Math Modeling

Allen Broughton

December 1, 2001

Chapter 1

Basic Probability

1.1 Introduction

The concept of probability is used to model situations which are random or uncertain in an individual case but show a quantifiable frequency over a large number of cases. In the most typical and most intuitive instance probability describes relative frequency. Suppose for instance that a perfectly fair six-sided die was thrown 600 times and the results were.

Die face	1	2	3	4	5	6	Total
Number of occurences	95	106	103	93	102	101	600
Frequency	0.158	0.177	0.172	0.155	0.170	0.168	1.000

The last row is the fraction to the throws that are a 1, 2, and so on. Even though in this experiment we cannot predict what the 435'th throw will be a one we do see that the fraction of 1's that we see is close to 0.167 the 1 in 6 chance we deduce by assuming that die faces are equally likely. Given a large experiment of 6,000 throws we might get something like

Die face	1	2	3	4	5	6	Total
Number of occurences	981	1010	1012	985	1035	977	6000
Frequency	0.164	0.168	0.169	0.164	0.173	0.163	1.000

and the frequencies are much closer to 0.1667 than before. As the experiments get larger and larger we expect that all the frequencies will get closer and closer to the theoretically frequency of 1/6. Thus an informal definition of the empirical or measured probability of an event is

$$\text{empirical probability} = \frac{\#\text{favorable cases}}{\#\text{ of cases considered}}$$

and the theoretical

$$\text{theoretical probability} = \text{limit}\ \frac{\#\text{favorable cases}}{\#\text{ of cases considered}}$$

as the number of cases become infinite. Normally, theoretical probability is found by some arguments such as the 1 of 6 equally likely cases, and then the theoretical probability is used to predict the number of favorable cases.

1.2 Definition and Rules of Probability

Definition 1 *Let S be some non-empty set of* outcomes*. The set S, called the* universe *or* sample space*, will often be finite but need not be so. A subset $A \subseteq S$ is called and* event*. A probability measure is a function P, such that for each event $A \subseteq S$ there is associated a probability $P(A)$ satisfying:*

$$\begin{aligned} 0 \leq P(A) \leq 1 \\ P(\emptyset) = 0,\ P(S) = 1,\ \textit{and} \\ \textit{if } A \cap B = \emptyset \textit{ then } P(A \cup B) = P(A) + P(B). \end{aligned}$$

The simplest case is that of a finite set of equally likely outcomes. In this case we simply define:

$$P(A) = \frac{\#A}{\#S}.$$

Example 2 *In the case of the die the set S may be thought of as $S = \{1, 2, 3, 4, 5, 6\}$ and the event "die face is even" would be the set $A = \{2, 4, 6\}$. Thus*

$$P(A) = \frac{\#\{2, 4, 6\}}{\#\{1, 2, 3, 4, 5, 6\}} = \frac{3}{6} = 0.5.$$

Example 3 *Now let S be the set of all possible throws of a pair of dice and A the event "die sum is 4". Now S may be thought of the set of all ordered pairs $\{(i, j) : 1 \leq i, j \leq 1\}$. Indeed suppose that the dice are red and green and that i is the number on the red die face and j is the number on the green die face. It follows that $\#S = 36$. Now $A = \{(1, 3), (2, 2), (3, 1)\}$, so*

$$P(A) = \frac{\#\{(1, 3), (2, 2), (3, 1)\}}{\#\{1, 2, 3, 4, 5, 6\}} = \frac{3}{36} = 1/12 = 0.083.$$

Some basic facts about probability are summarized in the following Proposition.

Example 4 *Let S be a region in the plane and for any set $A \subseteq S$, let*

$$P(A) = \frac{area(A)}{area(S)}.$$

If S has unit area then the formula is just $P(A) = area(A)$. Note that the set S is infinite. also the probability of a point or a smooth curve with no thickness is zero.

Proposition 5 *Let P be a probability measure on a set S, and let $A, B, A_1, A_2, \ldots$ be various subsets. Then the following hold:*

1. *If $A \subseteq B$ then $P(A) \leq P(B)$, i.e., P is monotone.*

2. *Let $A_1, A_2, ..., A_n$ be pairwise disjoint sets (mutually exclusive events), i.e., $A_i \cap A_j = \emptyset$. Then*

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n).$$

3. *If in addition $A_1 \cup A_2 \cup \cdots \cup A_n = S$, then the A_i form a partition of S (form a set of mutually exclusive, exhaustive events). In this case*

$$P(A_1) + P(A_2) + \cdots + P(A_n) = 1.$$

4. *If $A \subseteq S$ then the complement $\overline{A}$ is defined by $\overline{A} = S - A = \{x \in S : x \notin A\}$. Then we have:*

$$P(\overline{A}) = 1 - P(A).$$

We give an indication about why these may be true in the next section, The next proposition concerns non-disjoint unions.

Proposition 6 *Let P be a probability measure on a set S, and let $A_1, A_2, \ldots$ be various subsets. Then the following hold:*

$$\begin{aligned}
P(A_1 \cup A_2) &= P(A_1) + P(A_2) - P(A_1 \cap A_2). \\
P(A_1 \cup A_2 \cup A_3)) &= P(A_1) + P(A_2) + P(A_3) \\
&\quad - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) \\
&\quad + P(A_1 \cap A_2 \cap A_3). \\
P(A_1 \cup A_2 \cup \cdots \cup A_n) &= \sum_i P(A_i) - \sum_{i \neq j} P(A_i \cap A_j) + \\
&\quad \sum_{i \neq j \neq k} P(A_i \cap A_j \cap A_k) - \cdots + \\
&\quad (-1)^{n-1} P(A_1 \cap A_2 \cap \cdots \cap A_n).
\end{aligned}$$

1.3 Venn Diagrams

A convenient way to show the relationships among probabilities among Venn diagrams. In the Venn diagrams we will use the universe is shown by a box., and the various subsets are regions drawn in the box. If two regions have an intersection hen they are shown to overlap. From Example 4 we see that if S has unit area then the probability of a set is just its area.

Example 7 *Proof by Venn diagram. The proofs for all the statement of Proposition 5 are given in the Figure 1. The proof of and the case $n = 2$ and $n = 3$ of Proposition 6 are in Figure 2.*

blank page for Figure 1

blank page for Figure 2

1.4 Conditional probability and independence

Suppose that there are three red marbles and two green marbles in an urn. Suppose that two marbles are drawn from the urn. What is the probability of getting two reds under the following schemes, replace a marble after it is chosen, don't replace. One way to solve the problem is to number the red marbles $1, 2, 3$ and the green marbles $4, 5$. Then a possible sample space for the replacement model is the set of 25 equally likely ordered pairs $\{(i, j)1 \le i, j \le 5\}$ and for the non-replacement model is set of 20 equally likely ordered pairs $\{(i, j)1 \le i, j \le 5, i \ne j\}$. Let RR denote the event of choosing two reds in sequence, then with replacement we get

$$P_r(RR) = \frac{\#\{(i, j)1 \le i, j \le 3\}}{\#\{(i, j)1 \le i, j \le 5\}} = \frac{9}{25},$$

and without replacement we get

$$P_{nr}(RR) = \frac{\#\{(i, j)1 \le i, j \le 3, i \ne j\}}{\#\{(i, j)1 \le i, j \le 5, i \ne j\}} = \frac{6}{20},$$

