DECAY ESTIMATES FOR FOUR DIMENSIONAL SCHRÖDINGER, KLEIN-GORDON AND WAVE EQUATIONS WITH OBSTRUCTIONS AT ZERO ENERGY

WILLIAM R. GREEN AND EBRU TOPRAK

Abstract. We investigate dispersive estimates for the Schrödinger operator $H = -\Delta + V$ with V is a real-valued decaying potential when there are zero energy resonances and eigenvalues in four spatial dimensions. If there is a zero energy obstruction, we establish the low-energy expansion

$$e^{itH}\chi(H)P_{ac}(H) = O(1/(\log t))A_0 + O(1/t)A_1 + O((t\log t)^{-1})A_2 + O(t^{-1}(\log t)^{-2})A_3.$$

Here $A_0, A_1 : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$, while A_2, A_3 are operators between logarithmically weighted spaces, with A_0, A_1, A_2 finite rank operators, further the operators are independent of time. We show that similar expansions are valid for the solution operators to Klein-Gordon and wave equations. Finally, we show that under certain orthogonality conditions, if there is a zero energy eigenvalue one can recover the $|t|^{-2}$ bound as an operator from $L^1 \to L^\infty$. Hence, recovering the same dispersive bound as the free evolution in spite of the zero energy eigenvalue.

1. Introduction

The free Schrödinger evolution on $\mathbb{R}^n$, $e^{-it\Delta}$ maps $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ with norm bounded by $C_n|t|^{-n/2}$. This can be seen by the triangle inequality and the representation

$$e^{-it\Delta}f(x) = \frac{1}{(4\pi it)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-i|x-y|^2/4t}f(y)\,dy.$$

In this paper we study the dispersive properties of the operator e^{itH} where $H = -\Delta + V$ is a Schrödinger operator perturbed by a real-valued, decaying potential V. Formally, this defines the solution operator to the perturbed Schrödinger equation

$$iu_t + Hu = 0, \qquad u(x,0) = f(x). \tag{1}$$

That is, the solution to (1) may be expressed as $u(x,t) = e^{itH}f(x)$.

Quantifying the dispersive properties of the solution operator is a well-studied problem. In general, with P_{ac} the projection onto the absolutely continuous spectral subspace of

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$L^2(\mathbb{R}^n)$ associated to the Schrödinger operator H, the dispersive estimates are expressed as

$$\|e^{itH}P_{ac}(H)\|_{L^1\to L^\infty} \lesssim |t|^{-\frac{n}{2}}. \tag{2}$$

One requires the projection onto the absolutely continuous spectrum as the perturbed Schrödinger operator often possesses point spectrum, for which large time decay cannot occur. Under weak pointwise assumptions on the potential, say $|V(x)| \lesssim \langle x\rangle^{-\beta}$ for some $\beta > 1$, we have $\sigma(H) = \sigma_{ac}(H) \cup \sigma_p(H)$. Here the absolutely continuous spectrum $\sigma_{ac}(H) = [0, \infty)$, and the point spectrum consists of a finite collection of non-positive eigenvalues, [38]. One may alternatively seek to quantify the dispersive properties in terms of Strichartz norms, L^p bounds, or micro-local estimates. In this paper, we focus on proving point-wise bounds for the evolution when there are zero energy obstructions. The obstructions can be related to solutions to $H\psi = 0$. If $\psi \in L^2(\mathbb{R}^n)$, there is a zero energy eigenvalue, while $\psi \notin L^2(\mathbb{R}^n)$ is a resonance if it is in a different space which depends on the dimension. When $n = 4$, ψ is a resonance if $\langle\cdot\rangle^{0-}\psi \in L^2(\mathbb{R}^4)$.

Local dispersive estimates were first studied treating $e^{itH}P_{ac}$ as an operator between weighted $L^2(\mathbb{R}^n)$ spaces. The study was begun by Rauch in [37] on exponentially weighted spaces when $n = 3$. Jensen and Kato [29] for $n = 3$, and Jensen [27, 28] for $n > 3$ proved estimates on polynomially weighted L^2 spaces that decay at a rate of $|t|^{-\frac{n}{2}}$. Murata, [36], studied local dispersive estimates for a wide class of Schrödinger-like equations. In these works, it was shown that threshold obstructions can effect the time decay of the solution operator even though $P_{ac}(H)$ explicitly projects away from the zero energy eigenspace.

In recent years, there has been much interest in global dispersive estimates, in which one seeks to bound $e^{itH}P_{ac}(H)$ as an operator from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$. The study began with the seminal paper of Journé, Soffer and Sogge [32], with much of the recent work having its roots in the approach of Rodnianski and Schlag, [39]. If zero energy is regular, that is if there are no zero-energy eigenvalues of resonances, (2) has been established in all dimensions, see [44, 23, 41, 24, 7, 12, 25], a more thorough history may be found in [40].

When there is an obstruction at zero energy, either a resonance or an eigenvalue, in general, the time decay is slower. The effect of these threshold obstructions is dimension-specific, we note [23, 15, 46, 18, 13, 5, 14, 19, 20, 43] in which the effects were studied in all dimensions $n \geq 1$ in the sense of global dispersive estimates.

In this article we provide refined dispersive bounds for the Schrödinger operator with zero-energy resonances and/or eigenvalues in dimension $n = 4$. We recall the result of Erdoğan, Goldberg and the first author, [14], it was shown that if zero is not regular, one

has the low energy expansion

$$\|e^{itH}\chi(H)P_{ac}(H) - F_t\|_{L^1\to L^\infty} \lesssim |t|^{-1}, \qquad |t| > 2$$

where F_t is a finite rank operator satisfying $\|F_t\|_{L^1\to L^\infty} \lesssim (\log|t|)^{-1}$ for $|t| > 2$. We improve these results in several directions. First of all, we provide a more detailed expansion the evolution with at most three slowly decaying terms with an error term that is integrable for large time. In particular, we show that as $t \to \infty$, the non-integrable time decay is "small" in the sense of being attached to finite-rank operators.

