A statistical look at maps of the discrete logarithm

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Definitions

Functional Graph – A directed graph where the edges are determined by a transition function. In this case the function is

$$\varphi : x \to g^x \bmod p$$

Binary Functional Graph – A functional graph where the in-degree of each node is either 0 or 2. All the graphs studied were binary functional graphs.

Component - A connected set of nodes. The average number of components is measured for each prime modulus (e.g. 1.75 for p = 11)

Cyclic Nodes – Nodes that are part of a cycle, including nodes which loop back on themselves. The average cyclic nodes are measured for each prime (e.g. 3.25 for p = 11)

Average Cycle – The average cycle size as seen from a random node in a functional graph divided by the number of nodes in all the functional graphs for a given prime (e.g. 2.05 for p = 11)

Average Tail – The average distance to the cycle as seen from a random node in the graph. Cyclic nodes have a distance of 0. Computation is similar to that of the average cycle (e.g. 1.225 for p = 11)

Max Cycle – The largest cycle in a graph. The average is taken over all bases for a given p (e.g. 2.5 for p = 11)

Max Tail – The longest distance from a node to its cycle in a graph. Similar to max cycle (e.g. 2.75 for p = 11)

Methods

Generating functions:

$$\text{Binary Functional Graphs} = f(z) = e^{c(z)}$$

$$\text{Components} = c(z) = \ln\left(\frac{1}{1 - zb(z)}\right)$$

$$\text{Binary Trees} = b(z) = z + \frac{1}{2}zb(z)^2$$

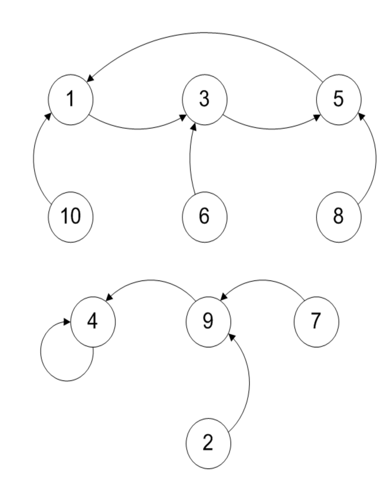
Marked generating function for total number of components:

Generating function fc $e^{-\frac{1}{2}u\ln(1-2z^2)}$ of components:

$$\text{Total Components} = \left.\frac{\partial}{\partial u} e^{u\,c(z)}\right|_{u=1} = -\frac{1}{2}\,\frac{\ln\left(1-2z^2\right)}{\sqrt{1-2z^2}}$$

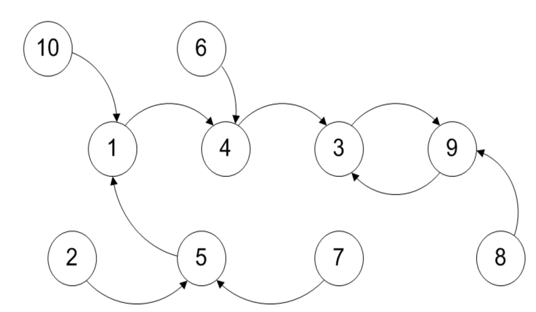
Example Graphs (mod 11)

g = 3



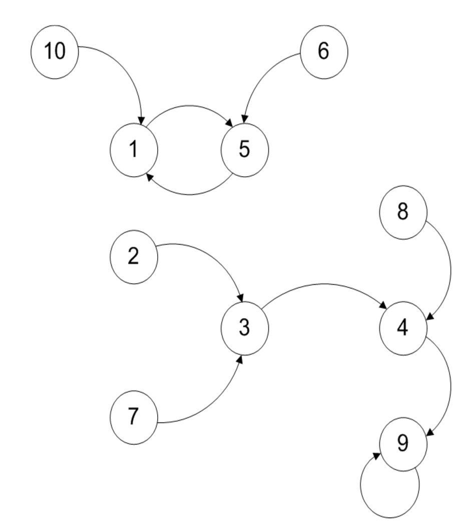
Components: 2
Cyclic nodes: 4
Average cycle: 2.2
Average tail: 0.8
Max cycle: 3
Max tail: 2

