ECE 300
Signals and Systems

Exam 2
26 October, 2009

This exam is closed-book in nature. You are not to use a calculator or computer during the exam. Do not write on the back of any page, use the extra pages at the end of the exam. **You must show your work to receive credit for a problem.**

Problem 1 ________/ 30  
Problem 2 ________/25  
Problem 3 ________/ 20  
Problem 4 ________/ 10  
Problem 5 ________/ 15

Exam 2 Total Score: ________/ 100

90-100 4
80-89 3
70-79 2  
60-69 1  
59 and below 0

*median = 70*
1. Impulse Response (30 points)
For each of the following systems, determine the impulse response \( h(t) \) between the input \( x(t) \) and output \( y(t) \). Be sure to include any necessary unit step functions. For full credit, simplify your answers as much as possible.

\[
\begin{align*}
\text{a) } y(t) &= \int_0^{t-2} e^{-(t-\lambda)} x(\lambda - 2) d\lambda + e^t x(t) \\
&= \begin{cases} 
\frac{t-2}{2} & t \geq 2 \\
\frac{t}{2} & t \geq 4
\end{cases} \\
\text{h}_1(t) &= e^{-(t-2)} \mathcal{U}(t-2) + e^t \delta(t) \\
\text{h}_2(t) &= e^{-(t-2)} \mathcal{U}(t-4) + \delta(t)
\end{align*}
\]

\[
\begin{align*}
\text{b) } 2y(t) + y(t) &= x(t-1) \\
\frac{d}{dt} \left( \frac{1}{2} x(t-1) \right) &= \frac{1}{2} e^{\frac{t}{2}} \mathcal{U}(t-1) \\
\text{h}_1(t) &= \frac{1}{2} e^{\frac{t}{2}} \mathcal{U}(t-1) \\
\text{h}_2(t) &= \frac{1}{2} e^{\frac{t}{2}} \mathcal{U}(t-1)
\end{align*}
\]

\[
\begin{align*}
\text{c) } \text{Determine the impulse response for the following system}
\end{align*}
\]

\[
\begin{align*}
\text{h}_1(t) &= 2u(t-3) \\
\text{h}_2(t) &= 2\delta(t+1)
\end{align*}
\]

\[
\begin{align*}
\text{h}(t) &= \text{h}_1(t) \ast \text{h}_2(t) \\
&= \int_0^t 2u(\lambda - 3) \frac{1}{2} e^{\frac{t-\lambda}{2}} \mathcal{U}(t-\lambda) d\lambda \\
&= \frac{1}{2} e^{\frac{t}{2}} \mathcal{U}(t-3)
\end{align*}
\]

d) If the response of a system to a step of amplitude \( A \) is given by

\[
s(t) = A[1 + e^{-\alpha t}]u(t)
\]

determine the unit impulse response of the system. (Do not just guess the answer, you will probably be wrong, and besides, you need to show your work!)

\[
\text{h}(t) = \frac{d}{dt} s(t) = \left( \frac{d}{dt} [1 + e^{-\alpha t}] \right) u(t) + \left[ 1 + e^{-\alpha t} \right] \frac{d}{dt} u(t)
\]

\[
= -\frac{1}{\alpha} e^{-\alpha t} u(t) + \left[ 1 + e^{-\alpha t} \right] \delta(t)
\]

\[
\text{h}(t) = -\frac{1}{\alpha} e^{-\alpha t} u(t) + 2 \delta(t)
\]
2. Fourier Series (25 points)

The periodic function $x(t)$ is defined over one period $(T_0 = 5 \text{ seconds})$ as

$$x(t) = \begin{cases} 2 & -2 \leq t \leq 1 \\ 0 & 1 \leq t \leq 3 \end{cases}$$

Determine the complex Fourier series coefficients, $c_k$, by evaluating the appropriate integral.

*Be sure to simplify your answer as much as possible and use a sinc function if appropriate.*

$$\omega_0 = \frac{2\pi}{5}$$

$$c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)e^{-j\omega_0 t} dt = \frac{2}{T_0} \left. \frac{e^{-j\omega_0 t}}{-j\omega_0} \right|_{-2}^{1}$$

$$= \frac{2}{\omega_0 T_0 (-j k)} \left[ e^{-j\omega_0} - e^{j\omega_0^2} \right] = \frac{2}{\kappa \pi} \left[ \frac{e^{j\omega_0^2} - e^{-j\omega_0^2}}{\omega_0^2} \right]$$

$$= \frac{2}{\kappa \pi} e^{j\omega_0^2/2} \left[ \frac{e^{j\omega_0^2/2} - e^{-j\omega_0^2/2}}{\omega_0^2} \right] = \frac{2}{\kappa \pi} e^{j\omega_0^2/2} \sin \left( \omega_0^2/2 \right)$$