where the subscript on P denote replacement or no replacement, when we need to distinguish. There is however an more effective way to compute the probability in either case. Let R_i denote the event of selecting a red on the i'th trial and G_i similarly defined. Note that $S = R_i \cup G_i$ as depicted in the Venn diagram in Figure 3 so that $P(G_i) = 1 - P(R_i)$. Also as depicted in the diagram we get $P(RR) = P(R_1 \cap R_2)$ and Lets calculate $P_r(RR)$ and $P_{nr}(RR)$ through a bit of simple but unmotivated algebra:

$$P(RR) = P(R_1 \cap R_2) = P(R_1)\frac{P(R_1 \cap R_2)}{P(R_1)}.$$

The value is that the two factors may be interpreted as easily calculated probabilities. First note that $P_r(R_1)$ and $P_{nr}(R_1)$ probability of one red marble out of 3 reds and 2 greens. Thus

$$P_r(R_1) = P_{nr}(R_1) = \frac{3}{5}.$$

Nothing new here. However we do encounter a difference when we select the second ball. The quantity $\frac{P(R_1 \cap R_2)}{P(R_1)}$ is the ratio of the doubly hatched area to the entire shade area. Thus, following Example 4, we may think of $\frac{P(R_1 \cap R_2)}{P(R_1)}$ as the restricted probability of the event $R_1 \cap R_2$ when we restrict our universe to $S = R_1$. This restricted probability is given a special name called the conditional probability with the notation

$$P(R_2|R_1) = \frac{P(R_1 \cap R_2)}{P(R_1)}.$$

The interpretation of conditional probability is what is the probability that event R_2 will occur (R chosen second) given that we know that R_1 must happen

(R chosen first) If both happen the we are in $R_1 \cap R_2$ but we are restricting our consideration to R_1. Given this interpretation it is easy to calculate the conditional probabilities in the two different cases

$$P_r(R_2|R_1) = \frac{3}{5}$$
$$P_{nr}(R_2|R_1) = \frac{2}{4} = \frac{1}{2}.$$

In the replacement case on the second choice there are still 3 reds and 2 greens. In the non-replacement case there are 2 reds and 2 greens. we easily get the table:

E	RR	RG	GR	GG
$P_r(E)$	$\frac{3}{5} \cdot \frac{3}{5} = \frac{9}{25}$	$\frac{3}{5} \cdot \frac{2}{5} = \frac{6}{25}$	$\frac{2}{5} \cdot \frac{3}{5} = \frac{6}{25}$	$\frac{2}{5} \cdot \frac{2}{5} = \frac{4}{25}$
$P_{nr}(E)$	$\frac{3}{5} \cdot \frac{2}{4} = \frac{6}{20}$	$\frac{3}{5} \cdot \frac{2}{4} = \frac{6}{20}$	$\frac{2}{5} \cdot \frac{3}{4} = \frac{6}{20}$	$\frac{2}{5} \cdot \frac{1}{4} = \frac{2}{20}$

Observe that these calculations match earlier calculations.

Definition 8 *Let P be a probability measure on a set S, and let $A, B, A_1, A_2, \dots A_n$ be various subsets. Then the conditional probability of A given that B has already occurred is:*

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

If events A_1 and A_2 have already occurred, the conditional probability of A_3 givenA_1 and A_2 is

$$P(A_3|A_1, A_2) = \frac{P(A_1 \cap A_2 \cap A_3)}{P(A_1 \cap A_2)}.$$

More generally,

$$P(A_n|A_1, \dots, A_{n-1}) = \frac{P(A_1 \cap A_2 \cap \dots \cap A_n)}{P(A_1 \cap A_2 \cap \dots \cap A_{n-1})}.$$

Proposition 9 *Let the terminology be as in definition 8, Then*

$$P(A \cap B) = P(B)P(A|B)$$
$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2|A_1)P(A_3|A_1, A_2)$$
$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1)P(A_2|A_1) \bullet \dots \bullet P(A_n|A_1, \dots, A_{n-1})$$

Proposition 10 *Law of total probability: Let P be a probability measure on a set S, and let $A, B, A_1, A_2, \dots, A_n$ be various subsets. Then*

$$P(B) = P(A_1)P(B|A_1) + \dots + P(A_n)P(B|A_n)$$

blank page for Figure 3 & 4

blank page for Figure 5 & 6

Definition 11 *Two events A and B are independent if The probability of B occurring does not depend on A and vice versa. Thus*

$$P(B|A) = P(B)$$

or

$$P(A \cap B) = P(A)P(B|A) = P(A)P(B)$$

Example 12 *The events R_1 and R_2 are independent in the scheme with replacement but not in the scheme without replacement.*

1.5 Bernouilli trials and binomial probabilities

Let us look at probabilities associated to repeated independent experiments. For example, the repeated tossing of a coin, selecting a marble from an urn (with replacement) a baseball player with repeated at bats, a series of games between two teams. Such a sequence of independent event with the same probability of success at each stage are called Bernouilli trials. Let us develop the formula for the probability by considering flipping a coin in n successive, independent trials. Suppose the coin is weighted and the probability of a head (success) at any stage is p the probability of a tail (failure) is $q = 1 - p$. Let H_j and T_j be the complementary events of choosing a head or tail during trial j. We will also assume that all the trials are independent that means for any two distinct i, j

$$\begin{aligned} P(H_i \cap H_j) &= P(H_i)P(H_j) = p^2 \\ P(H_i \cap T_j) &= P(H_i)P(T_j) = pq \\ P(T_i \cap H_j) &= P(H_i)P(T_j) = qp \\ P(T_i \cap T_j) &= P(H_i)P(T_j) = q^2 \end{aligned}$$

And for any triple of distinct i, j, k we have the eight relations

$$\begin{aligned} P(H_i \cap H_j \cap H_k) &= P(H_i)P(H_j)P(H_k) = p^3 \\ P(H_i \cap H_j \cap T_k) &= P(H_i)P(H_j)P(T_k) = p^2q \\ P(H_i \cap T_j \cap H_k) &= P(H_i)P(T_j)P(H_k) = pqp \\ &\cdots \\ P(H_i \cap T_j \cap H_k) &= P(T_i)P(T_j)P(T_k) = q^3 \end{aligned}$$

and so on for any selection of distinct indices. A sample space may be constructed of all ordered sequences of H's and T's with n elements in the sequence. The probability of a sequence is $p^s q^t$ where there are s H's and t T's and $s + t = n$. Thus for $n = 4$ the points of the sample space and their probabilities are:

points	probability
$HHHH$	p^4
$HHHT, HHTH, HTHH, THHH$	p^3q
$HHTT, HTHT, THHT, HTTH, THTH, TTHH$	p^2q^2
$HTTT, THTT, TTHT, TTTH$	pq^3
$TTTT$	q^4