g = 4



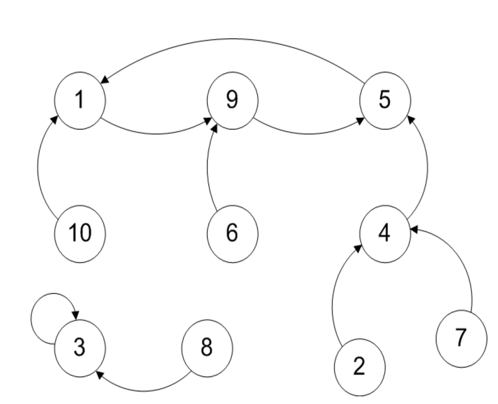
Components: 1
Cyclic nodes: 2
Average cycle: 2
Average tail: 2
Max cycle: 2
Max tail: 4

g = 5



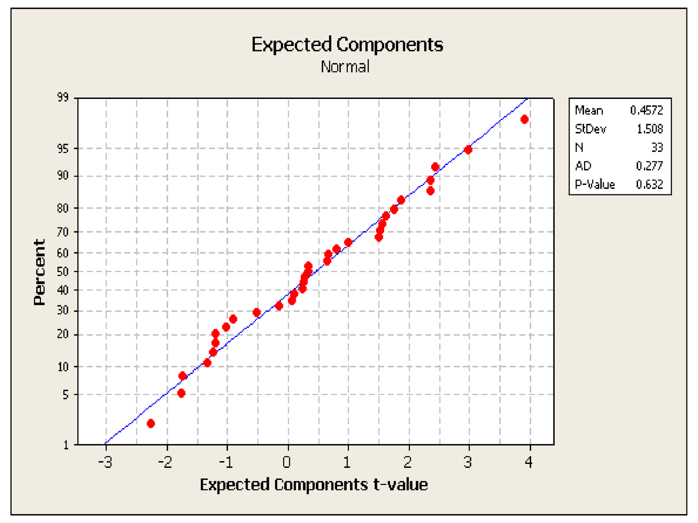
Components: 2
Cyclic nodes: 3
Average cycle: 1.4
Average tail: 1.3
Max cycle: 2
Max tail: 3

