1.4.9 $A \sim B$, so there is a 1-1 onto map, f, from A onto B.

For every $b \in B$, there is an $a \in A$ so that $f(a) = b$ (as f is onto). We define $g(b) = a$. Since f is 1-1, there is exactly one such a, so g defines a function.

For all $b_1, b_2 \in B$ with $b_1 \neq b_2$, we have unique elements $a_1, a_2 \in A$ so that $f(a_1) = b_1$ and $f(a_2) = b_2$. Since f is a function, $a_1 \neq a_2$. We see that $g(b_1) = a_1 \neq a_2 = g(b_2)$. Thus g is 1-1.

For all $a \in A$, there is an element $b \in B$ so that $f(a) = b$. Then $g(b) = a$. Thus g is onto.

Hence g is a 1-1, onto map from B to A, and $B \sim A$.

If $A \sim B$ and $B \sim C$, there exist 1-1, onto maps, $f : A \rightarrowtail\!\!\!\twoheadrightarrow B$ and $g : B \rightarrowtail\!\!\!\twoheadrightarrow C$. For every $a \in A$, define $h(a) = g(f(a))$.

For all $a, b \in A$ with $a \neq b$, we have $f(a) \neq f(b)$, as f is 1-1. But g is also 1-1, so $g(f(a)) \neq g(f(b))$, i.e. $h(a) \neq h(b)$. Thus h is 1-1.

For all $c \in C$, $\exists b \in B$ so that $g(b) = c$. There also is an $a \in A$ so that $f(a) = b$. Then $h(a) = g(f(a)) = c$, so h is onto.

Hence $A \sim C$.

2.3.1 Given any $\epsilon > 0$, take $N = 1$. For all $n \geq 1$, we have $|a_n - a| = |a - a| = 0 < \epsilon$. Thus $(a_n) \to a$.

2.3.7 (a_n) is bounded, so there is a real number M so that for all $n \in \mathbf{N}$, $|a_n| < M$.

$(b_n) \to 0$, so for any $\epsilon > 0$, there is a number N so that for all $n \geq N$, $|b_n - 0| < \epsilon/M$.

Then for all $n \geq N$, we have $|a_n b_n - 0| = |a_n b_n| = |a_n| \cdot |b_n| < M \cdot \frac{\epsilon}{M} = \epsilon$. Thus $(a_n b_n) \to 0$.

We may not use the limit theorem, because that only applies to convergent series, and (a_n) might not converge. (e.g. $(a_n) = (-1)^n$.)

2.3.11 Given $\epsilon > 0$,

There exists N_1 so that for all $n \geq N_1$, $|x_n - x| < \epsilon/2$.

Also, there exists N_2 so that $\frac{(|x_1 - x| + |x_2 - x| + \cdots |x_{N_1} - x|)}{N_2} < \epsilon/2$.

Let $N = \max(N_1, N_2)$. Then for $n \geq N$, we have

$$\begin{aligned}
|y_n - x| &= \left| \frac{x_1 - x + x_2 - x + \cdots x_n - x}{n} \right| \\
&\leq \frac{|x_1 - x|}{n} + \frac{|x_2 - x|}{n} + \cdots + \frac{|x_{N_1} - x|}{n} + \frac{|x_{N_1+1} - x|}{n} + \cdots + \frac{|x_n - x|}{n} \quad \text{(triangle inequality)} \\
&< \frac{\epsilon}{2} + \frac{\epsilon/2}{n} + \cdots + \frac{\epsilon/2}{n} \\
&< \frac{\epsilon}{2} + \frac{n - N}{n} \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

Therefore $(y_n) \to x$.

If $x_n = (-1)^{n+1}$, then x_n does not converge, but $y_n = \begin{cases} 1/n & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$, and $(y_n) \to 0$.

2.4.2 (a) We prove by induction that for all $n \in \mathbf{N}$, $x \in [0, 3]$.

Base case: $x_1 = 3$, so $x_1 \in [0, 3]$.

Induction Hypothesis: $x_n \in [0, 3]$.

Next case: Suppose that $0 \leq x_n \leq 3$. then $1 \leq 4 - x_n \leq 4$ and $\frac{1}{4} \leq \frac{1}{4 - x_n} \leq 1$. Thus $x_{n+1} \in [0, 3]$.

Hence (x_n) is bounded.

Next, we prove by induction that (x_n) is monotone decreasing.

Base case: We note that $x_1 = 3$, and $x_2 = 1$, so $x_1 > x_2$.

Induction Hypothesis: $x_n > x_{n+1}$.

Next case: Suppose that $x_n > x_{n+1}$. As $x_n \leq 3$, $0 < 4 - x_n < 4 - x_{n+1}$ and $\frac{1}{4 - x_n} > \frac{1}{4 - x_{n+1}}$, i.e.. $x_{n+1} > x_{n+2}$.

Therefore (x_n) is monotone decreasing.

As (x_n) is a bounded, monotone sequence, the monotone convergence theorem shows that (x_n) converges.

(b) given any $\epsilon > 0$, there exists an N so that for $n \geq N$, we have $|x_n - x| < \epsilon$. We note that when $n \geq N$, $n+1 > N$, so $|x_{n+1} - x| < \epsilon$. Thus, x_{n+1} converges to x.

(c) $\lim x_{n+1} = \lim \left(\frac{1}{4-x_n}\right) = \frac{\lim 1}{\lim 4 - \lim x_n}$, by the limit theorems(Theorem 2.2.3). Writing $x = \lim x_n$, we have $x = \frac{1}{4-x}$, so $x^2 - 4x + 1 = 0$ and $x = 2 \pm \sqrt{3}$. As $x_n \leq 3$, the order limit theorem (Theorem 2..3.4) shows that $x \leq 3$, and hence x and cannot be $2 + \sqrt{3}$. Therefore $x = 2 - \sqrt{3}$.

