Hw #2 - some sample solutions

1.5.9 (a) Countable. We map the function $f(0) = m$, $f(1) = n$ to the ordered pair (n, m). The inverse map takes (n, m) to $f(0) = m$, $f(1) = n$. We now observe that the image of our map is countable by using a diagonal count similar to that used to count the rationals;

$$\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & \cdots \\ (2,1) & (2,2) & (2,3) & (2,4) & \cdots \\ (3,1) & (3,2) & (3,3) & (3,4) & \cdots \\ (4,1) & (4,2) & (4,3) & (4,4) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix} \qquad \begin{matrix} 1 & 2 & 4 & 7 & \cdots \\ 3 & 5 & 8 & 12 & \cdots \\ 6 & 9 & 13 & 18 & \cdots \\ 10 & 14 & 19 & 25 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

(b) Uncountable. We map the a function to the sequence $(f(1), f(2), f(3), \ldots)$. This maps the functions into the set S described in exercise 1.5.4. Similarly, for any sequence $(a_1, a_2, a_3, \ldots)$ in S, we define $f(n) = a_n$ for all $n \in \mathbf{N}$. Thus we map the function onto S. Since S is uncountable, so is our set of functions.

(c) We represent the set $F \in P(\mathbf{N})$ sequence in $S = \{(a_1, a_2, a_3, \ldots) : a_n = 0 \text{ or } 1\}$ in the following way: $a_n = \begin{cases} 1 & \text{if } n \in F \\ 0 & \text{if } n \notin F \end{cases}$.

We call an antichain A *maximal* if for every $b \in P(\mathbf{N})$, there exists as set $a \in A$ so that either $b \subseteq a$ or $a \subseteq b$.

Consider the set of functions F which correspond to the elements of $T \subseteq S$, where T consists only of sequences which have inifinitely many 0's and infinitely many 1's, and T is a maximal antichain. We refer to our map as f, where $f : F \rightarrowtail\!\!\!\twoheadrightarrow T$.

Suppose that T is countable. We then list the elements of T as $\{t_n : n \in \mathbf{N}\}$, where $t_n = (t_{n1}, t_{n2}, t_{n3}, \ldots)$.

We construct $b \in S$ by $b_1 = 1 - t_{11}$. Let $n_1 =$ the smallest natural number so that $t_{1n_1} = 1 - t_{11}$, and define $b_2 = 1 - t_{1,n_1}$. Therefore $f^{-1}(b) \not\subseteq f^{-1}(t_1)$ and $f^{-1}(t_1) \not\subseteq f^{-1}(b)$.

Similary, we define $b_{2k-1} = 1 - a_{k,n_{k-1}-1}$, let n_k be the smallest number so that $t_{1n_k} = 1 - t_{1k_{k-1}}$, and define $b_{2k} = 1 - t_{1n_k}$.

Thus, for every $t_k \in T$ we have $f^{-1}(b) \not\subseteq f^{-1}(t_k)$ and $f^{-1}(t_k) \not\subseteq f^{-1}(b)$, contradicting the maximality of T.