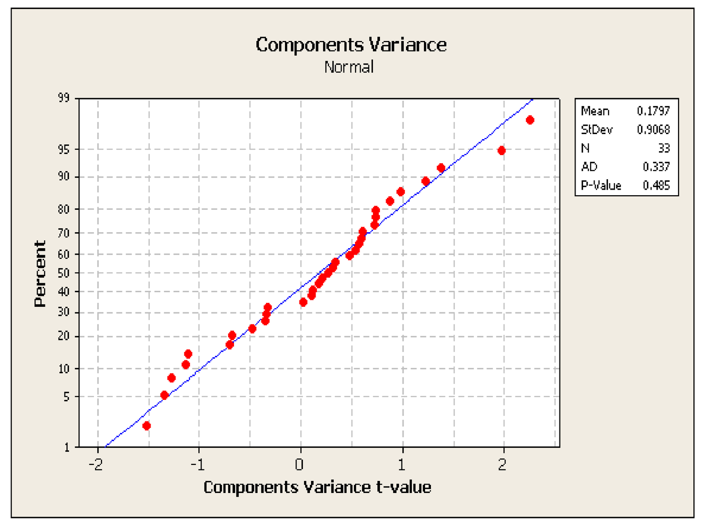
g = 9

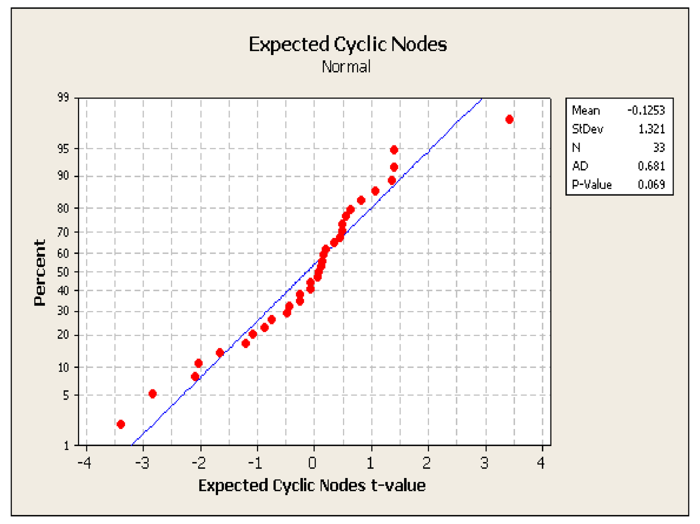


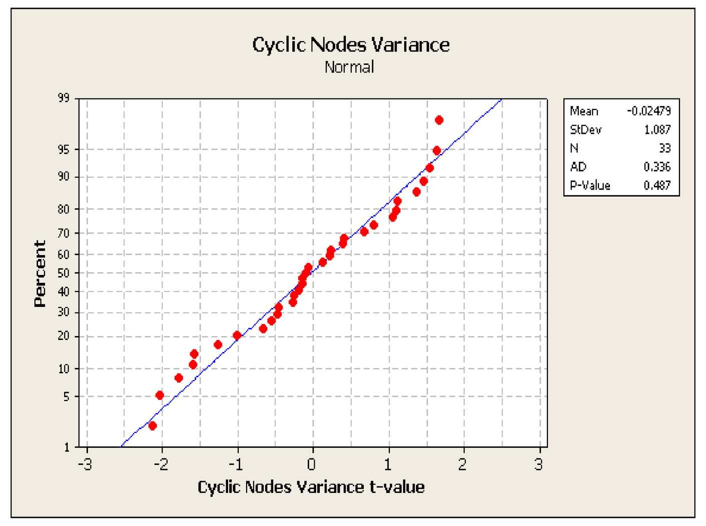
Components: 2
Cyclic nodes: 4
Average cycle: 2.6
Average tail: 0.8
Max cycle: 3
Max tail: 2

