Solving the Dirac Equation with the Unified Transform Method

Casey Garner*

Department of Mathematics
Rose-Hulman Institute of Technology
Terre Haute, IN 47803

Faculty advisor: William Green†
Department of Mathematics
Rose-Hulman Institute of Technology
Terre Haute, IN 47803

Abstract

In this article we use the Unified Transform Method to study boundary value problems for a hyperbolic system of partial differential equations from relativistic quantum mechanics. Specifically, we derive solutions to the Dirac equation in both the massive and massless cases on the half-line and the finite interval using this method.

1 Introduction

In this paper we study a family of boundary value problems for the Dirac equation in one spatial dimension,

$$\begin{cases} i\,\partial_t \Psi_1(x,t) = -i\,\partial_x \Psi_1(x,t) + m\Psi_2(x,t) \\ i\,\partial_t \Psi_2(x,t) = i\,\partial_x \Psi_2(x,t) + m\Psi_1(x,t) \end{cases} \qquad x,t \in \mathbb{R} \tag{1}$$

In particular we study the boundary value problems for this system on the half-line $x \in [0,\infty)$ and the finite interval $[0,L]$. We show the Unified Transform Method may be applied to arrive at explicit solutions determined by initial and boundary conditions.

The Dirac equation is a hyperbolic system of partial differential equations developed by Dirac to model quantum mechanical particles moving at relativistic speeds. Here $m \geq 0$

*garnercd@rose-hulman.edu
†green@rose-hulman.edu

is the mass of the quantum mechanical particle. We distinguish the massive $m > 0$ and massless $m = 0$ cases, as the underlying mathematical structure is more complicated for the massive case. We view the system as a relativistic modification of the Schrödinger equation. Due to the complicated structure of the Dirac equation, (1), compared to the Schrödinger or Klein-Gordon equation, the Dirac equation is far less studied. We refer the reader to the excellent text of Thaller, [13], for a more thorough introduction.

We employ the Unified Transform Method (UTM) which is also known as Fokas's Method in the literature. Fokas developed his method in the late 80's and early 90's while studying the linear version of the Korteweg-de Vries equation through an appropriate transform method [9]. Many of the key ideas of the UTM were presented in 1997 in [5]. The method has been shown to be adaptable to linear and certain non-linear problems, see for example [5, 6]. We note that boundary value problems for the Schrödinger equation have been well studied with the UTM, see [1, 7] among others. Just recently, [3] applied the method to a general class of systems of PDEs.

The UTM has many properties of interest. One, the UTM can handle complicated domains much easier than classical methods where different implementations are necessary for different domains. The Unified Transform Method has the ability to provide information about the number and type of boundary conditions necessary for well-posed problems. Additionally, the UTM has better computational properties that reduce the Gibbs phenomenon at the boundaries. For more on the numerical properties of the UTM see [4, 11, 12].

The UTM follows a general procedure when solving scalar PDEs and systems of linear PDEs. A list of the steps for the linear and system cases are provided in [2] and [3]. The main steps of the process include: computing the dispersion relations, determining the local relations, utilizing Green's Theorem to obtain global relations, applying the Fourier transform to compute an expression for the unknown variables, and lastly utilizing deformations in the complex plane and discrete symmetries to remove unknown constraints to arrive at a solution.

The paper is organized as follows. We first consider the Dirac equation on the half-line in Section 2. The massless and massive systems are considered in Subsections 2.1 and 2.2 respectively. We then consider the problem on the finite interval in Section 3. Again considering the massless and massive cases in Subsections 3.1 and 3.2.

2 Solving on the Half-Line

Consider the Dirac equation on the half-line with Dirichlet boundary conditions,

$$\begin{cases} i\,\partial_t \Psi_1(x,t) = -i\,\partial_x \Psi_1(x,t) + m\Psi_2(x,t), \\ i\,\partial_t \Psi_2(x,t) = i\,\partial_x \Psi_2(x,t) + m\Psi_1(x,t), \quad x \in [0,\infty),\, t \in (0,T], \\ \Psi_1(x,0) = \Psi_{1,0}(x), \quad \Psi_1(0,t) = b_1(t), \\ \Psi_2(x,0) = \Psi_{2,0}(x), \quad \Psi_2(0,t) = b_2(t), \end{cases} \tag{2}$$

where we assume $\Psi_1(x,t), \Psi_2(x,t) \to 0$ as $x \to \infty$ for all times t, T is a positive, finite time, and $m \geq 0$ is the mass of the system. To apply the Unified Transform Method to linear systems of partial differential equations, we follow the approach outlined in [3] and rewrite (2) in the form,

$$\partial_t\, Q + \Lambda(-i\partial_x)Q = 0. \tag{3}$$

Here Q is an N-dimensional vector and Λ is an $N \times N$ matrix-valued polynomial; $N = 2$ for the one-dimensional Dirac system. We now re-write (2) in matrix form as,

$$\partial_t \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} + \begin{pmatrix} \partial_x & mi \\ mi & -\,\partial_x \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 0. \tag{4}$$

Thus, from (3) we see

$$Q = \begin{pmatrix} \Psi_1(x,t) \\ \Psi_2(x,t) \end{pmatrix}, \quad \Lambda(-i\partial_x) = \begin{pmatrix} i\,[-i\partial_x] & mi \\ mi & -\,i\,[-i\partial_x] \end{pmatrix}. \tag{5}$$

The main distinction between the massive ($m > 0$) and massless ($m = 0$) systems is the level of complexity of the system of partial differential equations. When $m = 0$, Λ becomes a diagonal matrix and simplifies the solution process because the system of equations becomes decoupled. In the massless case, the problem reduces to two scalar partial differential equations; however, $m > 0$ ensures the system is coupled and Λ is not a diagonal matrix. For these reasons the massive system is more complicated than the massless; therefore, we solve the simpler massless case to introduce the technique before solving the massive case.