Then we see that

$$\begin{aligned} H_1 &= \{HHHH, HHHT, HHTH, HTHH, HHTT, HTHT, HTTH, HTTT\} \\ T_2 &= \{HTHH, HTHT, HTTH, TTHH, HTTT, TTHT, TTTH, TTTT\} \\ H_1 \cap T_2 &= \{HTHH, HTHT, HTTH, HTTT\} \end{aligned}$$

and

$$\begin{aligned} P(H_1) &= p^4 + 3p^3q + 3p^2q^2 + pq^3 = p(p^3 + 3p^2q + 3pq^2 + q^3) = \\ &= p(p+q)^3 = p \cdot 1^3 = p, \\ P(T_2) &= p^3q + 3p^2q^2 + 3pq^3 + q^4 = (p^3 + 3p^2q + 3pq^2 + q^3)q = q, \\ P(H_1 \cap T_2) &= p^3q + 2p^2q^2 + pq^3 = pq(p^2 + 2pq + q^2) = pq(p+q)^2 = pq. \end{aligned}$$

so that

$$P(H_1 \cap T_2) = P(H_1)P(T_2)$$

Thus the probability that we get s H's and t T's the number of ways we can write down s H's and t T's in some order. This in turn can be calculated as follows. Number the positions for the H's and T's , $1, 2, \ldots, n$. Let us select the positions for the s H's, once that is done the rest of the positions are filled in with T's. The first may be selected n ways, the second $n-1$ ways, and so on until there are $n-s+1$ choices for the last H. That is a total of

$$n \times (n-1) \times (n-2) \times \cdots \times (n-s+1) = \frac{n!}{(n-s)!}.$$

ways. Now in a given selection of s positions the first may have been selected in s ways the second in $s-1$ ways and so on. Thus we have overcounted by a factor of $s \times (s-1) \times \cdots \times 1 = s!$ ways and the number of ways of choosing constructing the sequence is

$$\binom{n}{s} = \frac{n!}{(n-s)!s!}.$$

The number $\binom{n}{s}$ is called a binomial coefficient and may be computed iteratively from Pascal's triangle, viz., $\binom{n}{s}$ is the $s+1$ entry of row $n+1$.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

The name, of course, comes from the binomial theorem.

For any two commuting real (or complex) quantities x and y

$$(x+y)^n = x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots \binom{n}{n-2} x^2 y^{n-2} + \binom{n}{n-1} x y^{n-1} + y^n.$$

The punch line of this section is the following theorem on binomial probabilities

Proposition 13 *Consider n independent Bernoulli trial with a probability p of success and q of failure. The the probability of s successes is*

$$\textit{probability of } s \textit{ successes} = \binom{n}{s} p^s q^{n-s}.$$

1.6 Decision trees and conditional probability

One of the applications of probability to modeling is reliability, which is prediction of failures. Sometimes the probabilities for a complex system can be most easily calculated by a decision or fault tree. The decision tree illustrates the two rules of conditional probability discussed earlier. Let's model a process that consists of a sequence of decisions or actions.

For an example consider the process of selecting 3 marbles without replacement form an urn. The tree starts with an node or root at the top as in Figure 7. Coming down from the top node are two branches representing the selections of red and green. From these two nodes are two more branches representing the selection of the second marble. We continue until the three marbles have been chosen experiment is over. Note that along the branch starting with the two greens, there is no more branching since only reds may be chosen. Any downward path along branches can be represented by a sequence of symbols, e.g., RRG would denote the downward path of selecting two reds, and then a green. Along each branch we write the conditional probability of the next decision. To make this more explicit let R_j and G_j be the events of choosing red and green at stage j. Thus the event of selecting the path RRG would be $R_1 \cap R_2 \cap G_3$. The two branches descending from the root node to the level 1 R-nodes and G-nodes would have the probabilities $P(R_1) = \frac{3}{5}$ and $P(G_1) = \frac{2}{5}$, respectively, written next to them. Following the path from the first R node to the second R-node we write the conditional probability $P(R_2|R_1) = \frac{2}{4}$ next to this branch. On the next branch down the RRG path we write $P(G_3|R_2, R_1) = \frac{2}{3}$. The probability for the event represented by the path RRG is the product of the conditional probabilities along the branches:

$$\begin{aligned} P(RRG) &= P(R_1 \cap R_2 \cap G_3) \\ &= P(R_1)P(R_2|R_1)P(G_3|R_2, R_1) \\ &= \frac{3}{5} \times \frac{2}{4} \times \frac{2}{3} = \frac{1}{5}. \end{aligned}$$

blank page for compatibility

blank page for Figure 7

1.7 Exercises

Exercise 14 *If a batter is hitting .300, the probability 0.3. In a game with 4 AB (at bats), how likely is that the player will get at least one hit?*

Exercise 15 *What are the breakdowns for number of hits; 0,1,2,3,4?*

Exercise 16 *How likely is it that a .300 hitter will have a 10 game hitting streak? a 20 game streak, assuming 4 AB's in each game.*

Exercise 17 *Do the previous exercises for all of the above for a .350 hitter?*

Exercise 18 *Assume that each of two teams are equally likely to win a game;*

Exercise 19 *What is the probability that a team leading the World Series 3-2 will win the series?*

Exercise 20 *How likely is that a team will sweep?*

Exercise 21 *What is the probability that the series will last 4,5,6,7 games?*

Exercise 22 *If a team takes a 1-0 lead, what's the probability that they will win the series?*

Exercise 23 *What if the home team wins* 55% *of the games and the series goes AABBBAA?*

Exercise 24 *Draw the complete tree diagram of conditional probabilities for choosing a marble without replacement from an urn containing three red and two green marbles. Using the tree determine*

- *the probability that the last marble is green,*
- *the probability that two greens are chosen in row during the entire exercise,*
- *the probability the three red marbles are chosen first.*

Exercise 25 *The tree is simple a visual way of enumerating all possible cases. Can you find quicker ways to compute the above probabilities, without enumeration or drawing the tree?*

Chapter 2

Random variables

Describing events as subsets the sample space as we have done until now can be a bit clumsy at times. Probability models are often best described in terms of random variables.

Definition 26 *A random variable $X : S \to \mathbb{R}$ is a real valued function on the sample space.*

Example 27 *Let S denote the sample space for the coin flipping experiment. Let $X_1 = 1$ if the first toss is a head and $X_1 = 0$ if the first toss is a tail. Define X_2, X_3, X_4 analogously. Thus for the outcome $HHTH \in S$ we have $X_1(HHTH) = 1, X_2(HHTH) = 1, X_3(HHTH) = 0, X_4(HHTH) = 1$.*

Example 28 *Consider the selection of marbles from the urn. Define $C_t = 1$ if red is chosen on selection t and $C_t = 0$ if green is selected. Thus $R_2 = \{\omega \in \Omega : W_2(\omega) = 1\}$. Here we would take as our sample space the set of all 120 permutations of the quintuple $(1, 2, 3, 4, 5)$. i.e., picking the numbered marbles in some order. Thus the point $\omega = (4, 1, 2, 3, 5)$ would give a color sequence $GRRRG$ and $C_2(\omega) = 1$.*

Example 29 *Let S denote the sample space for tossing a die, and X the face shown on a toss. Additional consider a game in which you lose \$1 for an odd die face and gain \$2 of an even die face. The random variable Y that represents the game's value to you is defined by*

Notation 30 *The subsets H_j and T_j can be described as*

$$H_j = \{\omega \in S : X_j = 1\}, T_j = \{\omega \in S : X_j = 0\}.$$

In turn the corresponding probabilities are described as

$$P(H_j) = P(\{\omega \in S : X_j = 1\}) = P(X_j = 1),$$
$$P(T_j) = P(\{\omega \in S : X_j = 0\}) = P(X_j = 0).$$

