Exam #2 - MA275 - Sample solutions

1 (A) The probability that the team that wins game 1 also wins games 2,3, and 4 is $(1/2)^3 = 1/8$.

(B) The series is an arrangement of 4 wins, `W`, and 3 losses, `L`, with the last two game as `WL`. For the other 5 games, there are $\binom{5}{3} = 10$ ways to arrange the three wins. This is out of a possible $2^5 = 32$ arrangements. Therefore the probability is $10/32 = 5/16$.

Note that the probability that the series lasts $4, 5, 6, 7$ games is $\frac{\binom{3}{3}}{2^3}, \frac{\binom{4}{3}}{2^4}, \frac{\binom{5}{3}}{2^5}, \frac{\binom{6}{3}}{2^6}$, resp.

2 ($\Rightarrow$) if $k|(A - B)$ then $A - B = kN$ for some $N \in \mathbf{Z}$.

By the division algorithm, $A = kq_A + r_A$, and $B = kq_B + r_B$. with $0 \leq r_A, r_B < k$. Then $A - B = k(q_A - q_B) + (r_A - r_B)$, where $-k < r_A - r_B < k$. Therefore $r_A - r_B = 0$, and $r_A = r_B$. Thus $k|a \Leftrightarrow k|B$.

($\Leftarrow$) If $k|A \Leftrightarrow k|B$, then $A = kq_A + r_A$, and $B = kq_B + r_B$, with $0 \leq r_A, r_B < k$. Then $A - B = k(q_A - q_B) = (r_A - r_B)$, so $(k|A \Leftrightarrow k|B) \Leftrightarrow k|(A - B)$.

3

$(\bar{A} \cup B) \cap (A \cap (A \cap B))$	Premise
$(B \cup \bar{A}) \cap (A \cap (A \cap B))$	Commutative Law
$(B \cup \bar{A}) \cap ((A \cap A) \cap B))$	Associative Law
$(B \cup \bar{A}) \cap (A \cap B)$	Idempotent
$[(B \cup \bar{A}) \cap A] \cap B$	Associative
$[A \cap (B \cup \bar{A})] \cap B$	Commutative
$[(A \cap B) \cup (A \cap \bar{A})] \cap B$	Distributive
$[(A \cap B) \cup \emptyset] \cap B$	Inverse
$[(A \cap B)] \cap B$	Identity
$A \cap (B \cap B)$	Associative
$A \cap B$	Idempotent

4 *Base Case*: $n = 1$. We have $\binom{2 \cdot 1}{1} = \binom{2}{1} = 2 < 4 = 4^1$.

IH: $(n = k)$ We have $\binom{2k}{k} < 4^k$.

Next case: $n = k + 1$. We see that

$$\begin{aligned}
\binom{2(k+1)}{k+1} &= \frac{(2k+2)(2k+1)}{(k+1)(k+1)} \binom{2k}{k} \\
&= 2\frac{2k+1}{k+1} \binom{2k}{k} \\
&< 2\frac{2k+1}{k+1} 4^k \quad \textbf{By IH} \\
&< 2\frac{2k+2}{k+1} 4^k = 2 \cdot 2 \cdot 4^k = 4^{k+1} = 4^n.
\end{aligned}$$

5 We prove this by mathematical induction on n.
Base case: $(n = 1)$ $3^1 - F_1 - F_3 = 3 - 1 - 2 = 0 = 5 \cdot 0$, so $5|(3^1 - F_1 - F_3)$.
IH: $5|3^k - F_k - F_{k+2}$
Next case: $n = k + 1$.

$$\begin{aligned}
3^{k+1} - F_{k+1} - F_{k+3} &= 3 \cdot 3^k - F_k - F_{k-1} - F_{k+1} - F_{k+2} \\
&= 3(3^k - F_k - F_{k+2}) + 2F_k - F_{k-1} + 2F_{k+2} - F_{k+1} \\
&= 3(5A) + 2F_k - F_{k-1} + 2F_{k+2} - F_{k+1} \quad \textbf{by IH} \\
&= 5(3A) + 2F_k - (F_{k+1} - F_k) + 2F_{k+2} - F_{k+1} \\
&= 5(3A) + 3F_k + 2F_{k+2} - 2F_{k+1} \\
&= 5(3A) + 3F_k + 2(F_{k+2} - F_{k+1}) \\
&= 5(3A) + 3F_k + 2(F_k) \\
&= 5(3A) + 5F_k \\
&= 5(3A + F_k)
\end{aligned}$$

Thus, $5|3^n - F_n - F_{n+2}$.

Alternate proof:
Base cases: $n = 1$; $3^1 - F_1 - F_3 = 3 - 1 - 2 = 0 = 5 \cdot 0$, so $5|(3^1 - F_1 - F_3)$.
$n = 2$; $3^2 - F_2 - F_4 = 9 - 1 - 3 = 5 = 5 \cdot 1$, so $5|(3^2 - F_2 - F_4)$.
IH: $5|3^{k-1} - F_{k-1} - F_{k+1}$ and $5|3^k - F_k - F_{k+2}$
Next case: $n = k + 1$.

$$\begin{aligned}
3^{k+1} - F_{k+1} - F_{k+3} &= 9 \cdot 3^{k-1} - F_k - F_{k-1} - F_{k+2} - F_{k+1} \\
&= 5 \cdot 3^{k-1} + (3^k - F_k - F_{k+2}) + (3^{k-1} - F_{k-1} - F_{k+1}) \\
&= 5 \cdot 3^{k-1} + 5A + 5B \qquad \textbf{by IH} \\
&= 5(3^{k-1} + A + B)
\end{aligned}$$

Thus, $5|3^n - F_n - F_{n+2}$.