2.1 The Massless System on the Half-line

To solve the massless system, we set $m = 0$ in (4) giving,

$$\partial_t \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} + \begin{pmatrix} \partial_x & 0 \\ 0 & -\partial_x \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = 0,$$

and

$$\Lambda(k) = \begin{pmatrix} ik & 0 \\ 0 & -ik \end{pmatrix}. \tag{6}$$

The first step to compute the solution to the massless system is calculating the dispersion relations. Computing the dispersion relations in the systems case is analogous to applying the Unified Transform Method on scalar PDEs, see [2, 3]. However, unlike the scalar case where the unknown variable is set equal to $e^{ikx-\omega(k)t}$ to compute the dispersion relation $w(k)$, we let

$$Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} e^{ikx-\omega(k)t}.$$

Then, in the system equivalent to solving for $w(k)$ after the substitution in the scalar case, we find $\omega(k)$ such that,

$$\det(\Lambda(k) - \omega(k)I) = 0.$$

Plugging in the substitution for Q after defining $\Lambda(k)$ as in (6), computing the determinant, and solving for $\omega(t)$, we determine the branches for the massless system are,

$$\Omega_{1,2} = \pm ik. \tag{7}$$

The next step in the scalar case is to determine the local relation. This remains the same for systems, but the computation is altered. In this situation, we must rewrite (3) in divergence form (see the appendix of [8]),

$$\left(e^{-ikxI+\Omega(k)t}A(k)Q\right)_t - \left(e^{-ikxI+\Omega(k)t}A(k)X(x,t,k)Q\right)_x = 0, \tag{8}$$

where $\Omega(k) = \operatorname{diag}(\Omega_1, ..., \Omega_N)$ is a diagonal matrix of the different branches of the system and $A(k)$ diagonalizes $\Lambda(k)$, that is,

$$\Lambda(k) = A^{-1}(k)\Omega(k)A(k).$$

Following [3], matrix $X(x,t,k)$ is defined by,

$$X(x,t,k) = \frac{i}{k + i\partial_x}\Big(\Lambda(k) - \Lambda(-i\partial_x)\Big).$$

For the massless Dirac equation, since $\Psi(k) = \Omega(k)$ and $A(k) = I$, we have,

$$X(x,t,k) = \frac{i}{k+i\partial_x}\begin{pmatrix} ik - \partial_x & 0 \\ 0 & -ik + \partial_x \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It should be noted the simple form of X comes from the fact when $m = 0$ the system is decoupled. This is not the case for the massive system. From (8), we now compute the local relations for the massless Dirac equation,

$$\begin{aligned} \left(e^{-ikx+\Omega_1 t}\Psi_1\right)_t - \left(e^{-ikx+\Omega_1 t}(-\Psi_1)\right)_x &= 0, \\ \left(e^{-ikx+\Omega_2 t}\Psi_2\right)_t - \left(e^{-ikx+\Omega_2 t}\ \Psi_2\right)_x &= 0. \end{aligned} \tag{9}$$

The next step is computing the global relations from each of the local relations. In [3] a generalized expression to compute the global relations was given; however, the unknown variables in this problem are sufficiently decoupled so the scalar process can be employed to determine the global relations for (9). Computing the global relations is done by integrating each local relation over the domain, $R = \{(x,t) : x \in [0,\infty), t \in (0,T]\}$, with Green's theorem.

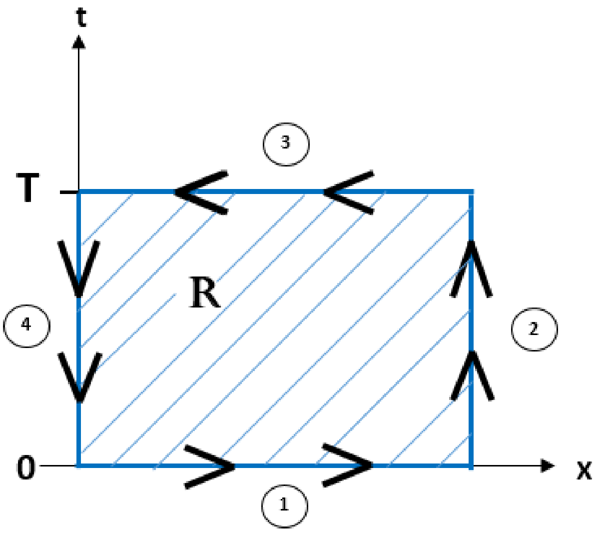

Figure 1: Domain of integration for computing the global relation with the labeled parameterizations for the necessary line integrals after application of Green's Theorem.

We first compute the global relation for Ψ_1. Integrate the first equation in (9) over R,

$$\iint_R \left[e^{-ikx+\Omega_1 t}\Psi_1\right]_t - \left[e^{-ikx+\Omega_1 t}(-\Psi_1)\right]_x dt\, dx = 0.$$

By invoking Green's Theorem we move the integration to the boundary of R,

$$\int_{\partial R} \left[e^{-ikx+\Omega_1 t}\Psi_1\right] dx + \left[e^{-ikx+\Omega_1 t}(-\Psi_1)\right] dt = 0.$$

Parameterizing the borders of the domain and integrating the respective line integrals leads to the global relation. The form of the global relation is given in terms of the Fourier transform of the unknown variable and its initial conditions. The contribution of the second line integral is zero because $\Psi_1(x,t) \to 0$ as $x \to \infty$ for all t. Thus, the first, second, and third terms in the global relation below come from the first, third, and fourth line integrals respectively.

$$\hat{\Psi}_1(k,0) - e^{\Omega_1 T}\hat{\Psi}_1(k,T) + g_0(\Omega_1,T) = 0,$$

where $\hat{\Psi}_1(k,t)$ is the Fourier transform of $\Psi_1(x,t)$ and we define the variables $g_0(\Omega_j,t)$, $j \in \{1,2\}$, which depend on the boundary conditions as,

$$g_0(\Omega_j,t) = \int_0^t e^{\Omega_j s}\Psi_j(0,s)ds = \int_0^t e^{\Omega_j s} b_j(s)ds. \tag{10}$$

It should be noted throughout this paper we define the Fourier transform and inverse transform as,

$$\mathcal{F}\{f(x)\} = \hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)dx,$$
$$\mathcal{F}^{-1}\{\hat{f}(k)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{f}(k)dk.$$