Results

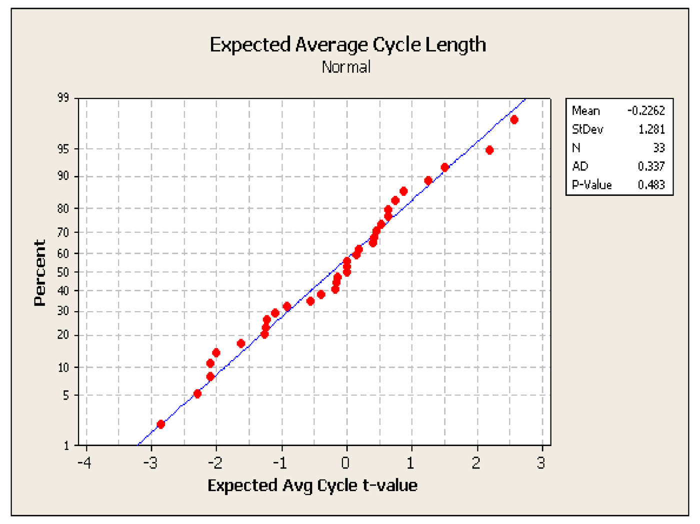




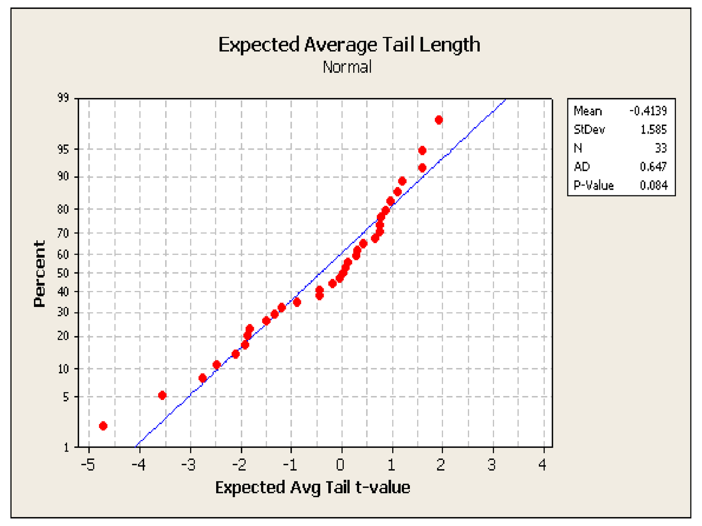




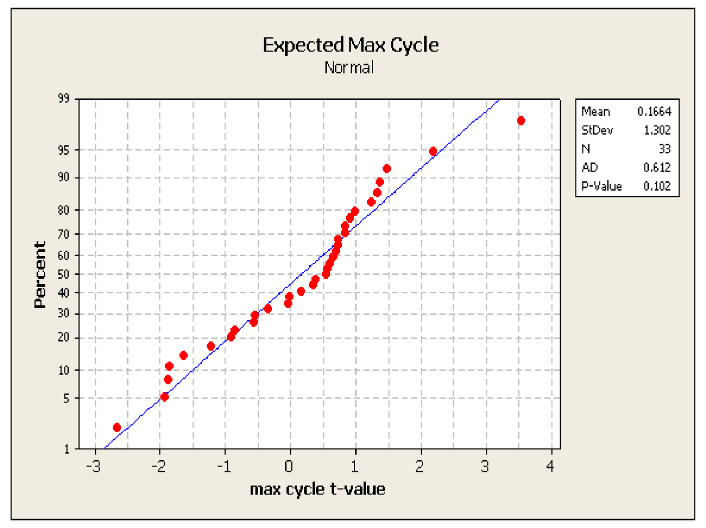
Sample Maximum Tail Statistics				
	Mean			Std. Dev.
p	Predicted	Observed	P-value	Observed
99923	547.605802	543.281073	0	163.809
99961	547.710225	541.005022	0	163.494
99971	547.737702	544.967041	0.002	165.249
99989	547.787156	542.47563	0	163.809
99991	547.792651	541.265167	0	163.805
100003	547.825617	543.876996	0	163.295
100019	547.86957	542.008421	0	163.79
100049	547.951971	544.38604	0.002	165.651
100069	548.006899	549.379291	0.318	165.926
100103	548.100263	540.966673	0	164.496



Sample Average Cycle Std. Dev.		
p	Predicted	Observed
99923	164.857	143.637
99961	164.888	140.91
99971	164.897	143.442
99989	164.911	144.856
99991	164.913	143.678
100003	164.923	143.686
100019	164.936	143.632
100049	164.961	142.121
100069	164.978	143.612
100103	165.006	142.79



Sample Average Tail Std. Dev.		
p	Predicted	Observed
99923	164.858	86.025
99961	164.89	85.319
99971	164.898	86.914
99989	164.913	85.661
99991	164.914	85.227
100003	164.924	85.261
100019	164.938	86.029
100049	164.962	86.398
100069	164.979	87.876
100103	165.007	86.234



Sample Max Cycle Std. Dev.	
p	Observed
99923	154.949
99961	152.417
99971	154.931
99989	156.06
99991	154.534
100003	154.874
100019	154.535
100049	153.792
100069	155.018
100103	154.1

Summary

- Many of the statistics gathered do not provide sufficient evidence to question the theory that modular exponentiation graphs are similar to random functional graphs.
- The observed variance in the average cycle and the average tail were significantly lower than the expected variance for a random binary functional graph.
- A few tests had surprisingly low p-values, but the normality tests indicate that these were just outliers.

Future Work

- Get theoretical values for maximum tail and maximum cycle variance.
- Analyze lower variances in average cycle length and average tail length to try and come up with a reason.
- Find an explanation for the lower maximum tail.