THE NATURAL FREQUENCY: MORE NATURAL AND MORE FREQUENT THAN EXPECTED

WILLIAM R. GREEN

Abstract. We show that the natural frequency of a mass-spring-damper system modeled by a constant coefficient second order differential equation occurs naturally and frequently when maximizing the amplitude of the steady-state response. Specifically, if any parameter other than the forcing frequency varies, the maximum amplitude of the response occurs at the natural frequency.

1. The natural frequency: more natural and more frequent than expected

A common problem in a first course in differential equations is to model a mass-spring-damper system under the influence of a periodic forcing function. Using Newton's laws of motion, one can arrive at the following differential equation to model the position $x(t)$ of an object with mass m, relative to its equilibrium by

$$mx'' + bx' + kx = F_0 \cos(\omega t). \tag{1}$$

The non-negative constants b and k are called the damping constant and spring constant, and the mass is under the influence of a periodic external force of strength F_0 and frequency ω. The natural frequency of a mass-spring-damper system, $\sqrt{k/m}$, occurs naturally and frequently when finding the maximum amplitude of solutions. It is well known that the maximum amplitude of an undamped system (when $b = 0$) occurs when ω is the natural frequency, and the amplitude of the velocity is maximized at the natural frequency regardless of damping. If any one parameter of the mass, spring constant or damping constant varies, the maximum amplitude occurs when the parameters m, k and ω satisfy $\omega = \sqrt{k/m}$, that is they occur at the natural frequency.

1.1. **The phenomenon of practical resonance.**

A standard application problem is to find the "resonant frequency," the value of ω at which the amplitude of the steady-state response of (1) is maximized. Typically, one first considers the case of a "pure resonance," when an undamped system is forced at its natural frequency. That is, when $b = 0$ and $\omega = \sqrt{k/m}$. In this case, we have

$$x_p(t) = \frac{F_0}{2\sqrt{km}}\, t \sin\left(\sqrt{\frac{k}{m}}\, t\right).$$

Here the amplitude of the particular solution grows without bound. We quickly dismiss this possibility, as real-world applications have non-zero damping, and study (1) with $b > 0$.

Date: June 6, 2019.

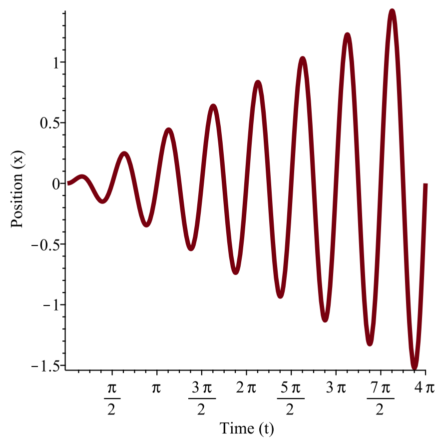

FIGURE 1. A pure resonance, here $m = 1$, $b = 0$, $k = 16$ with forcing $\cos(4t)$.

In this case the "steady-state" response is

$$x_p(t) = \frac{(k - m\omega^2)F_0}{b^2\omega^2 + (k - m\omega^2)^2}\cos(\omega t) + \frac{b\omega F_0}{b^2\omega^2 + (k - m\omega^2)^2}\sin(\omega t).$$

Recalling that the amplitude of $A\cos(\omega t) + B\sin(\omega t)$ is $\sqrt{A^2 + B^2}$, we have

$$\text{Amplitude of the steady-state response} = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}.$$

In usual applications, the mass, spring constant and damping constants are fixed and only the forcing frequency, ω, is allowed to vary. Then, the amplitude of the steady-state response of the damped system (1) is maximized at forcing frequency

$$\omega_{res} = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}},$$

provided $2km - b^2 > 0$. See for example [1, Section 3.9], [2, Section 5.6], or [3, Section 3.5].

1.2. **Variations on the practical resonance.** After finding the resonant frequency, I asked my class what would happen in a specific system if instead of varying the frequency ω, we let only the damping constant b vary. Instead of a peak for typical resonance curve, we saw much different behavior:
The maximum amplitude occurs when $b = 0$, since the presence of damping diminishes the motion of the mass. This observation leads to an even more natural result:

Theorem 1. *If we allow only one parameter out of m, b or k to vary, the amplitude of the steady-state response is maximized when the parameters satisfy the natural frequency relationship $\omega = \sqrt{k/m}$. To be precise,*

i) If only m is allowed to vary, the amplitude of the steady-state response is maximized when $m = k/\omega^2$.
ii) If only k is allowed to vary, the amplitude of the steady-state response is maximized when $k = m\omega^2$.