To state the results, we define the functions $\log^+(x) = \chi_{\{x>1\}}\log x$ and $w(x) = 1 + \log^+|x|$, and $\varphi(t)$ is a function that satisfies the bound $\varphi(t) = O(1/\log t)$ for $t > 2$. Further, χ is an smooth, even cut-off function supported in $[-2\lambda_1, 2\lambda_1]$ for a fixed sufficiently small $\lambda_1 > 0$ and it is equal to one if $|\lambda| \le \lambda_1$. We also define the logarithmically weighted L^p spaces

$$L^1_{w^k}(\mathbb{R}^4) = \{f : \int_{\mathbb{R}^4} w^k(x)|f(x)|\,dx < \infty\}, \qquad L^\infty_{w^{-k}}(\mathbb{R}^4) = \{g : \|w^{-k}(\cdot)g\|_\infty < \infty\}.$$

Theorem 1.1. *Suppose that $|V(x)| \lesssim \langle x\rangle^{-\beta-}$. If zero energy is not regular, for $t > 2$,*

$$e^{itH}\chi(H)P_{ac}(H) = \varphi(t)A_0 + O(1/t)A_1 + O((t\log t)^{-1})A_2 + O(t^{-1}(\log t)^{-2})A_4,$$

with $A_0 : L^1 \to L^\infty$ is a finite rank operator, $A_1 : L^1 \to L^\infty$, $A_2 : L^1_w \to L^\infty_{w^{-1}}$ finite rank operators, and $A_4 : L^1_{w^3} \to L^\infty_{w^{-3}}$, provided $\beta > 0$ is large enough. Furthermore, the operators A_1, A_2 are independent of time. In particular,

i) If there is a resonance but no eigenvalue at zero and $\beta > 4$, the above expansion is valid.

ii) If there is an eigenvalue but no resonance at zero and $\beta > 8$, the above expansion is valid with $A_0 = 0$.

iii) If there is a resonance and an eigenvalue at zero and $\beta > 8$, the above expansion is valid.

The polynomially weighted L^p spaces are defined by

$$L^{p,\sigma}(\mathbb{R}^n) = \{f : \|\langle x\rangle^\sigma f\|_p < \infty\}.$$

To prove this theorem, we employ an interpolation argument between the results of [14] and the three parts of the following theorem which we prove in Sections 3, 4 and 5 respectively.

Theorem 1.2. *Suppose that $|V(x)| \lesssim \langle x\rangle^{-\beta-}$.*

i) If there is a resonance but no eigenvalue at zero, then if $\beta > 4$ for $t > 2$,

$$e^{itH}\chi(H)P_{ac}(H) = \varphi(t)A_0 + O(1/t)A_1 + O((t\log t)^{-1})A_2 + O(t^{-1}(\log t)^{-2})A_3 + O(t^{-1-})A_4$$

with $A_0 : L^1 \to L^\infty$ a rank one operator, $A_1 : L^1 \to L^\infty$, $A_2 : L^1_w \to L^\infty_{w^{-1}}$ finite rank operators, $A_3 : L^1 \to L^\infty$ and $A_4 : L^{1,\frac{1}{2}} \to L^{\infty,-\frac{1}{2}}$.

ii) If there is an eigenvalue but no resonance at zero, then if $\beta > 8$, for $t > 2$,

$$e^{itH}\chi(H)P_{ac}(H) = O(1/t)A_1 + O(t^{-1-})A_4$$

with $A_1 : L^1 \to L^\infty$ a finite rank operators, and $A_4 : L^{1,\frac{1}{2}} \to L^{\infty,-\frac{1}{2}}$.

iii) If there is a resonance and an eigenvalue at zero, then if $\beta > 8$ for $t > 2$,

$$e^{itH}\chi(H)P_{ac}(H) = \varphi(t)A_0 + O(1/t)A_1 + O((t\log t)^{-1})A_2 + O(t^{-1}(\log t)^{-2})A_3 + O(t^{-1-})A_4$$

with $A_0, A_1, A_3 : L^1 \to L^\infty$, $A_2 : L^1_w \to L^\infty_{w^{-1}}$ finite rank operators, and $A_4 : L^{1,\frac{1}{2}} \to L^{\infty,-\frac{1}{2}}$.

Furthermore, the operators A_1, A_2, A_3 are independent of time.

The operators A_3, A_4 need not be finite rank. However, their contribution is integrable for large time. To establish Theorem 1.1, we recall that in [14], it was shown that

$$e^{itH}\chi(H)P_{ac}(H) = \varphi(t)A_0 + O(1/t), \tag{3}$$

with $A_0 : L^1 \to L^\infty$ finite rank, and the error term is understood as an operator mapping $L^1 \to L^\infty$ which is not finite rank. In Theorem 1.2, we show

$$e^{itH}P_{ac}(H) - \varphi(t)A_0 - \frac{w(x)B_1w(y)}{t\log t} - \frac{B_2}{t(\log t)^2} = O\left(\frac{\langle x\rangle^{\frac{1}{2}}\langle y\rangle^{\frac{1}{2}}}{t^{1+}}\right).$$

Here B_1, B_2 are finite rank and bounded independent of x, y. We can subtract them off of the bound (3) to conclude that

$$e^{itH}P_{ac}(H) - \varphi(t)A_0 - \frac{w(x)B_1w(y)}{t\log t} - \frac{B_2}{t(\log t)^2} = O\left(\frac{w(x)w(y)}{t}\right).$$

So that,

$$e^{itH}P_{ac}(H) - \varphi(t)A_0 - \frac{w(x)B_1w(y)}{t\log t} - \frac{B_2}{t(\log t)^2} = O\left(\frac{w(x)w(y)}{t}\min\left(1, \frac{\langle x\rangle^{\frac{1}{2}}\langle y\rangle^{\frac{1}{2}}}{t^{0+}}\right)\right).$$

Using $\min(1, \frac{a}{b}) \lesssim (\log a)^2/(\log b)^2$ if $a, b > 2$, we can obtain an error term bounded by $t^{-1}(\log t)^{-2}$ as an operator between logarithmically weighted spaces.

We can combine the low energy estimates, for which the spectral parameter λ is in a sufficiently small neighborhood of zero, proven above with large energy estimates from, for

example [45] or [7]. To apply these results, one requires additional regularity on the potential, which is expected from the counterexample constructed in [24]. This counterexample showed that the high energy portion of the evolution need not satisfy the desired dispersive bound if V does not have $\frac{n-3}{2}$ continuous derivatives when $n > 3$. The smoothness when $n = 4$ may be expressed in terms of a weighted Fourier transform as in [45], or explicitly requiring $V \in C^{\frac{1}{2}+}(\mathbb{R}^4)$ as in [7]. We note that the goal of [45] was not to prove dispersive bounds directly, but was instead concerned with the L^p-boundedness of the wave operators, which are defined by

$$W_\pm = s - \lim_{t\to\pm\infty} e^{itH} e^{it\Delta}.$$

The L^p boundedness of the wave operators allows one to deduce bounds on the perturbed operator from the free operator $H_0 = -\Delta$. That is, for any Borel function f,

$$f(H)P_{ac} = W_\pm f(-\Delta) W_\pm^*.$$

If $W_\pm$ is bounded on L^∞, then $W_\pm^*$ is bounded on L^1 and one can deduce the $|t|^{-\frac{n}{2}}$ dispersive bound for the evolution by using the natural bound for the free equation. It is known that the if zero is not regular, the range of p for which the wave operators are bounded shrinks from $1 \le p \le \infty$ when zero is regular to $\frac{4}{3} < p < 4$ when there is an eigenvalue but no resonance at zero, [31]. We expect that the range of p can be expanded to $1 \le p < 4$ in light of the recent works [47, 21, 48].[1]