Example 31 *Random variable can be combined in various ways. The sum $X_1 + \cdots + X_n$ counts the number successes in n Bernouilli trials. Thus for instance*

$$X_1(HHTH) + X_2(HHTH) + X_3(HHTH) + X_4(HHTH) = 1 + 1 + 0 + 1 = 3$$

the number of heads obtained. The theorem on binomial probabilities then becomes

$$P(X_1 + \cdots + X_n = s) = \binom{n}{s} p^s q^{n-s}.$$

Definition 32 ***Notation 33*** *Sometimes we use the symbol Ω to denote the sample space and ω to denote a point in it.*

2.1 Expectation

Often we will want to know the average number of occurrences of a given event or the average value of a given quantity on the sample space. The are two ways of calculating these means, an empirical way and a theoretical way. Consider the die tossing experiment in Section 1.1 and the game described in Example29. Let $\omega_1, \ldots, \omega_{600} \in \{1, 2, 3, 4, 5, 6\}$ be the 600 initial die tosses. The quantity $Y(\omega_t)$ will be the gain or loss on toss t. The average winnings equals

$$\begin{aligned}
&\frac{Y(\omega_1) + Y(\omega_2) + \cdots + Y(\omega_{600})}{600} \\
&= Y(1)\frac{\#\{\omega_t = 1\}}{600} + \cdots + Y(6)\frac{\#\{\omega_t = 6\}}{600} \\
&= -1 \times \frac{95}{600} + 2 \times \frac{106}{600} - 1 \times \frac{103}{600} + 2 \times \frac{93}{600} - 1 \times \frac{102}{600} + 2 \times \frac{101}{600} \\
&= 0.500.
\end{aligned}$$

For the 600 trials we would get

$$\begin{aligned}
&\frac{Y(\omega_1) + Y(\omega_2) + \cdots + Y(\omega_{600})}{6000} \\
&= Y(1)\frac{\#\{\omega_t = 1\}}{6000} + \cdots + Y(6)\frac{\#\{\omega_t = 6\}}{6000} \\
&= -1 \times \frac{981}{6000} + 2 \times \frac{1010}{6000} - 1 \times \frac{1012}{6000} + 2 \times \frac{985}{6000} - 1 \times \frac{1035}{6000} + 2 \times \frac{977}{6000} \\
&= 0.486.
\end{aligned}$$

For a large number N of trials, in which we let $N \to \infty$, we get

$$\begin{aligned}
&\frac{Y(\omega_1) + Y(\omega_2) + \cdots + Y(\omega_N)}{N} \\
&= Y(1)\frac{\#\{\omega_t = 1\}}{N} + \cdots + Y(6)\frac{\#\{\omega_t = 6\}}{N}.
\end{aligned}$$

As $N \to \infty$ this last quantity approaches $Y(1)P(Y=1) + \cdots + Y(6)P(Y=1)$ which equals

$$-1 \times \frac{1}{6} + 2 \times \frac{1}{6} - 1 \times \frac{1}{6} + 2 \times \frac{1}{6} - 1 \times \frac{1}{6} + 2 \times \frac{1}{6} = 0.500.$$

Following this example we define the *expectation* or *mean value* of a random variable.

Definition 34 *Let $X : \Omega \to \mathbb{R}$ with a discrete set of values $x_1, x_2, \ldots, x_n, \ldots$. Then the expectation or mean values of X is given by.*

$$E(X) = x_1 P(X = x_1) + x_2 P(X = x_2) + \cdots + x_2 P(X = x_2) + \cdots$$

taking a finite or infinite sum as appropriate. If the sample space Ω is finite (countable) then we have an alternative definition

$$E(X) = \sum_{\omega \in \Omega} X(\omega) P(\omega). \tag{2.1}$$

We think of the expectation as the limit of average of samples $\omega_1, \ldots, \omega_N \in \Omega$ form the sample space

$$E(X) = \lim_{N \to \infty} \frac{X(\omega_1) + X(\omega_2) + \cdots + X(\omega_N)}{N}. \tag{2.2}$$

The following is easily shown from either equation 2.1 or 2.2.

Proposition 35 *Let $X < Y$ be random variables on Ω and let a and b be scalars.*

$$E(aX + bY) = aE(X) + bE(Y).$$

Example 36 *The average number of successes in n Bernouilli trials with probability of heads is np. The number of success in n trials is given by the random variable $X_1 + \cdots + X_n$ in Example 31. Then we have $E(X_1 + \cdots + X_n) = E(X_1) + \cdots + E(X_n)$. But for each i $E(X_i) = 1 \cdot p + 0 \cdot .q = p$. I follows that $E(X_1 + \cdots + X_n) = nE(X_1) = np$.*

2.2 Stochastic Processes

In many of the example the points in our sample space correspond to a series of decisions, occurrences or trials. Often the outcome of the trials can be measured by a sequence of random variables. Such a sequence of random variables on a sample space $X_t : \Omega \to \mathbb{R}$, is called a *stochastic process.* We often carry out the probability modeling by using the stochastic process and do not concern our selves with the construction of the underlying sample space.

Example 37 *Here are the stochastic processes associated to some of the examples we have looked at so far:*

- *Flipping a coin* n *times and* $X_t = 1$ *if the outcome is a head on trial* t *and* 0 *for a tail.*
- *Repeatedly rolling a die and* $X_t =$ *dieface # at flip* t.
- *Choosing marble from the urn and letting* C_t *be the color number at selection* t *as described in Example 28.*

Chapter 3

Markov Chains

A system of differential equations tells us how a system changes over time. To model the probabilistic changes of a system over time we use a Markov chain. Let us start out with a game of chance. Suppose that a number of gamblers have fortunes from \$0 to \$5. The gamblers play in game that costs one dollar, if they win they get \$2. Thus in a single play of the game thee fortune can go up or down by \$1. Players with a fortune of \$0 (broke) can no longer play since there is no credit. Moreover, the players are disciplined, so that once the achieve a fortune of \$5 (flush) they quit. The probability of winning is always the same value p and the probability of losing is $q = 1 - p$. Players stay in until they are broke or flush. Various questions arise

- What percentage of gamblers go broke, or leave the game flush
- How long does the average gambler stay in.
- What is the change of a \$1 player going to maximum fortune in only 4 games.
- If you start with $\$k$ what is the probability of going broke.
- If the house modifies the chance of winning how do the answers to the above change?

Let us set up the mathematical machinery to answer these problems. We define the following:

$\mathcal{S}$ $=\{0, 1, 2, 3, 4, 5\}$ the set of states.

Ω = sample space, to be defined below, we may think of this as the set of all possible gamblers' histories of fortunes .

$X_t : \Omega \rightarrow \mathcal{S}$ a random variable that tells us a gamblers fortune at play t. The initial distribution of fortunes is given by X_0.