$$= \frac{2}{\kappa \pi} e^{j\pi/5} \sin \left( \frac{\pi}{2} \frac{3}{5} \frac{\omega_0^2}{2} \right) = \frac{2}{\kappa \pi} e^{j\pi/5} \sin \left( \frac{\pi}{2} \frac{3}{5} \frac{\omega_0^2}{2} \right)$$

$$= 2 e^{j\pi/5} \sin c \left( \frac{3\omega_0^2}{2} \right) = \frac{2e^{j\pi/5} \sin c \left( \frac{3\omega_0^2}{2} \right)}{\omega_0^2} = c_k$$
3. (20 points) A periodic signal has the Fourier series representation \( x(t) = \sum_{k=-\infty}^{k=\infty} c_k e^{jk\omega_0 t} \).

This signal is the input to an LTI system, and the (steady state) output of the system is

\[ y(t) = 4 + 4\cos(4t + 30^\circ) + 6\cos(6t + 30^\circ) \]

Fill in the following table:

| k | \( |c_k|^2 \) | \( |H(j\omega_0)| \) | \( \Delta c_k \) | \( \Delta H(j\omega_0) \) |
|---|---|---|---|---|
| 0 | 2 | 2 | 180° | 180° |
| 1 | 3 | 0 | -45° | |
| 2 | 1 | 2 | 45° | -150° |
| 3 | 0.5 | 0 | -30° | 0° |

\( \omega_0 = 2 \)

If you cannot determine a necessary value, leave the table entry blank.

\[ c_0 = 4 = c_0^\times H(j\omega) = (-2)(-2) = 4 \times 180° \]
\[ c_1 = 0 = c_1^\times H(j\omega) \Rightarrow |H(j\omega)| = 0 \]
\[ c_2 = 2 \times 30° = c_2^\times H(j\omega) = (1 \times 45°)(2 \times -150°) \]
\[ c_3 = 3 \times 30° = c_2^\times H(j\omega) = (0, \times -30°)(0 \times 60°) \]
4. **(10 points)** Assuming the system input \( x(t) = \sum_{k=-\infty}^{\infty} c_k^x e^{jkw_0t} \) and output \( y(t) = \sum_{k=-\infty}^{\infty} c_k^y e^{jkw_0t} \) are related through the LTI system \( y(t) + 2y(t-2) = 6x(t-3) \)

a) Determine the relationship between \( c_k^x \) and \( c_k^y \).

b) Determine the continuous transfer function \( H(j\omega) \) between the input and the output.

\[
\begin{align*}
\sum c_k^y \left[ jkw_0 e^{jkw_0t} + 2e^{jkw_0(t-2)} \right] &= \sum c_k^x \left[ 6 e^{jkw_0(t-3)} \right] \\
\sum c_k^y \left[ jkw_0 + 2e^{-jkw_0/2} \right] e^{jkw_0t} &= \sum c_k^x \left[ 6e^{-jkw_0/3} \right] e^{jkw_0t} \\
\end{align*}
\]

\[
\begin{align*}
c_k^y &= c_k^x \frac{6e^{-jkw_0/3}}{jkw_0 + 2e^{-jkw_0/2}} \\
\end{align*}
\]

\[
\begin{align*}
H(jkw_0) &= \frac{6e^{-jkw_0/3}}{jkw_0 + 2e^{-jkw_0/2}} \\
\end{align*}
\]

So \( H(j\omega) = \frac{6e^{-j\omega/3}}{j\omega + 2e^{-j\omega/2}} \)
5) **(15 points)** The periodic signal \( x(t) \) has the Fourier series representation

\[
x(t) = 2 + \sum_{k=-\infty}^{\infty} \frac{1}{1 + kj} e^{i3t}
\]

\( x(t) \) is the input to an LTI system (a high pass filter with transfer function

\[
H(j\omega) = \begin{cases} 
0 & |\omega| < 5 \\
3e^{-j2\omega} & |\omega| > 5
\end{cases}
\]

The steady state output of the system can be written as

\[
y(t) = ax(t - b) + c + d \cos(e(t - b) + f).
\]

Determine numerical values for the parameters \( a, b, d, e \) and \( f \)

\[
\begin{align*}
C_0 &= 3 \\
C_1 &= \frac{1}{1+j} = \frac{1}{\sqrt{2}} e^{-j45^\circ}
\end{align*}
\]

The filter removes \( k \geq 0 \) and \( k = 1 \) terms

\[
y_1(t) = x(t) - 3 - \frac{2}{\sqrt{2}} \cos(3t - 45^\circ)
\]

\[
\begin{align*}
\text{Am} \ y(t) &= 3y_1(t - 2) \\
y_1(t) &= 3x(t - 2) - 3 - \frac{6}{\sqrt{2}} \cos(3t - 2 - 45^\circ)
\end{align*}
\]