MA490-01 - Number Theory II - Galois Groups

The splitting field, **F**, for a polynomial $p(t)$ over the field **K** is the smallest field containing all the zeros of $p(t)$.

A **K**-automorphism of a field, **F**, containing the field **K** is an automorphism $\sigma : \mathbf{F} \to \mathbf{F}$ so that for all $k \in \mathbf{K}$, $\sigma(k) = k$.

The Galois Group Gal[**F** : **K**] is the group of all **K**-automorphisms of **F** under composition of maps.

For each of the following fields **K** and polynomials $p(t)$, do parts **A**, **B**, **C**, **D**.

1: $p(t) = t^2 - 4$, $\mathbf{K} = \mathbf{R}$
2: $p(t) = t^2 - 3$, $\mathbf{K} = \mathbf{R}$.
3: $p(t) = t^2 - 3$, $\mathbf{K} = \mathbf{Q}$.
4: $p(t) = t^2 + 1$, $\mathbf{K} = \mathbf{Q}$
5: $p(t) = t^2 + 1$, $\mathbf{K} = \mathbf{Z}/3\mathbf{Z}$
6: $p(t) = t^2 + 1$, $\mathbf{K} = \mathbf{Z}/5\mathbf{Z}$
7: $p(t) = t^4 - 5t^2 + 6$, $\mathbf{K} = \mathbf{Q}$
8: $p(t) = t^4 + 5t^2 + 6$, $\mathbf{K} = \mathbf{Q}$.
9: $p(t) = t^4 + 5t^2 + 6$, $\mathbf{K} = \mathbf{Z}/7\mathbf{Z}$.
10: $p(t) = t^7 - 3t^6 + 4t^3 - t - 1$, $\mathbf{K} = \mathbf{R}$.

A Let α be a root of $p(t)$. Describe the possible field extensions $\mathbf{K}(\alpha)$.
B Determine the degrees of each of the extension fields $[\mathbf{K}(\alpha) : \mathbf{K}]$.
C Determine the degree of the splitting field, **F**, of $p(t)$ over **K**.
D Describe the Galois Group Gal[**F** : **K**], where **F** is the splitting field of $p(t)$ over **K**.