We also note the half-line definition of the Fourier transform is utilized in computation because the values of $\Psi_1(x,t)$ and $\Psi_2(x,t)$ are assumed zero for all (x,t) outside the domain R. Following the same procedure for Ψ_2 leads to the second global relation,

$$\hat{\Psi}_2(k,0) - e^{\Omega_2 T}\hat{\Psi}_2(k,T) - g_0(\Omega_2,T) = 0.$$

We can replace T in the global relations with any $t \in [0,T]$ because both equations are valid for all feasible values of t. Thus, the global relations for the massless case are,

$$\begin{aligned} \hat{\Psi}_1(k,0) - e^{\Omega_1 t}\hat{\Psi}_1(k,t) + g_0(\Omega_1,t) = 0, \\ \hat{\Psi}_2(k,0) - e^{\Omega_2 t}\hat{\Psi}_2(k,t) - g_0(\Omega_2,t) = 0. \end{aligned} \tag{11}$$

The next step is to utilize the inverse Fourier transform to obtain expressions for $\Psi_1(x,t)$ and $\Psi_2(x,t)$ from (11). Multiplying the first equation in (11) by $e^{-\Omega_1 t}$ and the second by $e^{-\Omega_2 t}$, then algebraically solving for the Fourier transforms yields

$$\hat{\Psi}_1(k,t) = e^{-\Omega_1 t}\Big(\hat{\Psi}_{1,0}(k) + g_0(\Omega_1,t)\Big),$$
$$\hat{\Psi}_2(k,t) = e^{-\Omega_2 t}\Big(\hat{\Psi}_{2,0}(k) - g_0(\Omega_2,t)\Big).$$

Taking the inverse Fourier transforms of the above expressions and substituting in the dispersion relations (7),

$$\begin{aligned} \Psi_1(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-t)}\Big(\hat{\Psi}_{1,0}(k) + g_0(\Omega_1,t)\Big)dk, \\ \Psi_2(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x+t)}\Big(\hat{\Psi}_{2,0}(k) - g_0(\Omega_2,t)\Big)dk. \end{aligned} \tag{12}$$

Because (12) is overdetermined, our last step is to remove unnecessary boundary conditions. When applying Fokas's Method, normally one accomplishes this through contours.

In this case, the dispersion relations have zero non-trivial discrete symmetries, so we use analytic properties to simplify (12). Expanding (12) with the definitions for $g_0(\Omega_j, t)$, (10),

$$\begin{aligned}
\Psi_1(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-t)}\hat{\Psi}_{1,0}(k)\, dk \;+\; \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^t e^{ik(x-t+s)} b_1(s)\, ds\, dk, \\
\Psi_2(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x+t)}\hat{\Psi}_{2,0}(k)\, dk \;-\; \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^t e^{ik(x+t-s)} b_2(s)\, ds\, dk.
\end{aligned} \tag{13}$$

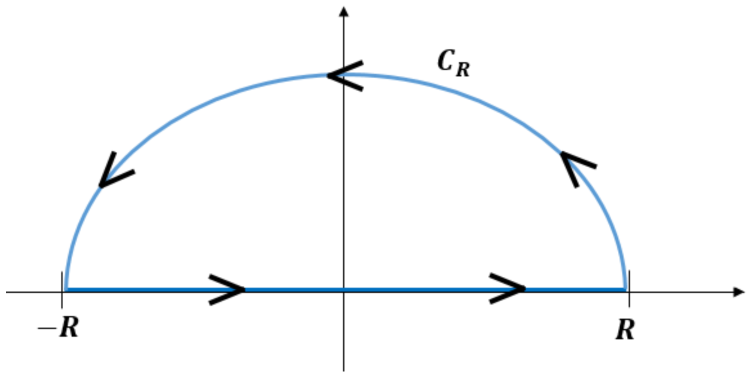

Figure 2: Contour used to remove the boundary conditions for $\Psi_2(x,t)$ in (13)

In the second equation of (13), we see the integrand for the second integral over the real line is analytic. Since $x+t-s>0$, if we integrate

$$\int_0^t e^{ik(x+t-s)} b_2(s)\, ds$$

along the contour $C = [-R, R] \cup C_R$ where C_R is the circular arc of radius R centered at the origin in $\mathbb{C}^+$, by Cauchy's Theorem,

$$\left[\int_{-R}^{R} + \int_{C_R}\right]\left(\int_0^t e^{ik(x+t-s)} b_2(s)\, ds\right) dk = 0.$$

Taking the limit as $R \to \infty$ and applying Jordan's Lemma, due to the exponential decay, the integral along C_R goes to 0; therefore,

$$\int_{-\infty}^{\infty}\int_0^t e^{ik(x+t-s)} b_2(s)\, ds\, dk = 0,$$

and using Fourier inversion,

$$\Psi_2(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x+t)}\hat{\Psi}_{2,0}(k)\, dk = \Psi_{2,0}(x+t).$$

To simplify $\Psi_1(x,t)$ in (13) we work in cases.

Case 1: If $x > t$, then $x - t + s > 0$ and using the previous arguments,

$$\Psi_1(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-t)}\hat{\Psi}_{1,0}(k)\, dk = \Psi_{1,0}(x-t).$$

Case 2: If $x < t$, then $x - t + s < 0$ and we cannot use contour C to eliminate the second integral defining $\Psi_1(x,t)$; however, using the distributional definition of the delta distribution we see,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{0}^{t} e^{ik(x-t+s)} b_1(s)\, ds\, dk \ = \ b_1(t-x).$$

So, for $x < t$,

$$\Psi_1(x,t) = b_1(t-x)$$

by the definition of the inverse Fourier transform and $\Psi_{1,0}(x-t) = 0$ since $x - t < 0$.

Remark: It should be noted the delta distribution argument in Case 2 could have been utilized for Case 1 instead of using contour C; however, this is only due to the simple dispersion relation (7) for the massless Dirac equation. Since invoking the delta distribution does not generalize to determining the massive solution we use the contour argument.

