Differential Equations and
Matrix Algebra II- Practice Test #1

Professor Broughton December , 2001

Name:______________ Section #______________ Box #______________

Instructions

- Answer all the questions directly on the test.
- Show all the necessary work and write your answers out neatly in English sentences. Use mathematical notation to express your answers, not *Maple* notation
- It is not necessary to use your computer to answer all of the questions but you can use it to obtain graphs, evaluate functions, solve equations, etc. If you use *Maple* be sure to say so by some sentence such as: Using *Maple* the above integral equals ....
- On your computer you may start off with one blank *Maple* worksheet only.
- On this practice test there are five questions. The real test will have 4 or 5 equally valued questions.

Question	Possible Points	Points Obtained
1	20	
2	20	
3	20	
4	20	
5	20	
Total		

1. Transforming systems

1.a Transform the following second order system a first order system. Write out the resulting system in vector-matrix format.

$$\begin{aligned} 2x'' &= -2cx' - 5x + 2y + \cos(25t) \\ 2y'' &= -cy' + 2x - 3y \end{aligned}$$

Set $u = x'$, $v = y'$

$$\begin{aligned} x' &= u \\ y' &= v \\ u' &= x'' = \frac{-2cx' - 5x + 2y + \cos(25t)}{2} \\ v' &= y'' = \frac{-cy' + 2x - 3y}{2} \end{aligned}$$

$$\begin{bmatrix} x' \\ y' \\ u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-5}{2} & 1 & -c & 0 \\ 1 & -\frac{3}{2} & 0 & \frac{-c}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{\cos(25t)}{2} \\ 0 \end{bmatrix}$$

1.b What method should be used to solve the equation

eigenvector method

2. Euler's Method

Consider the following non-linear first order DE.

$$x' = -2x^2 - 3y^2 + t$$
$$y' = -3x^2 - 2y^2 + t^3$$
$$x(0) = 1, y(0) = 1$$

Show how to find and approximation for $x(1)$ and $y(1)$ using two steps of Euler's method, by filling in the following table to the indicted precision.

n	t_n	x_n	y_n	x'_n	y'_n
0	0.000	1.000	1.000	-5.000	-5.000
1	0.500	-1.500	-1.500	-10.75	-11.125
2	1	-6.875	-7.0625		

3. First Order Systems

2.a Consider the following first order system.

$$x_1' = -x_1 + 3x_2$$
$$x_2' = -3x_1 + -x_2$$

Write out the general solution in vector format

$X' = AX$ where $A = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix}$. Eigenvectors: $\left\{ \begin{bmatrix} 1 \\ i \end{bmatrix} \right\} \leftrightarrow -1+3i, \left\{ \begin{bmatrix} 1 \\ -i \end{bmatrix} \right\} \leftrightarrow -1-3i$.

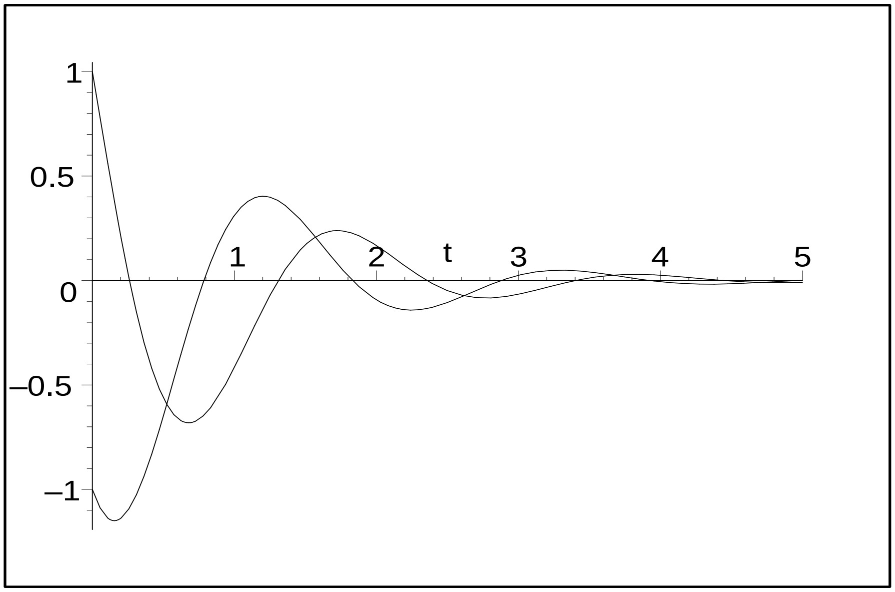
General Solution:

$$c_1 \exp((-1+3i)t) \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 \exp((-1-3i)t) \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} c_1 e^{(-1+3i)t} + c_2 e^{(-1-3i)t} \\ ic_1 e^{(-1+3i)t} - ic_2 e^{(-1-3i)t} \end{bmatrix}.$$

