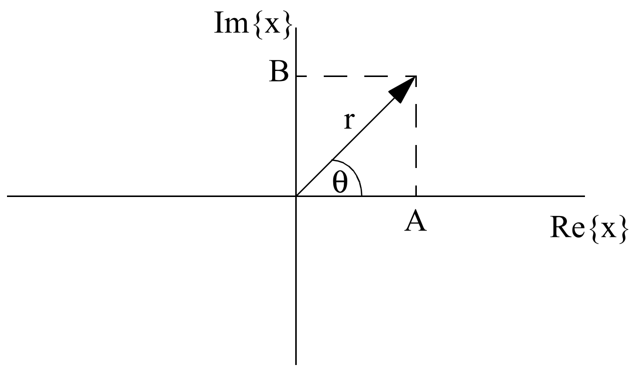
Complex Number Refresher

EE's use $j = \sqrt{-1}$ instead of i, so that we don't confuse current, which we often label as i, with imaginary numbers

A complex number $\underline{x} = A + jB$, note the underlining to denote a complex number.

- A bold font is also often used to denote complex numbers.
- it has a real part Re{x}=A
- it has an imaginary part Im{x}=B

Complex numbers can be drawn as vectors on the COMPLEX PLANE:



As a vector, we can speak of a complex number in terms of it's magnitude, r, and angle, θ.

- $\underline{x} = A + jB = r\angle\theta$
- $r = |\underline{x}|$

To convert between polar and cartesian coordinates:

- If you have cartesian form $\underline{x} = A + jB$
  - $r = |\underline{x}| = \sqrt{A^2 + B^2}$
  - $\theta = \text{atan}\left(\frac{B}{A}\right)$
- if you have polar form $\underline{x} = r\angle\theta$
  - $A = r\cos(\theta)$
  - $B = r\sin(\theta)$

There is another form that we use called Exponential Form

- $re^{j\theta} = r\cos(\theta) + jr\sin(\theta)$ see Euler's Formula on page 158 of textbook
- This is just a complex number with polar form, $r\angle\theta$, and cartesian form, $r\cos(\theta) + jr\sin(\theta)$

So now we have $\underline{x} = A + jB = r\angle\theta = re^{j\theta}$. But WHY do we care about these different forms?

ANSWER = Complex Arithmetic

When doing addition and subtraction of complex numbers use **Cartesian Form**:

- $\underline{x_1} = A + jB$ and $\underline{x_2} = C + jD$
  - $\underline{x_1} + \underline{x_2} = (A + C) + j(B + D)$
  - $\underline{x_1} - \underline{x_2} = (A - C) + j(B - D)$
- Example $\underline{x_1} = -3 + j6$ and $\underline{x_2} = 2 - j3$
  - $\underline{x_1} + \underline{x_2} = -1 + j3$
  - $\underline{x_1} - \underline{x_2} = -5 + j9$

When doing multiplication and division of complex numbers use **Polar Form**

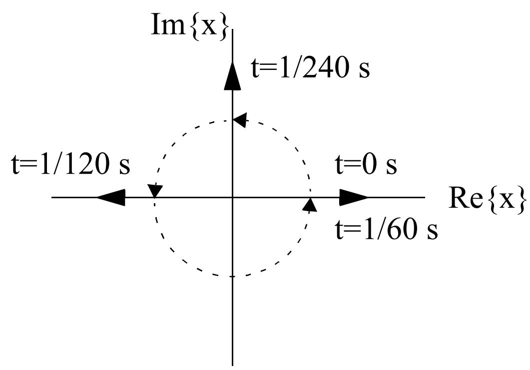
- $\underline{x_1} = r_1\angle\theta_1$ and $\underline{x_2} = r_2\angle\theta_2$
  - $\underline{x_1}\underline{x_2} = r_1 r_2\angle(\theta_1 + \theta_2) = r_1 e^{j\theta_1}(r_2 e^{j\theta_2}) = r_1 r_2 e^{j\theta_1} e^{j\theta_2} = r_1 r_2 e^{j(\theta_1 + \theta_2)}$
  - $\underline{x_1}/\underline{x_2} = \frac{r_1}{r_2}\angle(\theta_1 - \theta_2) = \frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}} = \frac{r_1}{r_2}(e^{j\theta_1} e^{-j\theta_2}) = \frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$
- Example $\underline{x_1} = 3\angle 10°$ and $\underline{x_2} = -2\angle -5°$
  - $\underline{x_1}\underline{x_2} = -6\angle 5°$
  - $\underline{x_1}/\underline{x_2} = -1.5\angle 15°$

But why do we care about the **Exponential Form**?