A second direction in which we improve the known results is to establish a global dispersive bound that matches the natural $|t|^{-2}$ decay of the free evolution even in the presence of a zero-energy eigenvalue. In [20, 19] $L^1 \to L^\infty$ dispersive bounds for the perturbed evolution were established in the presence of zero energy eigenvalues with the full $|t|^{-\frac{n}{2}}$ time decay, assuming additional orthogonality conditions between the zero energy eigenspace and the potential. In these papers, dimensions $n \ge 5$ were studied, where zero energy resonances do not occur. We show in Section 6 that such a bound holds in dimension $n = 4$. The more complicated structure of zero energy obstructions when $n = 4$ leads to significant technical difficulties in the analysis, even when there is not a zero energy resonance.

Theorem 1.3. *Assume that $|V(x)| \lesssim \langle x\rangle^{-12-}$, and that zero is an eigenvalue of $H = -\Delta + V$, but not a resonance. Further, suppose that $\int_{\mathbb{R}^4} V\psi\,dx = 0$ and $\int_{\mathbb{R}^4} x_j V\psi\,dx = 0$ for each $\psi \in$Null H and all $1 \le j \le 4$. Then,*

$$\|e^{itH}\chi(H)P_{ac}(H)\|_{L^1\to L^\infty} \lesssim |t|^{-2}.$$

[1]During the review period of this paper, the first author and Goldberg proved this result, see [22].

One can alternatively state the orthogonality hypotheses as $P_eVx, P_eV1 = 0$ where P_e is the projection onto the zero energy eigenspace.

The perturbed resolvent operators are defined by

$$R_V^{\pm}(\lambda^2) = R_V^{\pm}(\lambda^2 \pm i0) = \lim_{\epsilon \to 0^+} (H - (\lambda^2 \pm i\epsilon))^{-1}.$$

By Agmon's well-known limiting absorptions principle, [2], these limits are well-defined as bounded operators between weighted L^2 spaces. Treating $e^{itH}\chi(H)P_{ac}(H)$ as an element of functional calculus, Stone's formula yields the representation

$$e^{itH}\chi(H)P_{ac}(H)f(x) = \frac{1}{2\pi i}\int_0^\infty e^{it\lambda^2}\lambda\chi(\lambda)[R_V^+(\lambda^2) - R_V^-(\lambda^2)]f(x)\,d\lambda. \tag{4}$$

Here the difference of the resolvents provides the absolutely continuous spectral measure. As usual (cf. [39, 23, 41, 13, 14]) the proofs of Theorem 1.2 and Theorem 1.3 relies on the formula (4) and the expansion of the spectral density $[R_V^+(\lambda^2) - R_V^-(\lambda^2)]$ around zero energy which varies depending on which kind of obstruction one has at zero energy.

The final direction in which we improve the known results is to prove dispersive bounds for a wide class of wave-like equations. The Klein-Gordon equation

$$u_{tt} - \Delta u + m^2 u + Vu = 0, \qquad u(0) = f, \qquad \partial_t u(0) = g \tag{5}$$

is formally solved by

$$u(x,t) = \cos(t\sqrt{H+m^2})f(x) + \frac{\sin(t\sqrt{H+m^2})}{\sqrt{H+m^2}}g(x). \tag{6}$$

The formal solution is valid for the wave equation, when $m^2 = 0$. We restrict our attention to $m^2 \geq 0$ so that the unperturbed operator $-\Delta + m^2$ is positive.

In the free case, when $V = 0$, one has the natural dispersive bounds

$$\|u(\cdot,t)\|_\infty \lesssim |t|^{-\frac{3}{2}}(\|f\|_{W^{k+1,1}} + \|g\|_{W^{k,1}})$$

for $k > \frac{3}{2}$ in four dimensions. The derivative loss on the initial data is strictly a high-energy phenomenon, see [26, 14]. L^∞ bounds on solutions to the wave equation has been studied, [4, 3, 9, 11, 6, 26]. The dispersive nature of the Klein-Gordon has been studied in various senses [35, 33, 17]. The effect of threshold eigenvalues and resonances for wave-like equations has been studied in dimensions $n \leq 4$, see [34, 25, 14].

We prove low-energy dispersive bounds by taking advantage of the representation

$$\cos(t\sqrt{H+m^2})P_{ac} + \frac{\sin(t\sqrt{H+m^2})}{\sqrt{H+m^2}}P_{ac}$$

$$= \frac{1}{\pi i}\int_0^\infty \left(\cos(t\sqrt{\lambda^2+m^2}) + \frac{\sin(t\sqrt{\lambda^2+m^2})}{\sqrt{\lambda^2+m^2}}\right)\lambda[R_V^+(\lambda^2) - R_V^-(\lambda^2)]\,d\lambda.$$

This allows us to extend our analysis of the spectral measure and the perturbed resolvents we develop for the Schrödinger evolution in Sections 3, 4 and 5 below.

Theorem 1.4. *Let $m^2 > 0$ then the results of Theorem 1.2 are valid if the operator e^{itH} is replaced by either $\cos(t\sqrt{H+m^2})$ or $\frac{\sin(t\sqrt{H+m^2})}{\sqrt{H+m^2}}$.*

Note that Theorem 1.4 is stated for $m^2 > 0$. When $m^2 = 0$, (6) corresponds to the solution of the wave equation. This formula suggests a similar statement for the solution of the wave equation. However, the behavior of $\frac{\sin(t\sqrt{\lambda^2+m^2})}{\sqrt{\lambda^2+m^2}}$ when $m = 0$ for small λ is singular. In [14], to ensure integrability as $\lambda \to 0$ we had to use

$$\left|\frac{\sin(t\lambda)}{\lambda}\right| \lesssim |t| \quad \text{whereas} \quad \left|\frac{\sin(t\sqrt{\lambda^2+m^2})}{\sqrt{\lambda^2+m^2}}\right| \lesssim \frac{1}{m} \lesssim 1 \quad \text{for} \quad m > 0.$$

Considering this fact together with $|\cos(\lambda t)| \lesssim 1$, the results of Theorem 1.4 apply to the operator $\cos(t\sqrt{H})$, but the bounds for the sine operator must all be multiplied by t.

As in our analysis for the Schrödinger evolution, we consider only the low energy portion of the evolution. High energy bounds, with a loss of $\frac{5}{2}+$ derivatives of the initial data, for the wave equation are proven in [8] under the assumption that $V \in C^{\frac{1}{2}}(\mathbb{R}^4)$, and $|\partial_x^\alpha V(x)| \lesssim \langle x\rangle^{-5/2-}$ for $|\alpha| \le \frac{1}{2}$.