These simplifications give the solution to the massless Dirac equation on the half-line,

$$\Psi_1(x,t) = \begin{cases} \Psi_{1,0}(x-t), & x > t, \\ \\ b_1(t-x), & x < t \end{cases} \tag{14}$$

$$\Psi_2(x,t) = \Psi_{2,0}(x+t)$$

The form of these solutions were expected because the massless case on the half-line for (2) reduces to the transport equation for $\Psi_1(x,t)$ and $\Psi_2(x,t)$. The method of characteristics gives the solution to the transport equation as left and right moving waves agreeing with (14). The two different cases correspond with how the boundary conditions and initial conditions propagate. The boundary and initial conditions affect disjoint sections of the domain because information travels along the characteristics. In contrast to these simple solutions, the massive case is more challenging and requires additional work to remove unnecessary terms containing the Dirichlet boundary conditions.

2.2 Massive System $(m > 0)$

Now we solve the massive problem on the half-line. The mass term will now be some positive real number, so we are solving,

$$\partial_t \begin{pmatrix}\Psi_1 \\ \Psi_2\end{pmatrix} + \begin{pmatrix}\partial_x & im \\ im & -\partial_x\end{pmatrix}\begin{pmatrix}\Psi_1 \\ \Psi_2\end{pmatrix} = 0,$$

and

$$\Lambda(k) = \begin{pmatrix} ik & im \\ im & -ik \end{pmatrix}.$$

We begin again by computing the dispersion relation. The computation follows as before in the massless case. We find $\omega(k)$ such that,

$$\det(\Lambda(k) - \omega(k)I) = 0.$$

$\Lambda(k)$ is no longer a diagonal matrix which adds complexity to the dispersion relation. Performing the calculations, the branches for the massive system are,

$$\Omega_{1,2} = \pm i\sqrt{k^2 + m^2}. \tag{15}$$

Obtaining the local relation for the massive system still requires the system to be in divergence form given by,

$$\left(e^{-ikxI+\Omega(k)t} A(k)Q\right)_t - \left(e^{-ikxI+\Omega(k)t} A(k)X(x,t,k)Q\right)_x = 0.$$

The value of $X(x,t,k)$ does not change from the massless case, but the diagonalization of $\Lambda(k)$ is different due to the different branches. For the massive case,

$$A = \begin{pmatrix} im & \Omega_1 - ik \\ im & \Omega_2 - ik \end{pmatrix}, \quad X = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

With these definitions, after simplifying we obtain the local relations for $j \in \{1,2\}$,

$$\left(e^{-ikx+\Omega_j t}\,[\,(im)\Psi_1 + (\Omega_j - ik)\Psi_2\,]\right)_t - \left(e^{-ikx+\Omega_j t}\,[\,(-im)\Psi_1 + (\Omega_j - ik)\Psi_2\,]\right)_x = 0.$$

The global relations for the massive case are determined in the same manner as before in the massless case. We first integrate the local relations over the entire domain, R,

$$\iint_R \left(e^{-ikx+\Omega_j t}\,[\,(im)\Psi_1 + (\Omega_j - ik)\Psi_2\,]\right)_t - \left(e^{-ikx+\Omega_j t}\,[\,(-im)\Psi_1 + (\Omega_j - ik)\Psi_2\,]\right)_x dx\,dt = 0,$$

apply Green's Theorem to move the integration to the boundary,

$$\int_{\partial R} \left(e^{-ikx+\Omega_j t}\,[\,(im)\Psi_1 + (\Omega_j - ik)\Psi_2\,]\right)dx + \left(e^{-ikx+\Omega_j t}\,[\,(-im)\Psi_1 + (\Omega_j - ik)\Psi_2\,]\right)dt = 0,$$

and lastly parameterize the ∂R and use the properties of Ψ_1 and Ψ_2 to compute the various line integrals. Performing the integration allows us to the write the two global relations. The global relations for the massive case are,

$$(im)\hat{\Psi}_1(k,0) + (\Omega_j - ik)\hat{\Psi}_2(k,0) + (-im)e^{\Omega_j t}\hat{\Psi}_1(k,t) \\ + (ik - \Omega_j)e^{\Omega_j t}\hat{\Psi}_2(k,t) + (im)h_{0,1}(\Omega_j,t) + (ik - \Omega_j)h_{0,2}(\Omega_j,t) = 0, \quad (16)$$

for $j \in \{1, 2\}$ where T has been replaced with t and

$$h_{0,i}(\Omega_j,t) = \int_0^t e^{\Omega_j s}\Psi_i(0,s)ds = \int_0^t e^{\Omega_j s}b_i(s)ds. \quad (17)$$

In order to apply the inverse Fourier transform, we take the two global relations and solve for $\hat{\Psi}_1(k,t)$ and $\hat{\Psi}_2(k,t)$. We multiply equation (16), with $j = 1$ and $j = 2$, by $e^{-\Omega_1 t}$ and $e^{-\Omega_2 t}$ respectively to obtain,

$$\begin{aligned} (-im)\hat{\Psi}_1(k,t) + (ik - \Omega_1)\hat{\Psi}_2(k,t) + C_1(k,t)e^{-\Omega_1 t} &= 0, \\ (-im)\hat{\Psi}_1(k,t) + (ik - \Omega_2)\hat{\Psi}_2(k,t) + C_2(k,t)e^{-\Omega_2 t} &= 0, \end{aligned} \quad (18)$$

where

$$C_j(k,t) = (im)\hat{\Psi}_{1,0}(k) + (\Omega_j - ik)\hat{\Psi}_{2,0}(k) + (im)h_{0,1}(\Omega_j,t) + (ik - \Omega_j)h_{0,2}(\Omega_j,t), \quad j \in \{1,2\}.$$

Subtracting the second equation in (18) from the first, and solving for $\hat{\Psi}_2(k,t)$, using the fact $\Omega_2 - \Omega_1 = 2\Omega_2$, yields,

$$\hat{\Psi}_2(k,t) = \frac{1}{2\Omega_2}\Big(C_2(k,t)e^{-\Omega_2 t} - C_1(k,t)e^{-\Omega_1 t}\Big). \quad (19)$$