Here is a way to construct a sample space. suppose that a particular gambler starts with \$3 and wins, loses, loses, wins, wins, loses, loses, loses in the first 8 plays. Then the history of fortunes in the first 8 plays is $\{3, 4, 3, 2, 3.3, 2, 1, 0\}$ We can consider the gamblers fortune in future plays of the game to always be zero and so we can use the infinite sequence $\omega = \{3, 4, 3, 2, 3.3, 2, 1, 0, 0, 0, 0, \ldots\}$ to represent the gambler's history of fortunes. Let Ω be the set of all such fortunes $\omega = \{\omega_0, \omega_1, \omega_2, \ldots\}$ such that the states ω_t satisfies the following transition rules:

if $\omega_t = 0$ then $\omega_{t+1} = 0$
if $\omega_t = 5$ then $\omega_{t+1} = 5$
if $0 < \omega_t < 5$ then $\omega_{t+1} = \omega_t + 1$ with probability p
if $0 < \omega_t < 5$ then $\omega_{t+1} = \omega_t - 1$ with probability q

The state ω_0 is the initial fortune. The random variable X_t measures the fortune a time t so that

$$X_t = \omega_t.$$

The transition rules in the table are conveniently summarized by a labelled directed graph, as shown in Figure 8. Draw 6 nodes in a line and label them $0, 1, 2, 3, 4, 5$. Next for the nodes $k = 1, 2, 3, 4$ draw a directed transition arc from k to $j = k + 1$ labelled with the probability p. For the same nodes draw a directed transition arc from k to $j = k - 1$ labelled with the probability q. For the nodes 0 and 5 draw a directed transition loop back to itself labelled with probability 1.

Now let us consider calculating the distribution of fortunes after several plays of the game. Using the random variable X_0 we can define the column vector P_0 of initial probabilities by:

$$\begin{aligned} P_0 &= \left[\begin{array}{cccc} P(X_0 = 0) & P(X_0 = 1) & \cdots & P(X_0 = 5) \end{array}\right]^\top \\ &= \left[\begin{array}{cccc} p_0(0) & p_1(0) & \cdots & p_5(0) \end{array}\right] \end{aligned}$$

This is simply the list of percentages of gamblers with the various initial fortunes in the order $0, 1, 2, 3, 4, 5$. More generally, we may want to consider the percentages after t plays of the game

$$\begin{aligned} P_t &= \left[\begin{array}{cccc} P(X_t = 0) & P(X_t = 1) & \cdots & P(X_t = 5) \end{array}\right]^\top \\ &= \left[\begin{array}{cccc} p_0(t) & p_1(t) & \cdots & p_5(t) \end{array}\right]. \end{aligned}$$

We think of $p_j(t)$ as the time varying probability of being in state j.

blank page for Figure 8

One of the main questions in using Markov chains is to determine the behavior of P_t as t varies. To this end let us determine P_1 We allow one play of the game and condition on the value of X_0. Since we will use this conditioning argument many times let us set it up in the format of Proposition 10 the law of total probability. Let $A_k = \{\omega \in \Omega : X_0(\omega) = k\}$ i.e., the gamblers with initial fortunes k. Let $B_j = \{\omega \in \Omega : X_1(\omega) = j\}$. Then

$$\begin{aligned} P(B_j) &= P(B_j|A_0)P(A_0) + P(B_j|A_1)P(A_1) + \cdots + P(B_j|A_5)P(A_5) \\ &= P(X_1 = j|X_0 = 0)P(X_0 = 0) + \cdots + P(X_1 = j|X_0 = 5)P(X_0 = 5). \end{aligned} \tag{3.1}$$

In the second expression we have written everything out in terms of the stochastic process. For each j, k set

$$p_{j,k}(1) = P(X_1 = j|X_0 = k).$$

Then 3.1 may be written

$$p_j(1) = \sum_{k=0}^{5} p_{j,k}(1)p_k(0). \tag{3.2}$$

Let $P(1)$ be the matrix

$$P(1) = \begin{bmatrix} p_{0,0}(1) & p_{0,1}(1) & \cdots & p_{0,5}(1) \\ p_{1,0}(1) & p_{1,1}(1) & \cdots & p_{1,5}(1) \\ \vdots & \vdots & \ddots & \vdots \\ p_{5,0}(1) & p_{5,1}(1) & \cdots & p_{5,5}(1) \end{bmatrix}$$

The j, k entry is the conditional probability of going from k to fortune j. Using the graph and its interpretation we get

$$P(1) = \begin{bmatrix} 1 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 \\ 0 & p & 0 & q & 0 & 0 \\ 0 & 0 & p & 0 & q & 0 \\ 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & p & 1 \end{bmatrix}$$

From the form of equation 3.2 we determine that

$$P_1 = P(1)P_0 \tag{3.3}$$

a simple matrix multiplication. Moving along in the play, we further define

$$p_{j,k}(t) = P(X_t = j|X_{t-1} = k)$$

and

$$P(t) = \begin{bmatrix} p_{0,0}(t) & p_{0,1}(t) & \cdots & p_{0,5}(t) \\ p_{1,0}(t) & p_{1,1}(t) & \cdots & p_{1,5}(t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{5,0}(t) & p_{5,1}(t) & \cdots & p_{5,5}(t) \end{bmatrix},$$

The transition matrix at time t. In our case all of the transition matrices are the same

$$P(t) = \begin{bmatrix} 1 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 \\ 0 & p & 0 & q & 0 & 0 \\ 0 & 0 & p & 0 & q & 0 \\ 0 & 0 & 0 & p & 0 & 0 \\ 0 & 0 & 0 & 0 & p & 1 \end{bmatrix},$$

and we say that the Markov chain is stationary.

Then it is not hard to show that, similarly to the derivation of equation 3.3 we get

$$P_t = P(t)P_{t-1}$$

by conditioning on the variable X_{t-1} Combining these rules we get

$$\begin{aligned} P_2 &= P(2)P(1)P_0, \\ P_3 &= P(3)P(2)P(1)P_0, \\ P_t &= P(t)\cdots P(2)P(1)P_0. \end{aligned}$$

and in our specific case

$$P_t = P^t P_0.$$

Notation 38 *The numbering of the states of a Markov chain with n states is usually $1, 2, \ldots, n$. We will frequently use the notation U_t to represent the probability distribution at time t.*

3.1 Stochastic matrices

A probability vector is a vector on non-negative values that sums to 1. We can test to see if an vector represented by $n \times 1$ column matrix $U = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}^\top$ is a probability vector as follows. Let V be the $1 \times n$ row matrix all of whose entries are 1. Then$VU = [u_1 + u_2 + \cdots + u_n] = [1]$. If P is a transition matrix for a Markov chain then the columns sums are all 1 or what is the same $VP = V$. A matrix whose column sums are 1 is called a (column) stochastic matrix. To see why the column sums are 1 we argue as follows: The k'th column sum equals

$$P(X_{t+1} = 1 | X_t = k) + \cdots + P(X_{t+1} = n | X_t = k)$$

This is the same as

$$\frac{P(\{X_{t+1} = 1\} \cap \{X_{t+1} = k\})}{P(X_{t+1} = k)} + \cdots + \frac{P(\{X_{t+1} = n\} \cap \{X_{t+1} = k\})}{P(X_{t+1} = k)}$$

or

$$\frac{P(\{X_{t+1} = 1\} \cap \{X_{t+1} = k\}) + \cdots + P(\{X_{t+1} = n\} \cap \{X_{t+1} = k\})}{P(X_{t+1} = k)}$$

or

$$\frac{P(X_{t+1}=k)}{P(X_{t+1}=k)} = 1$$

It is traditional in Markov chain theory to represent probability vectors as row matrices an to multiply by the transition matrix on the right $U \to UP$. The translation from our set up to the traditional set up is $U \to U^\top$, $P \to P^\top$ $V \to V^\top$. The transition rules is $PU \to (PU)^\top = U^\top P^\top$. Finally the equation $VP = V$ translates to $V^\top P^\top = V^\top$ which say that the matrices in the traditional model are row stochastic.