2. Resolvent expansions around zero

Much of this discussion appears in [14], we include and expand upon it for completeness. The notation

$$f(\lambda) = \widetilde{O}(g(\lambda))$$

denotes

$$\frac{d^j}{d\lambda^j} f = O\big(\frac{d^j}{d\lambda^j} g\big), \quad j = 0, 1, 2, 3, ...$$

Unless otherwise specified, the notation refers only to derivatives with respect to the spectral variable λ. If the derivative bounds hold only for the first k derivatives we write $f = \widetilde{O}_k(g)$. If we write $f(\lambda) = \widetilde{O}_k(\lambda^j)$, it should be understood that differentiation is comparable to division by λ. That is, $|\frac{d^\ell}{d\lambda^\ell} f(\lambda)| \lesssim \lambda^{j-\ell}$, even for $\ell > j$. In this paper we use that notation for operators as well as scalar functions, the meaning should be clear from context.

Most properties of the low-energy expansion for $R_V^\pm(\lambda^2)$ are inherited in some way from the free resolvent $R_0^\pm(\lambda^2) = (-\Delta - (\lambda^2 \pm i0))^{-1}$. In this section we gather facts about $R_0^\pm(\lambda^2)$ and examine the relationship between $R_V^\pm(\lambda^2)$ and $R_0^\pm(\lambda^2)$.

Recall that the free resolvent in four dimensions has the integral kernel

$$R_0^{\pm}(\lambda^2)(x,y) = \pm\frac{i}{4}\frac{\lambda}{2\pi|x-y|}H_1^{\pm}(\lambda|x-y|) \tag{7}$$

where $H_1^{\pm}$ are the Hankel functions of order one:

$$H_1^{\pm}(z) = J_1(z) \pm iY_1(z). \tag{8}$$

From the series expansions for the Bessel functions, see [1], as $z \to 0$ we have

$$J_1(z) = \frac{1}{2}z - \frac{1}{16}z^3 + \widetilde{O}_2(z^5), \tag{9}$$

$$Y_1(z) = -\frac{2}{\pi z} + \frac{2}{\pi}\log(z/2)J_1(z) + b_1 z + b_2 z^3 + \widetilde{O}_2(z^5) \tag{10}$$

$$= -\frac{2}{\pi z} + \frac{1}{\pi}z\log(z/2) + b_1 z - \frac{1}{8\pi}z^3\log(z/2) + b_2 z^3 + \widetilde{O}_2(z^5\log z). \tag{11}$$

Here $b_1, b_2 \in \mathbb{R}$. Further, for $|z| \gtrsim 1$, we have the representation (see, *e.g.*, [1])

$$H_1^{\pm}(z) = e^{\pm iz}\omega_{\pm}(z), \quad |\omega_{\pm}^{(\ell)}(z)| \lesssim (1+|z|)^{-\frac{1}{2}-\ell}, \quad \ell = 0,1,2,\dots. \tag{12}$$

This implies, among the various expansions we develop, that (with $r = |x-y|$)

$$R_0^{\pm}(\lambda^2)(x,y) = r^{-2}\rho_-(\lambda r) + r^{-1}\lambda e^{\pm i\lambda r}\rho_+(\lambda r). \tag{13}$$

Here ρ_- is supported on $[0, \frac{1}{2}]$, ρ_+ is supported on $[\frac{1}{4}, \infty)$ satisfying the estimates $|\rho_-(z)| \lesssim 1$ and $\rho_+(z) = \widetilde{O}(z^{-\frac{1}{2}})$.

To obtain expansions for $R_V^{\pm}(\lambda^2)$ around zero energy we utilize the symmetric resolvent identity. Define $U(x) = 1$ if $V(x) \geq 0$ and $U(x) = -1$ if $V(x) < 0$, and let $v = |V|^{1/2}$, $w = Uv$ so that $V = Uv^2 = wv$. Then the formula

$$R_V^{\pm}(\lambda^2) = R_0^{\pm}(\lambda^2) - R_0^{\pm}(\lambda^2)vM^{\pm}(\lambda)^{-1}vR_0^{\pm}(\lambda^2), \tag{14}$$

is valid for $\Im(\lambda) > 0$, where $M^{\pm}(\lambda) = U + vR_0^{\pm}(\lambda^2)v$.

Note that the statements of Theorem 1.2 control operators from $L^1(\mathbb{R}^4)$ to $L^\infty(\mathbb{R}^4)$, while our analysis of $M^{\pm}(\lambda^2)$ and its inverse will be conducted in $L^2(\mathbb{R}^4)$. The free resolvents are not locally $L^2(\mathbb{R}^n)$ when $n > 3$. This requires us to iterate the standard resolvent identities, as we show that iterated resolvents have better local integrability. To use the symmetric resolvent identity, we need two resolvents on either side of $M^{\pm}(\lambda)^{-1}$. Accordingly, from the standard resolvent identity we have:

$$R_V^{\pm}(\lambda^2) = R_0^{\pm}(\lambda^2) - R_0^{\pm}(\lambda^2)VR_0^{\pm}(\lambda^2) + R_0^{\pm}(\lambda^2)VR_V^{\pm}(\lambda^2)VR_0^{\pm}(\lambda^2). \tag{15}$$

Combining this with (14), we have

$$R_V^{\pm}(\lambda^2) = R_0^{\pm}(\lambda^2) - R_0^{\pm}(\lambda^2)VR_0^{\pm}(\lambda^2) + R_0^{\pm}(\lambda^2)VR_0^{\pm}(\lambda^2)VR_0^{\pm}(\lambda^2) \tag{16}$$

$$- R_0^{\pm}(\lambda^2)VR_0^{\pm}(\lambda^2)vM^{\pm}(\lambda)^{-1}vR_0^{\pm}(\lambda^2)VR_0^{\pm}(\lambda^2). \tag{17}$$

Provided $V(x)$ decays sufficiently, $[R_0^{\pm}(\lambda^2)VR_0^{\pm}(\lambda^2)v](x,\cdot) \in L^2(\mathbb{R}^4)$ uniformly in x, and that $M^{\pm}(\lambda)$ is invertible in $L^2(\mathbb{R}^4)$. We recall the following lemma from [14],

Lemma 2.1. *If $|V(x)| \lesssim \langle x\rangle^{-\beta-}$ for some $\beta > 2$, then for any $\sigma > \max(\frac{1}{2}, 3-\beta)$ we have*

$$\sup_{x\in\mathbb{R}^4} \|[R_0^{\pm}(\lambda^2)VR_0^{\pm}(\lambda^2)](x,y)\|_{L_y^{2,-\sigma}} \lesssim \langle\lambda\rangle.$$

Consequently $\left\|R_0^{\pm}(\lambda^2)VR_0^{\pm}(\lambda^2)v\right\|_{L^2\to L^\infty} \lesssim \langle\lambda\rangle$.