We represent AC voltages and currents with sinusoids: $A\cos(\omega t)$

- $A\cos(\omega t) = \frac{A}{2}e^{j\omega t} + \frac{A}{2}e^{-j\omega t}$ see Euler's Formula on next page
- $Ae^{j\omega t}$ is a complex number in exponential form, but what does this mean?
- A is simply the magnitude of the vector, so what is $e^{j\omega t}$?
- If we convert this to polar form we have $e^{j\omega t} = 1\angle\omega t$, which is a unit vector with an angle of ωt. Let's assume that $f = 60\text{Hz}$ so that $\omega = 2\pi f = 120\pi$ and see what $e^{j\omega t} = 1\angle\omega t$ looks like in the complex plane...

t	ωt
0	0
1/240 s	$\pi/2$
1/120 s	π
1/60 s	2π

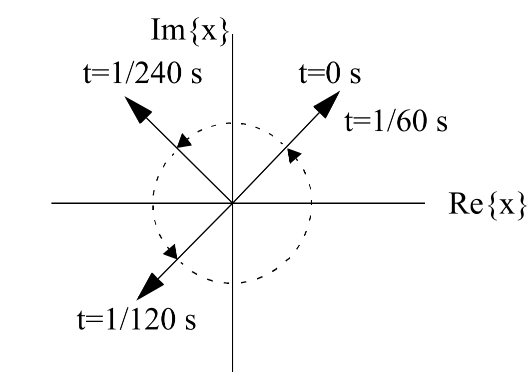


- So what we have is a vector that rotates around the complex plane at 60 times per second.

Sometimes the AC voltages and currents have a phase shift: $A\cos(\omega t + \theta)$

- $A\cos(\omega t + \theta) = \frac{A}{2}e^{j(\omega t + \theta)} + \frac{A}{2}e^{-j(\omega t + \theta)}$
- How does this relate to the example above?
- Let's assume the same frequency as before, but now we have a phase shift of $\theta = \pi/4$.

t	ωt+π/4
0	$\pi/4$
1/240 s	$3\pi/4$
1/120 s	$5\pi/4$
1/60 s	$9\pi/4$



- Produces the same rotating vectors as before except that at time t=0 s the vectors start at +45 degrees from the Real Axis.

Practice Exercises

1) Given three complex numbers, $\underline{w} = 7\angle 15°$, $\underline{z} = 6 - j2$, and $\underline{u} = 10e^{j2.2}$

a) Find $|\underline{z}|$

b) Evaluate $\frac{\underline{z}}{\underline{w} - \underline{u}}$ and place in exponential form.

c) solve for $\underline{s}$ if $\underline{z}(\underline{s}-\underline{w})=\underline{u}$ and express $\underline{s}$ in polar form.

2) Given that $\underline{z} = x + jy$ is a general complex number...

a) show that 1/$\underline{z}$ = x/(x2+y2) - j y/(x2+y2)

b) solve for the first time, t, when $Re\{|\underline{z}|e^{j\omega t}\} = 0$ if $\underline{z}$=2-j6 and ω=100.

Derivation of Euler's Formula

First do a series expansion of an exponential: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

Then make the exponent imaginary: $e^{j\theta} = 1 + (j\theta) + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \ldots$

Looking at each of the terms: $(j\theta)^2 = -\theta^2$ is a real number so all terms that are raised to an even power are real, and $(j\theta)^3 = -j\theta^3$ is an imaginary number so all terms that are raised to an odd power are imaginary

If we group the real and imaginary terms we have: $e^{j\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)$

It just so happens that the real term is the series expansion for $\cos(\theta)$ and the imaginary term is the series expansion for $\sin(\theta)$.

This results in Euler's Formula: $e^{j\theta} = \cos(\theta) + j\sin(\theta)$