3.2 Limits and speed of convergence

One of the key useful features of stationary Markov chains is that after some initial transient behavior the distributions settle down into a regular pattern. Indeed, in many cases, the limit $U_\infty = \lim_{t\to\infty} U_t$ exists. This concept is most easily explored by the means of eigenvalues, eigenvectors and diagonalization. We will not develop a general theory for Markov chains in what follows but confine ourselves to examples where the transition matrix is diagonalizable. We state without proof, the main theorem on the eigenvalues of a stochastic matrix.

Theorem 39 *Let P be a stochastic matrix. Then the eigenvalues λ of P satisfy $|\lambda| \leq 1$. There is always an eigenvalue of 1 and this is achieved for some probability eigenvector U, i.e. $PU = U$. If $U_\infty = \lim_{t\to\infty} U_t = \lim_{t\to\infty} P^t U_0$, exists then U_∞ is such an eigenvector, i.e., $PU_\infty = U_\infty$. Furthermore, if some power P^t of the transition matrix has all positive entries. then there is a unique probability vector$PU = U$ and $U = \lim_{t\to\infty} P^t U_0 for$ every possible initial distribution.*

Now suppose that $1 = \lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of P with $Y_1, Y_2, \ldots, Y_n$ the corresponding set of eigenvectors so that $PY_i = \lambda_i Y_i$. We can use these eigenvectors to get a useful representation of P. To this end let E_i be the diagonal matrix with a 1 at the (i,i) entry and zero elsewhere. For example in the 2×2 case $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Now let Q be the matrix

$$\begin{aligned} Q &= \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_n \end{bmatrix}, \\ D &= diag(\lambda_1, \lambda_2, \ldots, \lambda_n). = \lambda_1 E_1 + \lambda_2 E_2 + \cdots + \lambda_n E_n, \end{aligned}$$

e.g.,

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

in the 2×2 case. Now

$$\begin{aligned} PQ &= \begin{bmatrix} PY_1 & PY_2 & \cdots & PY_n \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 Y_1 & \lambda_2 Y_2 & \cdots & \lambda_n Y_n \end{bmatrix} \\ &= \begin{bmatrix} Y_1 & Y_2 & \cdots & Y_n \end{bmatrix} D \\ &= QD. \end{aligned}$$

If the eigenvectors are linearly independent then Q is invertible and $PQ = QD$ implies that $P = QDQ^{-1}$. Now

$$\begin{aligned} P &= QDQ^{-1} \\ &= Q(\lambda_1 E_1 + \lambda_2 E_2 + \cdots + \lambda_n E_n)Q^{-1} \\ &= \lambda_1 Q E_1 Q^{-1} + \lambda_2 Q E_2 Q^{-1} + \cdots + \lambda_n Q E_n Q^{-1}. \end{aligned}$$

For the 2×2 case if $Q^{-1} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$, for row vectors Z_1 and Z_2, then

$$\begin{aligned} P &= \lambda_1 Q E_1 Q^{-1} + \lambda_2 Q E_2 Q^{-1} \\ &= \lambda_1 Q E_1 E_1 Q^{-1} + \lambda_2 Q E_2 E_2 Q^{-1} \\ &= \lambda_1 \begin{bmatrix} Y_1 & Y_2 \end{bmatrix} E_1 E_1 \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} + \lambda_2 \begin{bmatrix} Y_1 & Y_2 \end{bmatrix} E_2 E_2 \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \\ &= \lambda_1 \begin{bmatrix} Y_1 & 0 \end{bmatrix} \begin{bmatrix} Z_1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 & Y_2 \end{bmatrix} \begin{bmatrix} 0 \\ Z_2 \end{bmatrix} \\ &= \lambda_1 Y_1 Z_1 + \lambda_2 Y_2 Z_2. \end{aligned}$$

3.2.1 Powers

Now let us consider power P^t. We observe that

$$\begin{aligned} P^2 &= QDQ^{-1}QDQ^{-1} = QDDQ^{-1} = QD^2Q^{-1}, \\ P^3 &= P^2P = QD^2Q^{-1}QDQ^{-1} = QD^2DQ^{-1} = QD^3Q^{-1} \end{aligned}$$

and inductively,

$$P^t = P^{t-1}P = QD^{t-1}Q^{-1}QDQ^{-1} = QD^{t-1}DQ^{-1} = QD^tQ^{-1}.$$

Now the value of this is that

$$\begin{aligned} D^t &= \begin{bmatrix} \lambda_1^t & 0 & \cdots & 0 \\ 0 & \lambda_2^t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^t \end{bmatrix} \\ &= \lambda_1^t E_1 + \lambda_2^t E_2 + \cdots + \lambda_n^t E_n \end{aligned}$$

and hence

$$\begin{aligned} P^t &= Q(\lambda_1^t E_1 + \lambda_2^t E_2 + \cdots + \lambda_n^t E_n)Q^{-1} \\ &= \lambda_1^t Q E_1 Q^{-1} + \lambda_2^t Q E_2 Q^{-1} + \cdots + \lambda_n^t Q E_n Q^{-1}. \end{aligned}$$

Now in a good case we may assume that the first few eigenvalues equal $\lambda_1 = \lambda_2 = \cdots = \lambda_s = 1$ and that the remaining eigenvalues satisfy $|\lambda_j| < 1$. Then we have:

$$P^t = QE_1Q^{-1} + \cdots + QE_sQ^{-1} + \lambda_{s+1}^t QE_{s+1}Q^{-1} \cdots + \lambda_n^t QE_nQ^{-1}$$

Then we may split P^t into steady state term + transient term $P^t = SS + TR$

$$\begin{aligned} SS &= QE_1Q^{-1} + \cdots + QE_sQ^{-1} \\ &= Q(E_1 + \cdots + E_s)Q^{-1} \\ TR &= \lambda_{s+1}^t QE_{s+1}Q^{-1} \cdots + \lambda_n^t QE_nQ^{-1}. \end{aligned}$$

The speed with which $TR \to 0$ depends on the largest absolute value among $|\lambda_{s+1}|, |\lambda_{s+2}|, \ldots, |\lambda_n|$. Rather than give a general statement of the rate of convergence, let us go through an illustrative 4×4 example. Let

$$P = \begin{bmatrix} 1 & .2 & 0 & 0 \\ 0 & 0 & .3 & 0 \\ 0 & .8 & 0 & 0 \\ 0 & 0 & .7 & 1 \end{bmatrix},$$

a matrix similar to those we obtained in the gambler's ruin problem. The eigenvalues and eigenvectors are:

$$1.0 \leftrightarrow \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$.4899 \leftrightarrow \left\{ \begin{bmatrix} -.2047 \\ .5222 \\ .8528 \\ -1.1703 \end{bmatrix} \right\}$$

$$-.4899 \leftrightarrow \left\{ \begin{bmatrix} -.0787 \\ .5863 \\ -.9543 \\ .44983 \end{bmatrix} \right\}$$

The matrix Q is given by:

$$Q = \begin{bmatrix} 1 & 0 & -.2047 & -.0787 \\ 0 & 0 & .5222 & .5863 \\ 0 & 0 & .8528 & -.9543 \\ 0 & 1 & -1.1703 & .44983 \end{bmatrix}$$

and

$$Q^{-1} = \begin{bmatrix} 1.0 & .2629 & .0791 & 0 \\ 0 & .7344 & .9226 & 1.0 \\ 0 & .9559 & .5873 & 0 \\ 0 & .8542 & -.5231 & 0 \end{bmatrix}$$