2.4.4 We observe that $x_{n+1} = \sqrt{2x_n}$.

We prove by induction that $x_n < 2$.

Base case: $x_1 = \sqrt{2} < 2$.

Induction Hypothesis: $x_n < 2$.

Next case: Since $x_n < 2$, $2x_n < 4$ and $x_{n+1} = \sqrt{2x_n} < 2$.

Thus (x_n) is bounded.

Next, we prove by induction that (x_n) is monotone increasing.

Base case: $x_1 = \sqrt{2} = \sqrt[4]{4}$ and $x_2 = \sqrt{2\sqrt{2}} = \sqrt[4]{8}$, so $x_1 < x_2$.

Induction Hypothesis: $x_n < x_{n+1}$.

Next case: If $x_n < x_{n+1}$ then $2x_n < 2x_{n+1}$ and $\sqrt{2x_n} < \sqrt{2x_{n+1}}$, i.e. $x_{n+1} < x_{n+2}$.

Thus (x_n) is a bounded monotone sequence, and converges by the monotone convergence theorem.

We have $x_{n+1}^2 = 2x_n$, so $\lim x_{n+1}^2 = \lim(2x_n)$, so $x^2 = 2x$. Hence $x = 0$ or $x = 2$. Since $x > x_1 = \sqrt{2}$, we must have $x = 2$

2.4.5 We shall prove that $x_n^2 > 2$ by mathematical induction.

Base case: We observe that $x_1^2 = 4 > 2$.

Induction Hypothesis: $x_n^2 > 2$.

Next case: If $x_n > \sqrt{2}$, then $\frac{x_n}{\sqrt{2}} - 1 > 0$ so $0 < \left(\frac{x_n}{\sqrt{2}-1}\right)^2 = \frac{x_n^2}{2} - 2\frac{x_n}{\sqrt{2}+1}$ and $\frac{x_n^2}{2} + 1 > 2\frac{x_n}{\sqrt{2}}$. Thus $\frac{x_n}{\sqrt{2}} + \frac{\sqrt{2}}{x_n} > 2$. Divide by $\sqrt{2}$ to see that $\frac{x_n}{2} + \frac{1}{x_n} > \sqrt{2}$. Therefore $x_{n+1} = \frac{1}{2}\left(\frac{x_n+2}{x_n}\right) > \sqrt{2}$.

To show that the sequence is decreasing, we observe that $x_n - x_{n+1} = x_n - \frac{x_n}{2} - \frac{1}{x_n} = \frac{x_n^2 - 2}{2x_n} > 0$.

Write $x = \lim x_n$. We then have $\lim x_{n+1} = \lim \left(\frac{1}{2}\left(x_n + \frac{2}{x_n}\right)\right)$, so $x = \frac{1}{2}\left(x + \frac{2}{x}\right) = \frac{x^2+2}{2x}$. Then $2x^2 = x^2 + 2$ and $x^2 = 2$. Since $x_n > \sqrt{2}$, $x \geq \sqrt{2}$, so $x \neq -\sqrt{2}$. Hence $x = \sqrt{2}$

2.4.6 (a) We observe that $y_n = \sup\{a_k : k \geq n\}$ is an upper bound for the subset $\{a_k : k \geq n+1\}$. Thus $y_{n+1} \leq y_n$, and (y_n) is a monotone sequence.

Since (a_n) is bounded, there is an M so that for all natural numbers n, $M \geq |a_n|$, $M \geq a_n \geq -M$. M is an upper bound for $\{a_k : k \geq n\}$, and $M \geq y_n$. Also, $y_n \geq a_n \geq -M$, so $|y_n| \leq M$, i.e. (y_n) is bounded.

As (y_n) is a bounded monotone sequence, the monotone convergence theorem shows that (y_n) converges.

(b) We define $x_n = \inf\{a_k : k \geq n\}$. An argument similar to that used in part (a) will show that (x_n) is a bounded monotone sequence, and will therefore converge.

(c) For every natural number N, we have $x_n \leq a_n \leq y_n$. Thus $x_n \leq y_n$. The order limit theorems show that $\lim x_n \leq \lim y_n$.

When $a_n = (-1)^n$ we have $\lim x_n = -1 < 1 = \lim y_n$.

(d) Write $x = \lim x_n$, and $y = \lim y_n$.

Suppose that $x = y$. then for all natural numbers n we have $x_n \leq a_n \leq y_n$. Given any $\epsilon > 0$, there exists an N so that for all $n \geq N$, we have $|x_n - x| < \epsilon$ and $|y_n - x| < \epsilon$.

Thus, $-\epsilon < x_n - x \leq a_n - x \leq y_n - x < \epsilon$, so $|a_n - x| < \epsilon$. Hence $(a_n) \to x$.

Suppose that (a_n) converges to a.

Given $\epsilon > 0$ there is an N so that for all $n \geq N$, $|a_n - a| < \epsilon$. Hence $a - \epsilon$ is a lower bound for (a_n). thus $a - \epsilon < x \leq a$. Hence, by theorem 1.2.6, $a \leq x$. Similarly, $a + \epsilon$ is an upper bound for (a_n). Thus $a + \epsilon > y \geq a$, and $a \geq y$.

We have $x \leq y \leq a \leq x$. Therefore $x = y$.