Adding both equations in (18), substituting (19), and solving for $\hat{\Psi}_1(k,t)$,

$$\hat{\Psi}_1(k,t) = \left(\frac{\Omega_2 i + k}{2m\Omega_1}\right)C_1(k,t)e^{-\Omega_1 t} + \left(\frac{\Omega_2 i - k}{2m\Omega_1}\right)C_2(k,t)e^{-\Omega_2 t}. \quad (20)$$

Taking the inverse Fourier transforms of (19) and (20) we obtain (21) and (22).

$$\Psi_1(x,t) = \frac{1}{4m\,\pi}\int_{-\infty}^{\infty}\left(\frac{\Omega_2 i + k}{\Omega_1}\right)C_1(k,t)e^{ikx - \Omega_1 t}dk \\ + \frac{1}{4m\,\pi}\int_{-\infty}^{\infty}\left(\frac{\Omega_2 i - k}{\Omega_1}\right)C_2(k,t)e^{ikx - \Omega_2 t}dk, \quad (21)$$

$$\Psi_2(x,t) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{e^{ikx-\Omega_2 t}}{\Omega_2}C_2(k,t)\,dk \; - \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{e^{ikx-\Omega_1 t}}{\Omega_2}C_1(k,t)\,dk. \quad (22)$$

Both equations are overdetermined because unnecessary boundary conditions have been specified. To remedy this we utilize the discrete symmetry of the dispersion relations, $k \to -k$. Note, $\Omega_{1,2}(k)$ are invariant under $w(k) = -k$ since $\Omega_{1,2}(k) = \Omega_{1,2}(w(k))$. Applying this transformation to (20) and (19) we have,

$$\hat{\Psi}_1(-k,t) = \left(\frac{\Omega_2 i - k}{2m\Omega_1}\right)C_1(-k,t)e^{-\Omega_1 t} + \left(\frac{\Omega_2 i + k}{2m\Omega_1}\right)C_2(-k,t)e^{-\Omega_2 t}, \tag{23}$$

$$\hat{\Psi}_2(-k,t) = \frac{1}{2\Omega_2}\left(C_2(-k,t)e^{-\Omega_2 t} - C_1(-k,t)e^{-\Omega_1 t}\right). \tag{24}$$

Taking the inverse Fourier transform of (23) and (24) and grouping all the terms on one-side yields expressions which sum to zero. Adding these expressions to (21) and (22) removes some of the boundary terms and yields the solution to the massive Dirac equation on the half-line,

$$\begin{aligned}\Psi_1(x,t) = {} & \frac{1}{4m\pi}\int_{-\infty}^{\infty}\frac{e^{ikx-\Omega_1 t}}{\Omega_1}\Big[m\Omega_1\big(\hat{\Psi}_{1,0}(k) - \hat{\Psi}_{1,0}(-k)\big) + imk\big(\hat{\Psi}_{1,0}(k) + \hat{\Psi}_{1,0}(-k)\big) \\ & \quad + \ (ik^2 + i\Omega_1^2)\big(\hat{\Psi}_{2,0}(-k) - \hat{\Psi}_{2,0}(k)\big) + (2imk)h_{0,1}(\Omega_1,t)\Big]dk \\ & + \frac{1}{4m\pi}\int_{-\infty}^{\infty}\frac{e^{ikx-\Omega_2 t}}{\Omega_1}\Big[m\Omega_1\big(\hat{\Psi}_{1,0}(k) - \hat{\Psi}_{1,0}(-k)\big) - imk\big(\hat{\Psi}_{1,0}(k) + \hat{\Psi}_{1,0}(-k)\big) \\ & \quad + (i\Omega_1\Omega_2 - ik^2)\big(\hat{\Psi}_{2,0}(-k) - \hat{\Psi}_{2,0}(k)\big) - (2imk)h_{0,1}(\Omega_2,t)\Big]dk \\ & + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{\Psi}_1(-k,t)\, dk \end{aligned} \tag{25}$$

$$\begin{aligned}\Psi_2(x,t) = {} & \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{e^{ikx-\Omega_2 t}}{\Omega_2}\Big[im\big(\hat{\Psi}_{1,0}(k) - \hat{\Psi}_{1,0}(-k)\big) + \Omega_2\big(\hat{\Psi}_{2,0}(k) - \hat{\Psi}_{2,0}(-k)\big) \\ & \quad - ik\big(\hat{\Psi}_{2,0}(k) - \hat{\Psi}_{2,0}(-k)\big) + (2ik)h_{0,2}(\Omega_2,t)\Big]\ dk \\ & - \frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{e^{ikx-\Omega_1 t}}{\Omega_2}\Big[im\big(\hat{\Psi}_{1,0}(k) - \hat{\Psi}_{1,0}(-k)\big) + \Omega_1\big(\hat{\Psi}_{2,0}(k) - \hat{\Psi}_{2,0}(-k)\big) \\ & \quad - ik\big(\hat{\Psi}_{2,0}(k) - \hat{\Psi}_{2,0}(-k)\big) + (2ik)h_{0,2}(\Omega_1,t)\Big]\ dk \\ & + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{\Psi}_2(-k,t)\, dk, \end{aligned} \tag{26}$$

where $h_{0,1}$ and $h_{0,2}$, defined in (17), contain the Dirichlet boundary conditions, and $\hat{\Psi}_{1,0}$

and $\hat{\Psi}_{2,0}$ are the Fourier transforms of the initial conditions for Ψ_1 and Ψ_2 respectively.

Making use of the discrete symmetry, $k \to -k$, removed $h_{0,1}$ from (26) and $h_{0,2}$ from (25); therefore, the solutions for Ψ_1 and Ψ_2 in (25) and (26) depend only on their respective boundary conditions. It is worth noting (25) and (26) are similar in form to the solutions of the Klein-Gordon equation in Section 7 of [3]. This meshes well with the interpretation of the Dirac equation as a square root of a diagonal system of Klein-Gordon equations.

3 Solving on the Finite Interval

We continue to generalize the solution to the Dirac equation by solving the system on the finite interval, $[0, L]$, for $L \in \mathbb{R}^+$ finite. We first solve the massless and then the massive systems. As before, the massive system has a more complex solution, but clear and expected consistency exists between the finite interval and half-line solutions.