Thus

$$\begin{aligned} SS &= \begin{bmatrix} 1 & 0 & -.205 & -.079 \\ 0 & 0 & .522 & .586 \\ 0 & 0 & .853 & -.954 \\ 0 & 1 & -1.170 & .450 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1.0 & .263 & .079 & 0 \\ 0 & .734 & .923 & 1.0 \\ 0 & .956 & .587 & 0 \\ 0 & .854 & -.523 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1.0 & .2629 & .0791 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & .7344 & .9226 & 1.0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} TR = (.4899)^t & \begin{bmatrix} 1 & 0 & -.205 & -.079 \\ 0 & 0 & .522 & .586 \\ 0 & 0 & .853 & -.954 \\ 0 & 1 & -1.170 & .450 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1.0 & .263 & .079 & 0 \\ 0 & .734 & .923 & 1.0 \\ 0 & .956 & .587 & 0 \\ 0 & .854 & -.523 & 0 \end{bmatrix} + \\ + (-.4899)^t & \begin{bmatrix} 1 & 0 & -.205 & -.079 \\ 0 & 0 & .522 & .586 \\ 0 & 0 & .853 & -.954 \\ 0 & 1 & -1.170 & .450 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1.0 & .263 & .079 & 0 \\ 0 & .734 & .923 & 1.0 \\ 0 & .956 & .587 & 0 \\ 0 & .854 & -.523 & 0 \end{bmatrix} \\ = (.4899)^t & \begin{bmatrix} 0 & -.196 & -.120 & 0 \\ 0 & .450 & .307 & 0 \\ 0 & .815 & .501 & 0 \\ 0 & -1.119 & -.687 & 0 \end{bmatrix} + (-.4899)^t \begin{bmatrix} 0 & -.0672 & .041 & 0 \\ 0 & .501 & -.307 & 0 \\ 0 & -.815 & .450 & 0 \\ 0 & .384 & -.235 & 0 \end{bmatrix} \end{aligned}$$

The transient part must then be smaller than

$$\begin{aligned} (.4899)^t & \begin{bmatrix} 0 & .196 & .120 & 0 \\ 0 & .499 & .307 & 0 \\ 0 & .815 & .501 & 0 \\ 0 & 1.119 & .687 & 0 \end{bmatrix} + (.4899)^t \begin{bmatrix} 0 & .0672 & .041 & 0 \\ 0 & .501 & .307 & 0 \\ 0 & .815 & .450 & 0 \\ 0 & .384 & .235 & 0 \end{bmatrix} \\ = (.4899)^t & \left(\begin{bmatrix} 0 & .196 & .120 & 0 \\ 0 & .499 & .307 & 0 \\ 0 & .815 & .501 & 0 \\ 0 & 1.119 & .687 & 0 \end{bmatrix} + \begin{bmatrix} 0 & .0672 & .041 & 0 \\ 0 & .501 & .307 & 0 \\ 0 & .815 & .450 & 0 \\ 0 & .384 & .235 & 0 \end{bmatrix} \right) \\ = (.4899)^t & \left(\begin{bmatrix} 0 & .263 & .161 & 0 \\ 0 & 1.0 & .613 & 0 \\ 0 & 1.630 & 1.00 & 0 \\ 0 & 1.503 & .923 & 0 \end{bmatrix} \right) \end{aligned}$$

Thus if we wish P^t to differ from SS by at most 10^{-s} then we need

$$\begin{aligned}(.4899)^t 1.630 &\leq 10^{-s}\\ \ln((.4899)^t 1.630) &\leq \ln(10^{-s})\\ -.7136t + .4886 &\leq -2.302s\\ t &\geq \frac{2.302s + .4886}{.7136} = 3.2259s + .6847.\end{aligned}$$

Thus we know how many iterations are required to achieve a given decimal place accuracy. For a more complex example we would consider an equation similar to $(.4899)^t 1.630 \leq 10^{-s}$ with the $(.4899)^t$ replaced by $|\lambda|^t$ where $|\lambda|$ is the largest absolute value smaller than 1.

3.3 Recurrent and transient states, transition payoffs

If we look at P^t for large t in the gambler's ruin problem we see that except for the two absorbing states the rows of the matrix are close to zero. Therefore the probability for transiting into these states, is small and goes to zero as $t \to \infty$. We call these states *transient* state. On the other hand suppose that for some state the corresponding row goes to a non zero limit. Then there is a positive probability $> \rho > 0$ of returning to that state for infinitely many values of t. We call such a state *recurrent*. A state is *absorbing* if it is impossible to leave once you enter. certainly recurrent states are absorbing. In the gamblers ruin problem all states except being broke or achieving maximum fortune are transient. In the Markov model of baseball 3 outs is an absorbing state, all others are transient.

Now consider some sort of score or payoff that is dependent on transitions. In the Markov model of baseball certain transitions result in a scored run. For instance the only way to transit from a man on first and no outs to no-one on base and no outs is to score two runs. Runs are not scored in transitions resulting in an out, nor is it possible to score from the absorbing state. In the gambler's ruin problem we may want to consider the expected number of plays before going broke. Thus we could consider a "score" in which you count a 1 if you play and 0 if you are out. As we count over all transitions and all iterations we total up the expected number of plays. If we average over all gambles we get the expected number of plays per gambler.

Let us consider the following simple Markov model pictured in Figure 9.

blank page for Figure 9

The transition matrix for this model is

$$P = \begin{bmatrix} .5 & .4 & .1 & .1 \\ .5 & .6 & .1 & .1 \\ 0 & 0 & .3 & .2 \\ 0 & 0 & .5 & .6 \end{bmatrix}.$$

Note that it is possible to go from states 3 and 4 to 1 and 3 but not back. Thus eventually one will always end up going back and fourth between states 1 and 2 but never return to states 3 and 4. Thus states 1 and 2 are recurrent and states 3 and 4 are transient. To corroborate this we look at the $\lim_{t\to\infty} P^t$:

$$\lim_{t\to\infty} P^t = \begin{bmatrix} 4/9 & 4/9 & 4/9 & 4/9 \\ 5/9 & 5/9 & 5/9 & 5/9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now consider a possible game model and payoffs for this process. States $1, 2, 3$ and 4 represents 4 types of games played. Each time a player plays one of games 3 or 4 you collect \$1. If a player switches to games 1 or 2 you collect a one time fee of \$200 (from the operator of games 1 and 2), but those players are lost to you forever. The question here is what is the expected payoff per player. Construct a matrix of payoffs R for this situation as follows:

$$R = \begin{bmatrix} 0 & 0 & 200 & 200 \\ 0 & 0 & 200 & 200 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The entry $R_{j,k}$ is the payoff you receive from a player transiting from state k to state j. The corresponding matrix to count the plays in the gambler's ruin problem with a maximum fortune of \$3 would be

$$R = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Not that in both cases the columns are zero for recurrent states.

