[
f(t_n) = f(t_{n+1}) - \frac{\Delta t}{1!} f'(t_{n+1}) + \frac{\Delta t^2}{2!} f''(t_{n+1}) - \frac{\Delta t^3}{3!} f'''(t_{n+1}) + \ldots
\]  

(6b)

We can use both (6a) and (6b) to obtain the “new” values of \( f(t_{n+1}) \) knowing the “old” value of \( f(t_n) \) and the derivative of \( f \) evaluated at either \( t_n \) or \( t_{n+1} \). Note that (6a,b) are infinite series and will need to be truncated to do any numerical work, hence we will be estimating the new value of \( f(t_{n+1}) \). Expressing (6a,b) compactly

**Forward TS (Explicit Euler):**

\[
f(t_{n+1}) = f(t_n) + \Delta t f'(t_n) + \mathcal{O}(\Delta t^2)
\]  

(6a)

**Backward TS (Implicit Euler):**

\[
f(t_{n+1}) = f(t_n) + \Delta t f'(t_{n+1}) + \mathcal{O}(\Delta t^2)
\]  

(6b)

What if we averaged the forward and backward TS?

**Trapezoid (Crank-Nicholson):**

\[
f(t_{n+1}) = f(t_n) + \frac{\Delta t}{2} (f'(t_{n+1}) + f'(t_n)) + \mathcal{O}(\Delta t^2)
\]  

(6c)
We can express all three schemes (6a-c) through an implicitness factor $\Theta$ yielding the Theta Taylor Series ($\Theta TS$)

$$f(t_{n+1}) = f(t_n) + \Delta t \Theta f'(t_{n+1}) + (1 - \Theta f'(t_n)) + O(\Delta t^2, \Delta t^3)$$  \hspace{1cm} (7)

where

- $\Theta = 0$ \hspace{0.5cm} forward TS / Explicit Euler
- $\Theta = 1$ \hspace{0.5cm} backward TS / Implicit Euler
- $\Theta = \frac{1}{2}$ \hspace{0.5cm} trapezoid rule / Crank - Nicholson

**Comments:**

- **Explicit**
  - no iteration required
  - incredibly small time steps

- **Implicit**
  - iteration
  - huge time steps
  - artificial dissipation (not time accurate, valid for steady state only)

- **Crank Nicholson**
  - iteration
  - big time steps
  - no artificial dissipation (time accurate)

Returning to (5), we have a matrix ODE of the form

$$GWS^n = [MASS]Q^n' + [DIFFA]Q^n - \{SRCS\} = \{0\}$$  \hspace{1cm} (8)

Applying the Theta Taylor Series (7)

$$\Theta TS = \{Q\}^{n+1}_x - \{Q\}^n_x + \Delta t \Theta \{Q\}^{n+1}_x + (1 - \Theta)\{Q\}^n_x + O(\Delta t^2, \Delta t^3)$$  \hspace{1cm} (9)

We must now figure out how to substitute (9) into (8) and then solve the resulting matrix statement.