2.b Suppose that $x_1(0) = 1$, $x_2(0) = -1$. Find an explicit real forms for $x_1(t)$ and $x_2(t)$ plot them below.

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}i \\ \frac{1}{2} - \frac{1}{2}i \end{bmatrix}$$

$$X(t) = (\frac{1}{2} + \frac{1}{2}i) \exp((-1+3i)t) \begin{bmatrix} 1 \\ i \end{bmatrix} + (\frac{1}{2} - \frac{1}{2}i) \exp((-1-3i)t) \begin{bmatrix} 1 \\ -i \end{bmatrix} =$$
$$= e^{-t} \begin{bmatrix} \left(\frac{1}{2} + \frac{1}{2}i\right)(\cos(3t) + i\sin(3t)) + \left(\frac{1}{2} - \frac{1}{2}i\right)(\cos(3t) - i\sin(3t)) \\ \left(-\frac{1}{2} + \frac{1}{2}i\right)(\cos(3t) + i\sin(3t)) + \left(-\frac{1}{2} - \frac{1}{2}i\right)(\cos(3t) - i\sin(3t)) \end{bmatrix}$$
$$e^{-t} \begin{bmatrix} \cos 3t - \sin 3t \\ -\cos 3t - \sin 3t \end{bmatrix}$$


1
0.5
0
–0.5
–1
1
2
t
3
4
5

4. Second order system

Three connected masses satisfy the following second order system $X'' = AX$. The eigenvalues and eigenvectors of A are:

$$-1 \leftrightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, -4 \leftrightarrow \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, -9 \leftrightarrow \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

4.a Write out the general solution for $X(t)$.

$$\begin{aligned} X(t) = a_1 \cos(t) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b_1 \sin(t) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \\ a_2 \cos(2t) \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} + b_2 \sin(2t) \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} + \\ a_3 \cos(3t) \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + b_3 \sin(3t) \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \end{aligned}$$

Suppose that the system is initially at rest, that the first mass is displaced one unit from equilibrium in the positive direction, and that the other masses are at equilibrium. Write out but do not solve the resulting system of equations for the coefficients in the general solution in part 4.a.

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 4 & 0 \\ 1 & -2 & -3 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

5. Salt Tank System

Problem 29 page 422 of the text.

$$x_1' = -\frac{1}{5}x_1 + \frac{2}{5}x_2$$
$$x_2' = \frac{1}{5}x_1 - \frac{2}{5}x_2$$

Matrix: $\left[\begin{array}{cc} -\frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & -\frac{2}{5} \end{array}\right]$, eigenvectors: $\left\{\left[\begin{array}{c} 2 \\ 1 \end{array}\right]\right\} \leftrightarrow 0, \left\{\left[\begin{array}{c} -1 \\ 1 \end{array}\right]\right\} \leftrightarrow -\frac{3}{5}$

$$X(t) = c_1 \left[\begin{array}{c} 2 \\ 1 \end{array}\right] + c_2 \exp(-\frac{3}{5}t) \left[\begin{array}{c} -1 \\ 1 \end{array}\right]$$

$$\left[\begin{array}{c} c_1 \\ c_2 \end{array}\right] = \left[\begin{array}{cc} 2 & -1 \\ 1 & 1 \end{array}\right]^{-1} \left[\begin{array}{c} 15 \\ 0 \end{array}\right] = \left[\begin{array}{c} 5 \\ -5 \end{array}\right]$$

$$X(t) = 5 \left[\begin{array}{c} 2 \\ 1 \end{array}\right] - 5\exp(-\frac{3}{5}t) \left[\begin{array}{c} -1 \\ 1 \end{array}\right] = \left[\begin{array}{c} 10 + 5e^{-\frac{3}{5}t} \\ 5 - 5e^{-\frac{3}{5}t} \end{array}\right]$$