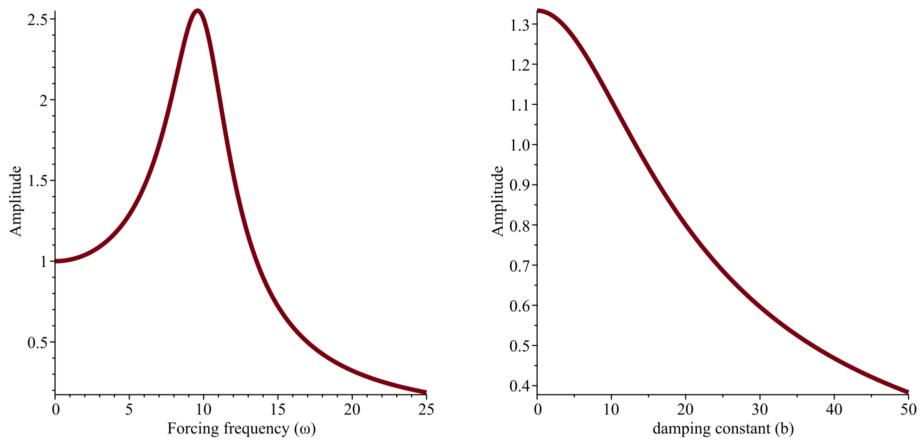

FIGURE 2. A typical resonance curve on the left, with $m = 1$, $b = 4$, $k = 100$ and $F_0 = 100$ and ω is varying. The peak corresponds to a practical resonance. On the right, b is allowed to vary, with $m = 1$, $k = 100$, $F_0 = 100$ and $\omega = 5$.

iii) If only b is allowed to vary, the amplitude of the steady-state response is maximized when $b = 0$. Further, the amplitude becomes unbounded if $\omega = \sqrt{k/m}$.

Proof. We begin with the case of varying the mass. Here, we consider the amplitude as a function of m, with F_0, k, b and ω as fixed constants. We wish to maximize

$$f(m) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}$$

We note that, the real-valued function $f(m)$ is maximized exactly when the non-negative quantity in the square root of the denominator is minimized. So we minimize

$$g(m) = (k - m\omega^2)^2 + b^2\omega^2$$

with respect to m. Then, $g'(m) = -2\omega^2(k - m\omega^2)$, and $g''(m) = 4\omega^4 m$. Here $g'(m) = 0$ when $k - m\omega^2 = 0$, or $m_{crit} = k/\omega^2$. To verify this is a minimum, substituting into the second derivative we see $g''(m_{crit}) = 4mk\omega^2 > 0$. This minimizes the denominator of the amplitude, hence the maximum amplitude occurs when $m = k/\omega^2$.

The case of varying k proceeds similarly by minimizing the denominator. We find that the amplitude is maximized when the function

$$h(k) = (k - m\omega^2)^2 + b^2\omega^2$$

is minimized with respect to k. Since $h'(k) = 2(k - m\omega^2)$ and $h''(k) = 2$, the maximum amplitude occurs when $k = m\omega^2$.

In the final case of varying the damping constant, we proceed similarly. In this case, the amplitude is maximized when the function

$$j(b) = (k - m\omega^2)^2 + b^2\omega^2$$

is minimized. Here $j'(b) = 2b\omega^2$ and $j''(b) = 2\omega^2$. At $b_{crit} = 0$, we have $j(0) = (k - m\omega^2)^2 > 0$ and the amplitude is

$$\text{Amplitude of the steady-state response}\Big|_{b=0} = \frac{F_0}{\sqrt{(k - m\omega^2)^2}}.$$

The amplitude is unbounded when $\omega = \sqrt{k/m}$. That is, we have rediscovered the pure resonance phenomenon.

In all cases, the amplitude of the steady-state response is maximized when the parameters are tuned to the natural frequency relationship $\omega = \sqrt{k/m}$.

□

2. Acknowledgment

The author wants to thank the referees and editors whose comments greatly improved the paper. The author is supported by Simons Foundation Grant 511825.

References

[1] W. E. Boyce and R. C. DiPrima, *Elementary Differential Equations and Boundary Value Problems.* John Wiley & Sons, Hoboken, 2005.

[2] C. H. Edwards and D. E. Penney, *Differential Equations & Linear Algebra.* Prentice Hall, Upper Saddle River, NJ, 2001.

[3] Noonburg, V. W., *Ordinary Differential Equations: from Calculus to dynamical systems*, MAA Press, Washington DC, 2014.

Department of Mathematics, Rose-Hulman Institute of Technology, Terre Haute, IN 47803, U.S.A.

Email address: green@rose-hulman.edu