The only change in the Dirac equation set-up from the previous section is the addition of another set Dirichlet boundary conditions at $x = L$ and $x \in [0, L]$. The finite Dirac equation is,

$$\begin{gathered} i\,\partial_t \Psi_1(x,t) = -i\,\partial_x \Psi_1(x,t) + m\Psi_2(x,t), \quad x \in [0,L], t \in (0,T], \\ i\,\partial_t \Psi_2(x,t) = i\,\partial_x \Psi_2(x,t) + m\Psi_1(x,t), \quad x \in [0,L], t \in (0,T], \\ \\ \Psi_1(x,0) = \Psi_{1,0}(x),\ \Psi_2(x,0) = \Psi_{2,0}(x), \quad x \in [0,L], \\ \Psi_1(0,t) = \alpha_1(t),\ \Psi_2(0,t) = \alpha_2(t), \quad t \in [0,T], \\ \Psi_1(L,t) = \beta_1(t),\ \Psi_2(L,t) = \beta_2(t), \quad t \in [0,T], \end{gathered} \tag{27}$$

where T is a positive, finite, time value, and $\alpha_i(t)$ and $\beta_i(t)$, $i \in \{1, 2\}$, are the Dirichlet boundary conditions for $\Psi_1(x,t)$ and $\Psi_2(x,t)$ at $x = 0$ and $x = L$ respectively.

3.1 Massless System $(m = 0)$

All of the previous work in the last section for the massless and massive systems on the half-line remain the same until the global relations are computed. The dispersion and local relations are not altered by the domain of the problem or the known boundary conditions; therefore, we begin the solution process for the massless Dirac equation on the finite interval at the computation of the global relations.

The same process is undertaken to compute the global relations. We integrate the local relation for $\Psi_1(x,t)$ over the domain, $R = \{(x,t)\ : x \in [0,L], t \in [0,T]\}$,

$$\iint_R \left[e^{-ikx+\Omega_1 t}\Psi_1\right]_t - \left[e^{-ikx+\Omega_1 t}(-\Psi_1)\right]_x dt\,dx = 0,$$

and apply Green's Theorem to move the integration to the boundary,

$$\int_{\partial R} \left[e^{-ikx+\Omega_1 t}\Psi_1\right] dx + \left[e^{-ikx+\Omega_1 t}(-\Psi_1)\right] dt = 0.$$

Performing the integration leads to the global relation,

$$\hat{\Psi}_1(k,0) - e^{-ikL}B_L(\Omega_1,t) - e^{\Omega_1 t}\hat{\Psi}_1(k,t) + B_0(\Omega_1,t) = 0, \tag{28}$$

where we define,

$$B_0(\Omega_j,t) = \int_0^t e^{\Omega_j s}\Psi_j(0,s)ds = \int_0^t e^{\Omega_j s}\alpha_j(s)ds, \tag{29}$$

$$B_L(\Omega_j,t) = \int_0^t e^{\Omega_j s}\Psi_j(L,s)ds = \int_0^t e^{\Omega_j s}\beta_j(s)ds. \tag{30}$$

Following the same procedure for the local relation of $\Psi_2(x,t)$ gives the second global relation,

$$\hat{\Psi}_2(k,0) + e^{-ikL}B_L(\Omega_2,t) - e^{\Omega_2 t}\hat{\Psi}_2(k,t) - B_0(\Omega_2,t) = 0. \tag{31}$$

We now solve for $\hat{\Psi}_1(k,t)$ and $\hat{\Psi}_2(k,t)$ in (28) and (31) and apply the inverse Fourier transform. Solving for the Fourier transforms,

$$\hat{\Psi}_1(k,t) = e^{-\Omega_1 t}\Big(\hat{\Psi}_1(k,0) - e^{-ikL}B_L(\Omega_1,t) + B_0(\Omega_1,t)\Big),$$
$$\hat{\Psi}_2(k,t) = e^{-\Omega_2 t}\Big(\hat{\Psi}_2(k,0) + e^{-ikL}B_L(\Omega_2,t) - B_0(\Omega_2,t)\Big),$$

and taking the inverse Fourier transform of these expressions leads to,

$$\Psi_1(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-t)}\Big(\hat{\Psi}_{1,0}(k) - e^{-ikL}B_L(\Omega_1,t) + B_0(\Omega_1,t)\Big)dk, \tag{32}$$

$$\Psi_2(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x+t)}\Big(\hat{\Psi}_{2,0}(k) + e^{-ikL}B_L(\Omega_2,t) - B_0(\Omega_2,t)\Big)dk. \tag{33}$$

As in the half-line, massless case, we can simplify (32) and (33) with regards to the boundary conditions to remove the over determining elements of the equations. Expanding (32) and (33) with definitions (29) and (30),

$$\Psi_1(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-t)}\hat{\Psi}_{1,0}(k)\,dk - \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^t e^{ik(x-t-L+s)}\beta_1(s)ds\,dk + \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^t e^{ik(x-t+s)}\alpha_1(s)\,ds\,dk \tag{34}$$

$$\Psi_2(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x+t)}\hat{\Psi}_{2,0}(k)\,dk + \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^t e^{ik(x+t-L-s)}\beta_2(s)ds\,dk - \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_0^t e^{ik(x+t-s)}\alpha_2(s)\,ds\,dk \tag{35}$$

Simplifying (34) first, since $x < L$ and $s < t$, this implies $x - t - L + s < 0$; therefore, using a similar contour to C, except with C_R in the lower half plane in $\mathbb{C}$, we can apply Cauchy's Theorem and Jordan's Lemma to show the integral vanishes due to the exponential decay and analytic properties of the integrand. The last integral in (34) is a previous case dealt with in Subsection 2.1, so utilizing the inverse Fourier transform,

$$\Psi_1(x,t) = \begin{cases} \Psi_{1,0}(x-t), & x > t, \\ \\ \alpha_1(t-x), & x < t. \end{cases}$$

In (35), the last integral was shown to vanish in Subsection 2.1 by applying contour C. The second integral in (35) is not signed because $t - s > 0$ and $x - L < 0$, so we use the definition of the delta distribution to obtain,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_0^t e^{ik(x+t-L-s)} \beta_2(s) ds \, dk = \beta_2(x+t-L).$$