To invert $M^{\pm}(\lambda)$ in L^2 under various spectral assumptions on the zero energy we need several different expansions for $M^{\pm}(\lambda)$. The following operators arise naturally in these expansions (see (9), (10)):

$$G_0 f(x) = -\frac{1}{4\pi^2}\int_{\mathbb{R}^4} \frac{f(y)}{|x-y|^2}\,dy = (-\Delta)^{-1}f(x), \tag{18}$$

$$G_1 f(x) = -\frac{1}{8\pi^2}\int_{\mathbb{R}^4} \log(|x-y|)f(y)\,dy, \tag{19}$$

$$G_j f(x) = \begin{cases} c_j \int_{\mathbb{R}^4} |x-y|^j f(y)\,dy & \text{for } j = 2,4,\dots \\ c_j \int_{\mathbb{R}^4} |x-y|^{j-1}\log(|x-y|)f(y)\,dy & \text{for } j = 3,5,\dots \end{cases} \tag{20}$$

Here c_j are certain real-valued constants, the exact values are unimportant for our analysis. We will use $G_j(x,y)$ to denote the integral kernel of the operator G_j. In addition, the following functions appear naturally,

$$g_j^+(\lambda) = \overline{g_j^-(\lambda)} = \lambda^{2j}(a_j\log(\lambda) + z_j), \qquad j = 1,2,3,\dots \tag{21}$$

Here $a_j \in \mathbb{R}\setminus\{0\}$ and $z_j \in \mathbb{C}\setminus\mathbb{R}$. In our expansions, we move any imaginary parts to these functions of the spectral variable so that the operators all have real-valued kernels. To gain a more detailed low energy expansion than in [14], we go further into the expansions of the resolvents, $M^{\pm}(\lambda)$ and $M^{\pm}(\lambda)^{-1}$ respectively.

We also define the operators

$$T := M^{\pm}(0) = U + vG_0v, \qquad P := \|V\|_1^{-1}v\langle v,\cdot\rangle. \tag{22}$$

Recall the definition of the Hilbert-Schmidt norm of an operator K with kernel $K(x,y)$,

$$\|K\|_{HS} := \left(\iint_{\mathbb{R}^{2n}} |K(x,y)|^2\,dx\,dy\right)^{\frac{1}{2}}$$

Lemma 2.2. *Assuming that $v(x) \lesssim \langle x \rangle^{-\beta}$. If $\beta > 2$, then we have*

$$M^{\pm}(\lambda) = T + M_0^{\pm}(\lambda), \qquad \sum_{j=0}^{1} \| \sup_{0<\lambda<\lambda_1} \lambda^{-2+j+} \partial_\lambda^j M_0^{\pm}(\lambda) \|_{HS} \lesssim 1, \tag{23}$$

and

$$M^{\pm}(\lambda) = T + \|V\|_1 g_1^{\pm}(\lambda) P + \lambda^2 v G_1 v + M_1^{\pm}(\lambda), \qquad \sum_{j=0}^{2} \| \sup_{0<\lambda<\lambda_1} \lambda^{-2+j-} \partial_\lambda^j M_1^{\pm}(\lambda) \|_{HS} \lesssim 1. \tag{24}$$

If $\beta > 4$, we have

$$M^{\pm}(\lambda) = T + \|V\|_1 g_1^{\pm}(\lambda) P + \lambda^2 v G_1 v + g_2^{\pm}(\lambda) v G_2 v + \lambda^4 v G_3 v + M_2^{\pm}(\lambda), \qquad \sum_{j=0}^{2} \| \sup_{0<\lambda<\lambda_1} \lambda^{-4+j-} \partial_\lambda^j M_2^{\pm}(\lambda) \|_{HS} \lesssim 1. \tag{25}$$

If $\beta > 6$, we have

$$M^{\pm}(\lambda) = T + \|V\|_1 g_1^{\pm}(\lambda) P + \lambda^2 v G_1 v + g_2^{\pm}(\lambda) v G_2 v + \lambda^4 v G_3 v + g_3^{\pm}(\lambda) v G_4 v + \lambda^6 v G_5 v + M_3^{\pm}(\lambda), \qquad \sum_{j=0}^{2} \| \sup_{0<\lambda<\lambda_1} \lambda^{-6+j-} \partial_\lambda^j M_3^{\pm}(\lambda) \|_{HS} \lesssim 1. \tag{26}$$

Proof. Using the notation introduced in (18)–(21) in (7), (9), and (10), we obtain (for $\lambda|x-y| \ll 1$)

$$R_0^{\pm}(\lambda^2)(x,y) = G_0(x,y) + \widetilde{O}_2(\lambda^2(1 + \log(\lambda|x-y|))) \tag{27}$$

$$R_0^{\pm}(\lambda^2)(x,y) = G_0(x,y) + g_1^{\pm}(\lambda) + \lambda^2 G_1(x,y) + \widetilde{O}_1(\lambda^4 |x-y|^2 \log(\lambda|x-y|)). \tag{28}$$

$$R_0^{\pm}(\lambda^2)(x,y) = G_0(x,y) + g_1^{\pm}(\lambda) + \lambda^2 G_1(x,y) + g_2^{\pm}(\lambda) G_2(x,y) + \lambda^4 G_3(x,y) + \widetilde{O}_2(\lambda^6 |x-y|^4 \log(\lambda|x-y|)). \tag{29}$$

$$R_0^{\pm}(\lambda^2)(x,y) = G_0(x,y) + g_1^{\pm}(\lambda) + \lambda^2 G_1(x,y) + g_2^{\pm}(\lambda) G_2(x,y) + \lambda^4 G_3(x,y) + g_3^{\pm}(\lambda) G_4(x,y) + \lambda^6 G_5(x,y) + \widetilde{O}_2(\lambda^8 |x-y|^6 \log(\lambda|x-y|)). \tag{30}$$

The $\widetilde{O}_2$ notation refers to derivatives with respect to λ in all cases.