Now consider an arbitrary Markov process with n states, a transition matrix P, initial distribution U_0 and a matrix of transition payoffs R. Later we shall suppose that the states are labeled so that recurrent states are first. We may write the transition matrix as

$$P = \begin{bmatrix} Q_1 & V \\ 0 & Q_2 \end{bmatrix}$$

Where Q_1 and Q_2 are the transition matrices among recurrent and transient states respectively and V is a matrix representing the flow from transient to recurrent states. Also assume for this case that the matrix of payoffs has the form

$$R = \begin{bmatrix} 0 & S \end{bmatrix}$$

where the 0 represents no payoff for transitions from recurrent states to recurrent states. Finally split the initial distribution between the recurrent and transient states as:

$$U_0 = \begin{bmatrix} Y_0 \\ Z_0 \end{bmatrix}.$$

Finally let T be the $n \times 1$ matrix $\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$.

Now let us consider the payoff for the first transition. The transition $k \to j$ has payoff $R_{j,k}$ and probability $P(X_1 = j, X_0 = k) = P(X_1 = j|X_0 = k)P(X_0 = k) = P_{j,k}U_0(k)$. The expected payoff from this transition is then $R_{j,k}P_{j,k}U_0(k)$. where $U_0(k)$ is the k'th entry of U_0 Thus the expected payoff, obtained by summing over all transitions is

$$\sum_{j=1}^{n}\sum_{k=1}^{n} R_{j,k}P_{j,k}U_0(k).$$

For subsequent transitions we note that the transition $k \to j$ at the t'th stage has probability

$$P(X_t = j, X_{t-1} = k) = P(X_t = j|X_{t-1} = k)P(X_{t-1} = k) = P_{j,k}U_{t-1}(k).$$

Thus the t'th expected payoff is

$$\sum_{j=1}^{n}\sum_{k=1}^{n} R_{j,k}P_{j,k}U_{t-1}(k).$$

It will be helpful to express the above as a matrix product. Let B be the matrix defined by:

$$B_{i,j} = R_{j,k}P_{j,k},$$

the entry-wise "product" of R and P. In our example

$$B = \begin{bmatrix} 0 & 0 & 200 \times .1 & 200 \times .1 \\ 0 & 0 & 200 \times .1 & 200 \times .1 \\ 0 & 0 & 1 \times .3 & 1 \times .2 \\ 0 & 0 & 1 \times .5 & 1 \times .6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 20.0 & 20.0 \\ 0 & 0 & 20.0 & 20.0 \\ 0 & 0 & .3 & .2 \\ 0 & 0 & .5 & .6 \end{bmatrix}$$

Then the quantity $\sum_{k=1}^{n} R_{j,k}P_{j,k}U_{t-1}(k)$ is the j'th element of the vector BU_{t-1}. Thus $\sum_{j=1}^{n}\sum_{k=1}^{n} R_{j,k}P_{j,k}U_{t-1}(k)$ is the sum of the entries of the vector BU_{t-1} and so

$$\sum_{j=1}^{n}\sum_{k=1}^{n} R_{j,k}P_{j,k}U_{t-1}(k) = TBU_{t-1}.$$

Thus we may compute the total expected payoff over all iterations and over all transitions as:

$$\text{total payoff } = \sum_{t=1}^{\infty} TBU_{t-1} = \sum_{t=0}^{\infty} TBU_t = \sum_{t=0}^{\infty} TBP^t U_0.$$

Now it would be tempting to write

$$\text{total payoff } = TB\left(\sum_{t=0}^{\infty} P^t\right) U_0 = TB\left(I-P\right)^{-1} U_0,$$

except that the matrix series is not convergent. However, note that:

$$\begin{aligned}
P^2 &= \begin{bmatrix} Q_1 & V \\ 0 & Q_2 \end{bmatrix}\begin{bmatrix} Q_1 & V \\ 0 & Q_2 \end{bmatrix} = \begin{bmatrix} Q_1^2 & Q_1V+VQ_2 \\ 0 & Q_2^2 \end{bmatrix} \\
P^3 &= \begin{bmatrix} Q_1^2 & Q_1V+VQ_2 \\ 0 & Q_2^2 \end{bmatrix}\begin{bmatrix} Q_1 & V \\ 0 & Q_2 \end{bmatrix} = \begin{bmatrix} Q_1^3 & Q_1^2V+Q_1VQ_2+VQ_2^2 \\ 0 & Q_2^3 \end{bmatrix} \\
&\dots \\
P^t &= \begin{bmatrix} Q_1^t & V_t \\ 0 & Q_2^t \end{bmatrix}, V_t = Q_1^tV + Q_1^{t-1}VQ_2 + \cdots + VQ_2^t.
\end{aligned}$$

By using the other block structure of our matrices we have

$$B = \begin{bmatrix} 0 & B_1 \end{bmatrix}$$

because of the structure of the matrix of payoffs. Thus

$$\begin{aligned}
BP^tU_0 &= \begin{bmatrix} 0 & B_1 \end{bmatrix}\begin{bmatrix} Q_1^t & V_t \\ 0 & Q_2^t \end{bmatrix}\begin{bmatrix} Y_0 \\ Z_0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & B_1Q_2^t \end{bmatrix}\begin{bmatrix} Y_0 \\ Z_0 \end{bmatrix} \\
&= B_1Q_2^tZ_0
\end{aligned}$$

as block matrices. But now, since Q_2 is transient, i.e., $\lim_{t\to\infty} Q_2^t, so$ all its eigenvalues are less than 1 in modulus. It then follows$I - Q_2$ is invertible and that:

$$\begin{aligned}
\sum_{t=0}^{\infty} Q_2^t &= \lim_{N\to\infty} \sum_{t=0}^{N} Q_2^t \\
&= \lim_{N\to\infty} (I-Q_2)^{-1}(I-Q_2^{N+1}) \\
&= (I-Q_2)^{-1}.
\end{aligned}$$

Thus we get our final formula:

$$\text{total payoff } = TB_1\left(\sum_{t=0}^{\infty} Q_2^t\right) Z_0 = TB_1 \lim_{N\to\infty} (I-Q_2)^{-1} Z_0,$$

In out particular case:

$$B_1 = \begin{bmatrix} 20.0 & 20.0 \\ 20.0 & 20.0 \\ .3 & .2 \\ .5 & .6 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} .3 & .2 \\ .5 & .6 \end{bmatrix}$$

$$(I - Q_2)^{-1} = \begin{bmatrix} .7 & -.2 \\ -.5 & .4 \end{bmatrix}^{-1} = \begin{bmatrix} 2.2222 & 1.1111 \\ 2.7778 & 3.8889 \end{bmatrix}$$

$$Z_0 = \begin{bmatrix} .25 \\ .25 \end{bmatrix}$$

assuming a uniform distribution initially. The total payoff is then:

$$\begin{aligned} \text{total payoff } &= TB_1(I - Q_2)^{-1}Z_0 \\ &= \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 20.0 & 20.0 \\ 20.0 & 20.0 \\ .3 & .2 \\ .5 & .6 \end{bmatrix} \begin{bmatrix} 2.2222 & 1.1111 \\ 2.7778 & 3.8889 \end{bmatrix} \begin{bmatrix} .25 \\ .25 \end{bmatrix} = \\ & 102.0. \end{aligned}$$

3.4 Exercises

Exercise 40 *In the gamblers' ruin problem start with a uniform distribution of fortunes and* 10 *as the maximum fortune. What is the probability distribution of going broke, as a function of* p*, the probability of winning. Any of an analytic, tabular or graphical solution will work.*

Exercise 41 *Again consider the gamblers ruin problem. Compute the expected number of plays as a function of* p*. Again, any of an analytic, tabular or graphical solution will work.*