When $x+t-L < 0$, $\beta_2(x+t-L) = 0$ because the boundary condition is only defined for positive time values. Therefore, applying the inverse Fourier transform to these cases,

$$\Psi_2(x,t) = \begin{cases} \Psi_{2,0}(x+t), & x+t < L, \\ \\ \beta_2(x+t-L), & x+t > L, \end{cases}$$

where we have $\Psi_{2,0}(x+t) = 0$ when $x + t > L$ because these arguments are outside the domain where initial conditions are specified. Therefore, the solution to the massless Dirac equation on the finite interval $[0, L]$ is,

$$\Psi_1(x,t) = \begin{cases} \Psi_{1,0}(x-t), & x > t, \\ \\ \alpha_1(t-x), & x < t \end{cases} \tag{36}$$

$$\Psi_2(x,t) = \begin{cases} \Psi_{2,0}(x+t), & x+t < L, \\ \\ \beta_2(x+t-L), & x+t > L. \end{cases}$$

Each different case aligns with how information travels along the characteristics. $\Psi_1(x,t)$ does not need the left boundary specified because it is determined by the initial and right boundary conditions traveling to the left. This is also true for $\Psi_2(x,t)$ except the right boundary condition is determined by the initial and left boundary conditions propagating rightward.

3.2 **Massive System** $(m > 0)$

For the massive system, we again start at the global relations because the dispersion and local relations are not affected by restraining the domain of the PDEs to a finite region in the plane. These properties are only dictated by local information and thus are not altered by the global situation of the partial differential equations.

Using the same process to compute the global relation as before, integrate the local relations over the domain, $R = \{(x,t) \ : x \in [0,L], t \in [0,T]\}$,

$$\iint_R \left(e^{-ikx+\Omega_j t}\,[\,(im)\Psi_1 + (\Omega_j - ik)\Psi_2\,]\right)_t - \left(e^{-ikx+\Omega_j t}\,[\,(-im)\Psi_1 + (\Omega_j - ik)\Psi_2\,]\right)_x dx\,dt = 0,$$

and apply Green's Theorem to move the integration to the boundary,

$$\int_{\partial R} \left(e^{-ikx+\Omega_j t}\,[\,(im)\Psi_1 + (\Omega_j - ik)\Psi_2\,]\right)dx + \left(e^{-ikx+\Omega_j t}\,[\,(-im)\Psi_1 + (\Omega_j - ik)\Psi_2\,]\right)dt = 0.$$

Performing the necessary line integrals gives the following global relations,

$$\begin{aligned}(im)\hat{\Psi}_1(k,0) &+ (\Omega_j - ik)\hat{\Psi}_2(k,0)\\ &+ (-im)e^{-ikL}B_{L,1}(\Omega_j,t) + (\Omega_j - ik)e^{-ikL}B_{L,2}(\Omega_j,t)\\ &+ (-im)e^{\Omega_j t}\hat{\Psi}_1(k,t) + (ik - \Omega_j)e^{\Omega_j t}\hat{\Psi}_2(k,t)\\ &+ (im)B_{0,1}(\Omega_j,t) + (ik - \Omega_j)B_{0,2}(\Omega_j,t) = 0,\end{aligned} \tag{37}$$

for $j \in \{1,2\}$ where we define,

$$B_{0,i}(\Omega_j,t) = \int_0^t e^{\Omega_j s}\Psi_i(0,s)ds = \int_0^t e^{\Omega_j s}\alpha_i(s)ds, \tag{38}$$

$$B_{L,i}(\Omega_j,t) = \int_0^t e^{\Omega_j s}\Psi_i(L,s)ds = \int_0^t e^{\Omega_j s}\beta_i(s)ds. \tag{39}$$

Comparing these global relations to the ones for the massive case on the half-line, many similarities are noted with the addition of two extra terms and the new definitions for the integrals on the left and right boundaries. These similarities ease the algebra necessary to obtain valid expressions for $\Psi_1(x,t)$ and $\Psi_2(x,t)$.

We now use the global relations and solve for $\hat{\Psi}_1(k,t)$ and $\hat{\Psi}_2(k,t)$ to apply the inverse Fourier transform. This was accomplished on the half-line by multiplying the global relations by different exponentials and adding and subtracting the respective expressions. Due to the symmetry between the half-line and finite interval cases, the same steps as before can be taken, but new definitions for $C_1(k,t)$ and $C_2(k,t)$ in (32) and (33) are needed to account for the additional terms left from integrating over the domain. On the finite interval, the new definitions for these expressions are,

$$\begin{aligned}\tilde{C}_j(k,t) = (im)\hat{\Psi}_{1,0}(k) \ + \ &(\Omega_j - ik)\hat{\Psi}_{2,0}(k)\\ &- (im)e^{-ikL}B_{L,1}(\Omega_j,t) \ + \ (\Omega_j - ik)e^{-ikL}B_{L,2}(\Omega_j,t)\\ &+ \ (im)B_{0,1}(\Omega_j,t) \ + \ (ik-\Omega_j)B_{0,2}(\Omega_j,t), \quad j \in \{1,2\}.\end{aligned}$$

Similar to the half-line, applying the discrete symmetry $k \to -k$ enables us to eliminate unnecessary boundary terms and obtain the solution on the finite interval. Working through the same argument, we obtain our solutions (40) and (41). We were able to remove the left boundary condition of Ψ_2, contained in $B_{0,2}$ from (40) and the left boundary condition of Ψ_1, in $B_{0,1}$ from (41).