In light of these expansions and using the notation in (22), we define $M_j^{\pm}(\lambda)$ by the identities

$$M^{\pm}(\lambda) = U + v R_0^{\pm}(\lambda^2) v = T + M_0^{\pm}(\lambda). \tag{31}$$

$$M_0^{\pm}(\lambda) = \|V\|_1 g_1^{\pm}(\lambda) P + \lambda^2 v G_1 v + M_1^{\pm}(\lambda). \tag{32}$$

$$M_1^{\pm}(\lambda) = g_2^{\pm}(\lambda) v G_2 v + \lambda^4 v G_3 v + M_2^{\pm}(\lambda). \tag{33}$$

$$M_2^{\pm}(\lambda) = g_3^{\pm}(\lambda) v G_4 v + \lambda^6 v G_5 v + M_3^{\pm}(\lambda). \tag{34}$$

When $\lambda|x-y| \ll 1$, the bounds follow from (27), (28), (29), and (30). On the other hand, when $\lambda|x-y| \gtrsim 1$, we use (7) and (12) to see (for any $\alpha \geq 0$ and $k = 0, 1, 2$)

$$|\partial_\lambda^k R_0(\lambda^2)(x,y)| = \left| \partial_\lambda^k \left[\frac{\lambda e^{i\lambda|x-y|} \omega(\lambda|x-y|)}{|x-y|} \right] \right| \lesssim (\lambda|x-y|)^{\frac{1}{2}+\alpha} |x-y|^{k-2}. \tag{35}$$

Using (27), (31), and (35), and choosing $\alpha = \frac{3}{2} - k$, we see that

$$\begin{aligned} M_0(\lambda)(x,y) &= \begin{cases} v(x)v(y) \log|x-y| \widetilde{O}_1(\lambda^{2-}), & \lambda|x-y| \ll 1 \\ v(x)v(y)[G_0(x,y) + \widetilde{O}_1(\lambda^2)], & \lambda|x-y| \gtrsim 1 \end{cases} \\ &= v(x)v(y)(1 + \log|x-y|)\widetilde{O}_1(\lambda^{2-}). \end{aligned}$$

This yields the bounds in (23) as $v(x) \lesssim \langle x \rangle^{-2-}$.

The other assertions of the lemma follow from similar arguments. Taking $\alpha = \frac{3}{2} - k+$ in (35) and using

$$\widetilde{O}_2(\lambda^4 |x-y|^2 \log(\lambda|x-y|)) = \widetilde{O}_2(\lambda^2 (\lambda|x-y|)^{0+}), \quad \text{for } \lambda|x-y| \ll 1$$

we obtain (24), whereas taking $\alpha = \frac{7}{2} - k-$ in (35) and using

$$\widetilde{O}_2(\lambda^6 |x-y|^4 \log(\lambda|x-y|)) = \widetilde{O}_2(\lambda^2 (\lambda|x-y|)^{2+}), \quad \text{for } \lambda|x-y| \ll 1$$

we obtain (25). Finally, an operator with integral kernel $v(x)|x-y|^\gamma v(y)$ or $v(x)|x-y|^\gamma \log|x-y| v(y)$, $\gamma > 0$, is Hilbert-Schmidt provided $\beta > 2 + \gamma$.

□

The following corollary is useful.

Corollary 2.3. *We have the expansion*

$$R_0^{\pm}(\lambda^2) V R_0^{\pm}(\lambda^2)(x,y) = K_0 + \widetilde{E}_0^{\pm}(\lambda)(x,y),$$

here the operators K_j have real-valued kernels. Furthermore, the error term $\widetilde{E}_0^{\pm}(\lambda)$ satisfies

$$\widetilde{E}_0^{\pm}(\lambda)(x,y) = (1 + \log^- |x - \cdot| + \log^- |\cdot - y|)\widetilde{O}_1(\lambda^{2-}).$$

Furthermore, if one wishes to have 2 derivatives, the extended expansion

$$\widetilde{E}_0^{\pm}(\lambda)(x,y) = g_1^{\pm}(\lambda) K_1 + \lambda^{n-2} K_2 + \widetilde{E}_1^{\pm}(\lambda)(x,y),$$

satisfies the bound

$$\widetilde{E}_1^{\pm}(\lambda)(x,y) = \langle x \rangle^{\frac{1}{2}} \langle y \rangle^{\frac{1}{2}} \widetilde{O}_2(\lambda^{\frac{5}{2}}).$$

Proof. This follows from the expansions for $R_0^{\pm}(\lambda^2)$ in Lemma 2.2. For the iterated resolvents, the desired bounds come from simply multiplying out the terms. In particular,

$$K_0 = G_0 V G_0, \qquad K_1 = 1 V G_0 + G_0 V 1$$

where 1 is the operator with integral kernel a scalar multiple of $1(x,y) = 1$, and

$$K_2 = G_0 V G_1 + G_1 V G_0.$$

These are all real-valued, operators.

□

Remark 2.4. *The spatially weighted bound* $|\partial_\lambda^2 \widetilde{E}_1^{\pm}(\lambda)(x,y)| \lesssim \langle x \rangle^{\frac{1}{2}} \lambda^{\frac{1}{2}}$ *is only needed both derivatives act on the leading resolvent,* $R_0^{\pm}(\lambda^2)(x,z_1)$*, in the product. Similarly, the upper bound* $\langle y \rangle^{\frac{1}{2}} \lambda^{\frac{1}{2}}$ *is only needed if all derivatives act on the lagging resolvent,* $R_0^{\pm}(\lambda^2)(z_1,y)$*, in the product. All other expressions that arise would be consistent with* $\widetilde{E}_1^{\pm}(\lambda)$ *belonging to the class* $\widetilde{O}_2(\lambda^{2-})$*. A very similar bound holds for* $R_0^{\pm}(\lambda^2)$ *rather than iterated resolvents.*

One can see that the invertibility of $M^{\pm}(\lambda)$ as an operator on L^2 for small λ depends upon the invertibility of the operator T on L^2, see (22). We now recall the definition of resonances at zero energy, following [30, 14]. This definition and subsequent discussion also appear in [14].

Definition 2.5. *(1) We say zero is a regular point of the spectrum of* $H = -\Delta + V$ *provided* $T = U + vG_0v$ *is invertible on* $L^2(\mathbb{R}^4)$*.*

(2) Assume that zero is not a regular point of the spectrum. Let S_1 *be the Riesz projection onto the kernel of* T *as an operator on* $L^2(\mathbb{R}^4)$*. Then* $T + S_1$ *is invertible on* $L^2(\mathbb{R}^4)$*. Accordingly, we define* $D_0 = (T + S_1)^{-1}$ *as an operator on* $L^2(\mathbb{R}^4)$*. We say there is a resonance of the first kind at zero if the operator* $T_1 := S_1 P S_1$ *is invertible on* $S_1 L^2(\mathbb{R}^4)$*.*

(3) Assume that T_1 *is not invertible on* $S_1 L^2(\mathbb{R}^4)$*. Let* S_2 *be the Riesz projection onto the kernel of* T_1 *as an operator on* $S_1 L^2(\mathbb{R}^4)$*. Then* $T_1 + S_2$ *is invertible on* $S_1 L^2(\mathbb{R}^4)$*. We say there is a resonance of the second kind at zero if* $S_2 = S_1$*. If* $S_1 \neq S_2$*, we say there is a resonance of the third kind.*

