Automorphic Numbers

A number is called automorphic modulo m, $0 \leq a \leq m-1$ and $a^2 \equiv a(\text{mod } m)$. Thus 6 is automorphic modulo 10 since $6^2 \equiv 6(\text{mod } 10)$. notice that this is equivalent to saying that if the last digit of the base m representation of a number is a then the last digit of the base m representation of the square of the number is also a.

A number is automorphic base m if a is a non-negative integer satisfying $a < m^n$ and $a^2 \equiv a(\text{mod } m^n)$. Thus the final n digits of the base m representation of a remain unchanged when a is squared. For example, $76^2 \equiv 76(\text{mod } 100)$ so the last two digits of any number ending in 76 remain unchanged when the number is squared.

1. For each integer $m = 1, 2, \ldots, 10$ find all automorphic numbers modulo m. Do you see any patterns? If so, what are they?

2. Find all automorphic numbers modulo 10^n for $n = 1, 2, 3, 4, 5, 6$. What patterns do you notice?

3. Show that if a is an automorphic number modulo m, then so is $m + 1 - a$.

4. Find all automorphic numbers modulo 7^n for $n = 1, 2, 3, \ldots, 10$. Do you notice a pattern? Can you prove that this pattern persists?

5 Find a base b so that there are more than four automorphic numbers modulo b^n for $n = 1, 2, 3, 4, 5$.

6. Find all moduli m with k automorphic numbers for $k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots$. Describe any patterns that you find. Prove that these patterns persist.