$$\begin{aligned}\Psi_1(x,t) = \frac{1}{4m\pi}\int_{-\infty}^{\infty} \frac{e^{ikx-\Omega_1 t}}{\Omega_1}\big[m\Omega_1\big(\hat{\Psi}_{1,0}(k) - \hat{\Psi}_{1,0}(-k) + B_{L,1}(\Omega_1,t)\big(e^{ikL} - e^{-ikL}\big)\big)\\ + imk\big(\hat{\Psi}_{1,0}(k) + \hat{\Psi}_{1,0}(-k) - B_{L,1}(\Omega_1,t)\big(e^{ikL} + e^{-ikL}\big)\big)\\ + \ (i\Omega_1^2 + ik^2)\big(\hat{\Psi}_{2,0}(-k) - \hat{\Psi}_{2,0}(k) + B_{L,2}(\Omega_1,t)\big(e^{ikL} - e^{-ikL}\big)\big) + (2imk)B_{0,1}(\Omega_1,t)\big]\,dk\\ + \frac{1}{4m\pi}\int_{-\infty}^{\infty} \frac{e^{ikx-\Omega_2 t}}{\Omega_1}\big[m\Omega_1\big(\hat{\Psi}_{1,0}(k) - \hat{\Psi}_{1,0}(-k) + B_{L,1}(\Omega_2,t)\big(e^{ikL} - e^{-ikL}\big)\big)\\ - imk\big(\hat{\Psi}_{1,0}(k) + \hat{\Psi}_{1,0}(-k) - B_{L,1}(\Omega_2,t)\big(e^{ikL} + e^{-ikL}\big)\big)\\ + \ (i\Omega_1\Omega_2 - ik^2)\big(\hat{\Psi}_{2,0}(-k) - \hat{\Psi}_{2,0}(k) + B_{L,2}(\Omega_2,t)\big(e^{ikL} - e^{-ikL}\big)\big) - (2imk)B_{0,1}(\Omega_2,t)\big]\,dk\\ + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{\Psi}_1(-k,t)\,dk \qquad (40)\end{aligned}$$

$$\Psi_2(x,t) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{e^{ikx-\Omega_2 t}}{\Omega_2}\big[im\big(\hat{\Psi}_{1,0}(k) - \hat{\Psi}_{1,0}(-k)\big) + \Omega_2\big(\hat{\Psi}_{2,0}(k) - \hat{\Psi}_{2,0}(-k)\big)$$
$$- ik\big(\hat{\Psi}_{2,0}(k) + \hat{\Psi}_{2,0}(-k)\big) + imB_{L,1}(\Omega_2,t)\big(e^{ikL} - e^{-ikL}\big)$$
$$+ \Omega_2 B_{L,2}(\Omega_2,t)\big(e^{-ikL} - e^{ikL}\big) - ikB_{L,2}(\Omega_2,t)\big(e^{ikL} + e^{-ikL}\big) + (2ik)B_{0,2}(\Omega_2,t)\big]\, dk$$

$$- \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{e^{ikx-\Omega_1 t}}{\Omega_2}\big[im\big(\hat{\Psi}_{1,0}(k) - \hat{\Psi}_{1,0}(-k)\big) + \Omega_2\big(\hat{\Psi}_{2,0}(k) - \hat{\Psi}_{2,0}(-k)\big)$$
$$- ik\big(\hat{\Psi}_{2,0}(k) + \hat{\Psi}_{2,0}(-k)\big) + imB_{L,1}(\Omega_2,t)\big(e^{ikL} - e^{-ikL}\big)$$
$$+ \Omega_2 B_{L,2}(\Omega_2,t)\big(e^{-ikL} - e^{ikL}\big) - ikB_{L,2}(\Omega_2,t)\big(e^{ikL} + e^{-ikL}\big) + (2ik)B_{0,2}(\Omega_2,t)\big]\, dkk$$

$$+ \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\hat{\Psi}_2(-k,t)\, dk. \tag{41}$$

4 Acknowledgements

First, I want to give the utmost thanks to my undergraduate thesis advisor William Green for his guidance and mentorship. In addition, I want to thank Dionyssios Mantzavinos for providing feedback on this problem. The insights of both advanced my undergraduate thesis immensely.

References

[1] A. Boutet de Monvel, A. S. Fokas, D. Shepelsky *Analysis of the Global Relation for the Nonlinear Schrödinger Equation on the Half-line*, Lett. Math. Phys. 65 3 (2003), 199-212.

[2] B. Deconinck, Q. Guo, E. Shilzerman, and V. Vasan, *Fokas's Unified Transform Method for Linear Systems,* Quart. Appl. Math. 76 (2018), 463–488 .

[3] B. Deconinck, T. Trogdon, and V. Vasan, *The method of Fokas for solving linear partial differential equations,* SIAM Rev. **56** (2014), no 1, 159-186, DOI 10.1137/110821871.

[4] N. Flyer and A.S. Fokas, *A hybrid analytical-numerical method for solving evolution partial differential equations. I. the half-line*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 464 (2008), pp. 1823-1849.

[5] A. S. Fokas *A unified transform method for solving linear and certain nonlinear PDEs*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 453 (1997), 1411–1443.

[6] A.S. Fokas, *A Unified Approach to Boundary Value Problems,* CBMS-NSF Regional Conf. Ser. in Appl. Math. 78, SIAM, Philadelphia, PA, 2008.

[7] A. S. Fokas, I. M. Gel'fand *Integrability of Linear and Nonlinear Evolution Equations and the Associated Nonlinear Fourier Transforms*, Lett. Math. Phys. 32 (1994), 189–210.

[8] A. S. Fokas, A. R. Its *The Linearization of the Initial-Boundary Value Problem of the Nonlinear Schrödinger Equation*, SIAM J. Appl. Math. 27 3 (1996), 738–764.

[9] A.S. Fokas and B. Pelloni, *Boundary value problems for Boussinesq type systems,* Math. Phys. Anal. Geom. **8** (2005), no. 1, 59-96, DOI 10.1007/s11040-004-1650-6. MR2136653

[10] E.-N.G. Grylonakis, C.K. Filelis-Papadopoulos, G.A. Gravvanis, A.S. Fokas. *An iterative spatial-stepping numerical method for linear elliptic PDEs using the Unified Transform.* Journal of Computational and Applied Mathematics 352, pages 194-209.

[11] A.G. Sifalakis, A.S. Fokas, S.R. Fulton, Y.G. Saridakis. *The generalize Dirichlet-Neumann map for linear elliptic PDEs and its numerical implementation.* Journal of Computational and Applied Mathematics 219, pages 9-34.

[12] Thaller, B. *The Dirac equation.* Texts and Monographs in Physics. Springer-Verlag, Berlin, 1992.