Remarks. i) To relate the projections S_j to the type of obstruction, we use the characterization proven in [14]. In particular, $S_1 - S_2 \neq 0$ corresponds to the existence of a resonance at zero energy, and $S_2 \neq 0$ corresponds to the existence of an eigenvalue at zero energy. A resonance of the first kind indicates that there is a resonance at zero only, for a resonance

of the second kind there is an eigenvalue at zero only, and a resonance of the third kind means there is both a resonance and an eigenvalue at zero energy. For technical reasons, we need to employ different tools to invert $M^{\pm}(\lambda)$ for the different types of resonances. It is well-known that different types of resonances at zero energy lead to different expansions for $M^{\pm}(\lambda)^{-1}$ in other dimensions, see [15, 13, 14]. Accordingly, we will develop different expansions for $M^{\pm}(\lambda)^{-1}$ in the following sections.

ii) Noting that T is self-adjoint, S_1 is the orthogonal projection onto the kernel of T, and we have (with $D_0 = (T + S_1)^{-1}$)

$$S_1 D_0 = D_0 S_1 = S_1.$$

This statement also valid for S_2 and $(T_1 + S_2)^{-1}$.

iii) S_1 and S_2 are finite-rank projections in all cases. This follows by the observation that T is a compact perturbation of the invertible operator U, and invoking the Fredholm alternative. See Section 8 below for a full characterization of the spectral subspaces of L^2 associated to $H = -\Delta + V$.

Definition 2.6. *We say an operator $K : L^2(\mathbb{R}^4) \to L^2(\mathbb{R}^4)$ with kernel $K(\cdot,\cdot)$ is absolutely bounded if the operator with kernel $|K(\cdot,\cdot)|$ is bounded from $L^2(\mathbb{R}^4)$ to $L^2(\mathbb{R}^4)$.*

Note that Hilbert-Schmidt and finite rank operators are absolutely bounded. In [14], it was proven that

Lemma 2.7. *The operator D_0 is absolutely bounded on L^2.*

To invert $M^{\pm}(\lambda) = U + vR_0^{\pm}(\lambda^2)v$ for small λ, we use Lemma 2.1 in [30].

Lemma 2.8. *Let A be a closed operator on a Hilbert space $\mathcal{H}$ and S a projection. Suppose $A + S$ has a bounded inverse. Then A has a bounded inverse if and only if*

$$B := S - S(A + S)^{-1}S$$

has a bounded inverse in $S\mathcal{H}$, and in this case

$$A^{-1} = (A + S)^{-1} + (A + S)^{-1}SB^{-1}S(A + S)^{-1}.$$

We will apply this lemma with $A = M^{\pm}(\lambda)$ and $S = S_1$, the orthogonal projection onto the kernel of T. Thus, we need to show that $M^{\pm}(\lambda) + S_1$ has a bounded inverse in $L^2(\mathbb{R}^4)$ and

$$B_{\pm}(\lambda) = S_1 - S_1(M^{\pm}(\lambda) + S_1)^{-1}S_1 \tag{36}$$

has a bounded inverse in $S_1L^2(\mathbb{R}^4)$.

The invertibility of the operator $B_\pm$ is established using different techniques, which depend on the type of resonance at zero energy. We recall Lemma 2.9 in [14].

Lemma 2.9. *Suppose that zero is not a regular point of the spectrum of $H = -\Delta + V$, and let S_1 be the corresponding Riesz projection. Then for sufficiently small $\lambda_1 > 0$, the operators $M^\pm(\lambda) + S_1$ are invertible for all $0 < \lambda < \lambda_1$ as bounded operators on $L^2(\mathbb{R}^4)$. Further, one has (with $\widetilde{g}_1^\pm(\lambda) = \|V\|_1 g_1^\pm(\lambda)$)*

$$(M^\pm(\lambda) + S_1)^{-1} = D_0 - \widetilde{g}_1(\lambda) D_0 P D_0 - \lambda^2 D_0 v G_1 v D_0 + \widetilde{O}_2(\lambda^{2+}) \tag{37}$$

$$= D_0 + \widetilde{O}_2(\lambda^{2-}) \tag{38}$$

as an absolutely bounded operator on $L^2(\mathbb{R}^4)$ provided $v(x) \lesssim \langle x\rangle^{-2-}$.

Corollary 2.10. *We have the following expansion(s) for $B^\pm(\lambda)$. If $v(x) \lesssim \langle x\rangle^{-2-}$,*

$$B^\pm(\lambda) = -\widetilde{g}_1(\lambda) S_1 P S_1 - \lambda^2 S_1 v G_1 v S_1 + \widetilde{O}_2(\lambda^{2+}) \tag{39}$$

$$= \widetilde{O}_2(\lambda^{2-}) \tag{40}$$

The contribution of (16), the finite terms of the Born series, to the Stone formula (4) is controlled by the following lemma. The lemma, which is Proposition 3.1 in [20], was proven in great generality and applies in our case by taking $n = 4$. A similar proposition is proven for odd n in [19].

Lemma 2.11. *If $|V(x)| \lesssim \langle x\rangle^{-3-}$, then the following bound holds.*

$$\sup_{x,y\in\mathbb{R}^4} \left| \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \left[\sum_{k=0}^{2m+1} (-1)^k \{ R_0^+ (V R_0^+)^k - R_0^- (V R_0^-)^k \} \right] (\lambda^2)(x,y)\, d\lambda \right| \lesssim |t|^{-2}.$$

This allows us now to focus only on the more singular part of the resolvent expansion in (15). As the expansion for (17) depends on the type of obstruction at zero energy, we control its contribution to the Stone formula separately in the next sections.

Remark 2.12. *We note that if zero is regular, if $|V(x)| \lesssim \langle x\rangle^{-4-}$ the low energy dispersive bound*

$$\|e^{itH} \chi(H) P_{ac}(H)\|_{L^1 \to L^\infty} \lesssim |t|^{-2}$$

holds. This can be done by using the expansion for $M^\pm(\lambda)^{-1}$ obtained in Lemma 2.9, specifically (37) is valid when $S_1 = 0$. The time decay is obtained by an analysis that mirrors that of the Born series.

