Hw #1 - some sample solutions

1.2.1 (a) Suppose that $\sqrt{3}$ is rational. Then there exist $p, q \in \mathbf{Z}$ so that p and q have no factors in common and $\sqrt{3} = p/q$. Squaring both sides, we have $3 = p^2/q^2$ and $3q^2 = p^2$. Therefore, 3 divides p^2. This can happen only if 3 divides p. Hence we may write $p = 3m$ for some $m \in \mathbf{Z}$. We then have $3q^2 = (3m)^2$, i.e., $3q^2 = 9m^2$. Thus, $q^2 = 3m^2$. This means that 3 divides q^2. Thus 3 divides q. We have now shown that 3 divides both p and q, but this contradicts the fact that p and q have no factors in common.

Yes. Suppose that $\sqrt{6}$ is rational. Then there exist $p, q \in \mathbf{Z}$ so that p and q have no factors in common and $\sqrt{6} = p/q$. Squaring both sides, we have $6 = p^2/q^2$ and $6q^2 = p^2$. Therefore, 6 divides p^2. This can happen only if 6 divides p. Hence we may write $p = 6m$ for some $m \in \mathbf{Z}$. We then have $6q^2 = (6m)^2$, i.e., $6q^2 = 36m^2$. Thus, $q^2 = 6m^2$. This means that 6 divides q^2. Thus 6 divides q. We have now shown that 6 divides both p and q, but this contradicts the fact that p and q have no factors in common.

(b) The proof breaks down at the stage when we attempt to say 4 divides p^2, so 4 divides p. We may only say that 4 divides p^2 implies that 2 divides p. From this we will not be able to get a common divisor for q.

1.2.3 Note that we may use the following facts from section 1.2:

$x \in A \cup B \leftrightarrow x \in A$ or $x \in B$,

$x \in A \cap B \leftrightarrow x \in A$ and $x \in B$,

$A^c = \{x \in \mathbf{R} : x \notin A\}$.

(a) Consider x so that $x \in (A \cap B)^c$. Since $(A \cap B)^c = \{y \in \mathbf{R} : y \notin A \cap B\}$, $x \notin A \cap B$. This means that $x \notin A$ or $x \notin B$. i.e., $x \in A^c$ or $x \in B^c$. But this happens if and only if $x \in (A^c \cup B^c)$. Therefore $(A \cap B)^c \subseteq (A^c \cup B^c)$.

(b) Consider x so that $x \in (A^c \cup B^c)$. This is equivalent to $x \in A^c$ or $x \in B^c$. i.e., $x \notin A$ or $a \notin B$. But this is $x \notin A \cap B$, so $x \in (A \cap B)^c$. Therefore $(A^c \cup B^c) \subseteq (A \cap B)^c$.

(c) Consider x so that $x \in (A \cup B)^c$. Since $(A \cup B)^c = \{y \in \mathbf{R} : y \notin A \cup B\}$, $x \notin A \cup B$. This means that $x \notin A$ and $x \notin B$. i.e., $x \in A^c$ and $x \in B^c$. But this happens if and only if $x \in (A^c \cap B^c)$. Therefore $(A \cup B)^c \subseteq (A^c \cap B^c)$.

Consider x so that $x \in (A^c \cap B^c)$. This is equivalent to $x \in A^c$ and $x \in B^c$. i.e., $x \notin A$ and $a \notin B$. But this is $x \notin A \cup B$, so $x \in (A \cup B)^c$. Therefore $(A^c \cap B^c) \subseteq (A \cup B)^c$.

Thus we have shown that $(A^c \cap B^c) = (A \cup B)^c$.

1.2.10 (a) Base case: $y_1 = 1 < 4$.

Induction Hypothesis: For some k, $y_k < 4$.

Next Case: $y_{k+1} = (3y_n + 4)/4 = 3y_n/4 + 1$. By the induction hypothesis we have $y_n < 4$, so $3y_n/4 < 3$. Therefore $3y_n/4 + 1 < 4$. Thus $y_{k+1} < 4$.

(b) Base case: $n = 1$; $y_1 = 1$ and $y_2 = (3y_1 + 4)/4 = 7/4$. Therefore $y_1 < y_2$.

Induction Hypothesis: For some k, $y_k < y_{k+1}$.

Next Case: $n = k + 1$; By the induction hypothesis, $y_k < y_{k+1}$, so $3y_k/4 < 3y_{k+1}/4$, and $(3y_k + 4)/4 < (3y_{k+1} + 4)/4$. i.e. $y_{k+1} < y_{k+2}$.

1.2.12