Groups, Rings, and Fields – MA 478

A *Group* is closed, associative, has an identity, and every element has an inverse.

1 Let G be the four element set $\{(1,1),(1,-1),(-1,1),(-1,-1)\}$,
and let $(a,b) \oplus (c,d) = (ac, bd)$. Prove that G is a group.

2 Lt $i^2 = -1$. Prove that the set of complex numbers $\{+1,-1,+i,-i\}$ is a group under multiplication.

3 Prove that the elements $\{1,3,5,7\}$ under multiplication modulo 8 form a group.

4 Prove that the set consisting of all 2×2 matrices with determinant 1 along with the operation matrix multiplication form a group.

5 Let $\{a,b,c,d\}$ be all elements in a group with binary operation $\oplus$. Construct all possible "addition" tables. Note that changing the spelling of the elements does not count as a fundamentally different group.

A *Ring* is a commutative group under $\oplus$, and is closed and associative over $\otimes$, with a distributive law for $\otimes$ over $\oplus$. The additive identity is referred to as the 'zero' of the ring.

A *Field* is a ring in which the non-zero elements under $\otimes$ form a commutative group.

6 Prove that the set $\{0,3,6,9\}$ is a ring under addition and multiplication modulo 12.

7 Prove that the set of rational numbers forms a field.

8 Prove that the set of residues modulo 31 forms a field.

9 Prove that the set of residues modulo 35 does not form a field. Is it a ring?

10 An element, a, is a *torsion element* of a ring if $a^n = a \otimes a \otimes \cdots \otimes a = 0$ for some positive integer n. Prove that in a commutative ring, if a and b are torsion elements, then $a \oplus b$ is a torsion element. (all such elements are called *nilpotent*.)

11 Let R be a ring. Then the *polynomial ring* $R[x]$ is the ring consisting of polynomials in x with coefficients contained in R. Show that $\mathbf{Z}[x]$ is a ring, where $\mathbf{Z}$ is the ordinary (rational) integers.