1 (A) $\lim_{x\to 0} \dfrac{\sin^2(x)}{1-\cos(x)}$

With $f(x) = \frac{\sin^2(x)}{1-\cos(x)}$, we see that $f(0) = \frac{0}{1} = 0$. Since $f(x)$ is continuous at $x = 0$, $\lim_{x\to 0} f(x) = f(0)$.

(B) Write $L = \lim_{x\to 0} \dfrac{\cos^2(x)}{1-\sin(x)}$

$\lim_{x\to 0} \cos^2(x) = 0$, and $\lim_{x\to 0} 1 - \sin(x) = 0$. Therefore we may apply L'Hopital's rule to get $\lim_{x\to 0} \dfrac{\cos^2(x)}{1-\sin(x)} = \lim_{x\to 0} \dfrac{-2\cos(x)\sin(x)}{-\cos(x)}$.

We may algebraically simplify the expression inside the limit to see that $L = \lim_{x\to 0} 2\sin(x)$. As $2\sin(x)$ is continuous at $x = 0$, $L = 2\sin(0) = 0$.

(C) Write $L = \lim_{x\to\infty} \frac{x^{100}}{e^x}$.

The function $g(x) = x^{1/100}$ is continuous on $[0, \infty)$. Therefore $\left(\lim_{x\to\infty} \frac{x^{100}}{e^x}\right)^{1/100} = \lim_{x\to\infty} \left(\frac{x^{100}}{e^x}\right)^{1/100}$, i.e. $L^{1/100} = \lim_{x\to\infty} \frac{x}{e^{x/100}}$.

$\lim_{x\to\infty} x = \infty$, and $\lim_{x\to\infty} e^{x/100} = \infty$. Therefore we may apply L'Hopital's rule to get $L^{1/100} = \lim_{x\to\infty} \frac{1}{e^{x/100}/100}$.

We may algebraically simplify the expression inside the limit to see that $L^{1/100} = \lim_{x\to\infty} 100e^{-x/100}$, which $L^{1/100} = 0$. Hence $L = 0$.

(D) $\lim_{x\to\infty} \frac{(\ln(x))^{100}}{x}$

The continuity of $g(x) = x^{1/100}$ may be used again to deduce the limit. As an alternative method, we consider $x = e^u$. We have $u = \ln(x)$, and $\lim_{x\to\infty} u = \infty$.

$\lim_{x\to\infty} f(x) = L$ mean that

We note that if the limit $\lim_{x\to\infty} f(x) = L$, then for any given $\epsilon > 0$, there exists $N > 0$ so that $n \geq N \Rightarrow |f(x) - L| < \epsilon$. We also note that for any function $g(u)$ that increases without bound as $u \to \infty$ we have given any $N > 0$, there exists $N_2 > 0$ so that $n \geq N_2 \Rightarrow g(u) > N$. Thus, $n \geq N_2$ means that $|f(g(u)) - L| < \epsilon$, i.e. $\lim_{u\to\infty} f(g(u)) = \lim_{x\to\infty} f(x)$.

We then observe that $\lim_{u\to\infty} \frac{u^{100}}{e^u} = \lim_{x\to\infty} \frac{(\ln(x))^{100}}{x}$.

Thus, by part (C), $\lim_{x\to\infty} \frac{(\ln(x))^{100}}{x} = 0$.

(E) Write $L = \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$.

We apply the algebraic limit theorem to write $L = \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} \div \frac{x}{x} = \lim_{x\to\infty} \frac{\sqrt{1+1/x^2}}{1}$.

Apply the continuity of $\sqrt{x}$ to see that $L = \sqrt{\lim_{x\to\infty} 1 + 1/x^2}$.

Apply the algebraic limit theorems to see that $L = \sqrt{\lim_{x\to\infty} 1 + \lim_{x\to\infty} x^{-2}}$

and apply the known limits of a constant, and $\lim_{x\to\infty} x^p$ to see that $L = \sqrt{1+0} = 1$.

Alternate methods: Since x^2 is continuous everywhere, $L^2 = \lim_{x\to\infty} \frac{x^2+1}{x^2} = \lim_{x\to\infty} 1 + \frac{1}{x^2}$. By the algebraic limit theorems this is $\lim_{x\to\infty} 1 + \lim_{x\to\infty} \frac{1}{x^2}$. The continuity of the constant function shows that the first limit is 1. The archimedean principle and the ordering of integers shows that the second limit is 0. Thus, $L = 1 + 0 = 1$.

L'Hopital's Rule: $\lim_{x\to\infty} \sqrt{x^2+1} = \infty$ and $\lim_{x\to\infty} x = \infty$. Therefore we may apply L'Hopital's rule. This gives us $L = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$, which is $\frac{1}{L}$, by the algebraic limit theorems. Thus $L = \frac{1}{L}$ and $L = \pm 1$. Since $\sqrt{x^2+1} > x > 0$, L must be non-negative. Hence $L = 1$.

Composition of functions: Substitute $x = \tan(\theta)$. Then $L = \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x} = \lim_{\theta\to\pi/2} \frac{\sec(\theta)}{\tan(\theta)} = \lim_{\theta\to\pi/2} \frac{1}{\sin(\theta)}$. By the algebraic limit theorems, this is $L = \frac{\lim_{\theta\to\pi/2} 1}{\lim_{\theta\to\pi/2} \sin(\theta)}$. The continuity of 1 and $\sin(\theta)$ shows that this simplifies to $L = \frac{1}{\sin(\pi/2)} = 1$.