Exam #3 - MA275 - Sample solutions

1 (A) We apply the Euclidean algorithm to see that the gcd is 7 and then trace back up to find a representation of 7:

$$\begin{array}{lll} 1001 & = 9 \cdot 105 + 56 & 7 = 2 \cdot (1001 - 9 \cdot 105) - 1 \cdot 105 = 2 \cdot 1001 - 19 \cdot 105 \\ 105 & = 1 \cdot 56 + 49 & 7 = 1 \cdot 56 - 1(105 - 56) = 2 \cdot 56 - 105 \\ 56 & = 1 \cdot 49 + 7 & 7 = 1 \cdot 56 - 1 \cdot 49 \\ 49 & = 7 \cdot 7 + 0 & \end{array}$$

Thuus one solution is $(n, m) = (-19, 2)$.

The homogenous solution is $105n + 1001m = 0 \to 105n = -1001m$, so $\frac{n}{m} = \frac{-1001}{105} = \frac{-1001/7}{105/7} = \frac{-143}{15} = \frac{-143k}{15k}$, $(n, m) = (15k, -143k)$.

Therefore the general solution is $(n, m) = (-19 + 15k, 2 - 143k)$.

(B) We multiply our particular solution from part (A) by 6 to obtain $(n, m) = (-114, 12)$. Add the homogeneous solution to get the general solution

$(n, m) - (-114 + 15k, 12 - 143k)$.

2 (A) once for each divisor $100 = 2^2 \cdot 5^2$ has $(2 + 1)(3 + 1) = 9$ divisors, ther fore it will be turned over **9** times.

(B) Coins with an odd number of divisors. 2 and 3 are primes, they have an even number of divisors. 1 has one divisor, 4 has three divisors. The first two coins that end tails up are **1,4**.

(C) Coins with an odd number of divisors. If the number n, is not a perfect quare, then the flip from divisor d is canceled by the flip from ddivsor n/d and the coin ends up heads.

If the number n is a perfect square, then the flip from divisor $\sqrt{n}$ is never cancelled (since $n/\sqrt{n} = \sqrt{n}$) so the coin ends tails up. The positive squares less than or equal to 100 are **1,4,9,16,25,36,49,64,81,100**.

(D) The number of divisors of $2^a 3^b 5^c$ is $(a + 1)(b + 1)(c + 1)$.

$2^6 < 100 < 2^7$, so the power with the most flips is 2^6, which has 7 flips.

For numbers of the form $2^a 3^b$;

$2^5 \cdot 3 < 100 < 2^6 \cdot 3$. $2^5 \cdot 3 = 96$ has $6 \cdot 2 = 12$ flips.

$2^3 \cdot 3^2 < 100 < 2^4 \cdot 3^2$. $2^3 \cdot 3^2 = 72$ has $4 \cdot 3 = 12$ flips.

$2^1 \cdot 3^3 < 100 < 2^2 \cdot 3^3$. $2^1 \cdot 3^3 = 54$ has $2 \cdot 4 = 8$ flips.

For numbers of the form $2^a 3^b 5^c$;

$2^2 \cdot 3 \cdot 5 < 100 < 2^3 \cdot 3 \cdot 5$. $2^2 \cdot 3 \cdot 5 = 60$ has $3 \cdot 2 \cdot 2 = 12$ divisors.

$2 \cdot 3^2 \cdot 5 < 100 < 2^2 \cdot 3^2 \cdot 5$, $2 \cdot 3^2 \cdot 5 = 90$ has $2 \cdot 3 \cdot 2 = 12$ divisors.

$2 \cdot 3 \cdot 5^2 = 150 > 100$.

The coins turned over twelve times are coins **60,72,90,96**.

3 We proceed by mathematical induction.
Base case: $n = 1$; $\sum_{j=1}^{1} \frac{F_{j-1}}{2^j} = \frac{F_0}{2^1} = \frac{0}{2} = 0 = 1 - 1 = 1 - \frac{2}{2^1} = 1 - \frac{f_3}{2^1}$.
IH: $n = k$; We have $\sum_{j=1}^{k} \frac{F_{j-1}}{2^j} = 1 - \frac{F_{k+2}}{2^k}$.
Next case: $n - k + 1$; We see that

$$\begin{aligned}
\sum_{j=1}^{k+1} \frac{F_{j-1}}{2^j} &= \frac{F_k}{2^{k+1}} + \sum_{j=1}^{k} \frac{F_{j-1}}{2^j} \\
&= \frac{F_k}{2^{k+1}} + 1 - \frac{F_{k+2}}{2^k} \quad \textbf{by IH} \\
&= 1 - \frac{2F_{k+2} - F_k}{2^{k+1}} \\
&= 1 - \frac{F_{k+2} + (F_{k+2} - F_k)}{2^{k+1}} \\
&= 1 - \frac{F_{k+2} + F_{k+1}}{2^{k+1}} \\
&= 1 - \frac{F_{k+3}}{2^{k+1}} \\
&= 1 - \frac{F_{n+2}}{2^n}
\end{aligned}$$

4 (A) $|A \times B| = 48$, so there are 2^{48} subsets of $A \times B$, i.e. there are 2^{48} binary relations between A and B.

(B) Each element of A may map to any of the 8 elements of B so there are 8^6 functions.

(C) The first element of A map map to any of the 8 elements of B. The next element may map top any of the 7 remaining elements of B. Thus there are $8!/2!$ injective maps.

(D) Let x_i represent the number of elements of A mapping to element i of B. Then $x_1 + x_2 + \cdots x_8 = 6$. There are $\binom{8+6-1}{7} = \binom{13}{7}$ such maps.

(E) We choose 6of the 8 elements of B and list then in increasing order to find the strictly increasing function. Thus there are $\binom{8}{6}$ such functions.