3. Resonance of the first kind

In this section we develop the tools necessary to prove the first claim of Theorem 1.2 when there is a resonance of the first kind. That is, there is a resonance but no eigenvalue at zero energy. Further, $S_1 \neq 0$ and $S_2 = 0$, and S_1 is of rank one by Corollary 8.3. In particular, we prove

Theorem 3.1. *Suppose that $|V(x)| \lesssim \langle x\rangle^{-4-}$. If there is a resonance of the first kind at zero, then for $t > 2$,*

$$e^{itH}\chi(H)P_{ac}(H) = \varphi(t)P_r + O(1/t)A_1 + O((t\log t)^{-1})A_2 + O(t^{-1}(\log t)^{-2})A_3 + O(t^{-1-})A_4$$

with $P_r = G_0VG_0vS_1vG_0VG_0 : L^1 \to L^\infty$ a rank one operator, $A_1 : L^1 \to L^\infty$, $A_2 : L^1_w \to L^\infty_{w^{-1}}$ finite rank operators, $A_3 : L^1 \to L^\infty$ and $A_4 : L^{1,\frac{1}{2}} \to L^{\infty,-\frac{1}{2}}$.

We note here that the operator P_r can be viewed as a sort of projection onto the canonical resonance function $\psi \in L^{2,0-}(\mathbb{R}^4)$, which is chosen from the one-dimensional resonance space so that $\langle v\psi, v\psi\rangle = 1$. From the representation $P_r = G_0VG_0vS_1vG_0VG_0$, using Lemma 8.12 and the fact that S_1 is rank one, we can see that P_r is a bounded operator from L^1 to L^∞.

We recall Lemma 3.2 in [14],

Lemma 3.2. *If $\mathcal{E}(\lambda) = \widetilde{O}_1((\lambda\log\lambda)^{-2})$, then*

$$\left|\int_0^\infty e^{it\lambda^2}\lambda\chi(\lambda)\mathcal{E}(\lambda)\,d\lambda\right| \lesssim \frac{1}{\log t}, \qquad t > 2.$$

This lemma essentially defines the function $\varphi(t)$ in the statement of Theorem 1.2. To invert $M^\pm(\lambda)$ using Lemma 2.8, we need to compute $B_\pm(\lambda)$, we use Lemma 3.3 in [14].

Lemma 3.3. *In the case of a resonance of the first kind at zero, under the hypotheses of Theorem 3.1 the operator $B_\pm(\lambda)$ is invertible for small λ and*

$$B_\pm(\lambda)^{-1} = f^\pm(\lambda)S_1, \tag{41}$$

where

$$f^+(\lambda) = \frac{1}{\lambda^2}\frac{1}{a\log\lambda + z + \widetilde{O}_2(\lambda^{0+})} = \overline{f^-(\lambda)} \tag{42}$$

for some $a \in \mathbb{R}/\{0\}$ and $z \in \mathbb{C}/\mathbb{R}$.

In particular we note that for $0 < \lambda < \lambda_1$,

$$f^+(\lambda) - f^-(\lambda) = \frac{1}{\lambda^2}\left(\frac{(a\log\lambda + z) - (a\log\lambda + \overline{z}) + \widetilde{O}_1(\lambda^{0+})}{(a\log\lambda + z)(a\log\lambda + \overline{z}) + \widetilde{O}_1(\lambda^{0+})}\right) = \widetilde{O}_1((\lambda\log\lambda)^{-2}). \tag{43}$$

We are now ready to use Lemma 2.8 to obtain an expansion for $M^{\pm}(\lambda)^{-1}$. This expansion is longer than the corresponding expansion in [14], which allows us to give a more detailed long-time expansion for the evolution.

Proposition 3.4. *If there is a resonance of the first kind at zero, then for small λ*

$$M^{\pm}(\lambda)^{-1} = f^{\pm}(\lambda)S_1 + K_0 + f_1^{\pm}(\lambda)K_1 + f_2^{\pm}(\lambda)K_2 + \widetilde{O}_2\left(\frac{1}{(\log\lambda)^3}\right),$$

where K_j are λ independent, finite rank, absolutely bounded operators, and

$$f_j^{\pm}(\lambda) = \widetilde{O}_2\left(\frac{1}{(\log\lambda)^j}\right), \qquad j = 1, 2.$$

Proof. Using Lemma 2.8 and (41), we see that

$$\begin{aligned} M^{\pm}(\lambda)^{-1} &= (M^{\pm}(\lambda) + S_1)^{-1} + (M^{\pm}(\lambda) + S_1)^{-1}S_1B_{\pm}(\lambda)^{-1}S_1(M^{\pm}(\lambda) + S_1)^{-1} \\ &= (M^{\pm}(\lambda) + S_1)^{-1} + f^{\pm}(\lambda)(M^{\pm}(\lambda) + S_1)^{-1}S_1(M^{\pm}(\lambda) + S_1)^{-1}. \end{aligned}$$

The representation (38) in Lemma 2.9 takes care of the first summand. Using (37), and $S_1D_0 = D_0S_1 = S_1$, we have

$$\begin{aligned} (M^{\pm}(\lambda) + S_1)^{-1}S_1 &= S_1 - \widetilde{g}_1^{\pm}(\lambda)D_0PS_1 - \lambda^2 D_0vG_1vS_1 + \widetilde{O}_2(\lambda^{2+}), \\ S_1(M^{\pm}(\lambda) + S_1)^{-1} &= S_1 - \widetilde{g}_1^{\pm}(\lambda)S_1PD_0 - \lambda^2 S_1vG_1vD_0 + \widetilde{O}_2(\lambda^{2+}). \end{aligned}$$

When an error term of size $\widetilde{O}_2(\lambda^{2+})$ interacts with $f^{\pm}(\lambda)$, the product satisfies $\widetilde{O}_2(\lambda^{2+})f^{\pm}(\lambda) = \widetilde{O}_2(\lambda^{0+})$, which satisfies the desired error bound for small λ. For the remaining terms

$$-\widetilde{g}_1^{\pm}(\lambda)f^{\pm}(\lambda)[D_0PS_1 + S_1PD_0] - \lambda^2 f^{\pm}(\lambda)[D_0vG_1vS_1 + S_1vG_1vD_0]$$

For small λ, by a Taylor expansion,

$$\begin{aligned} \widetilde{g}_1^{\pm}(\lambda)f^{\pm}(\lambda) &= \frac{\widetilde{g}_1^{\pm}(\lambda)}{c_1\widetilde{g}_1^{\pm}(\lambda) + c_2\lambda^2 + \widetilde{O}_2(\lambda^{2+})} = \left(\frac{1}{c_1 + \frac{c_2}{a_2\log\lambda + z_2^{\pm}} + \widetilde{O}_2(\lambda^{0+})}\right) \\ &= a_0 + \frac{a_1}{\log\lambda + z_2^{\pm}} + \frac{a_2}{(\log\lambda + z_2^{\pm})^2} + \widetilde{O}_2((\log\lambda)^{-3}), \text{ and} \qquad (44) \\ \lambda^2 f^{\pm}(\lambda) &= \frac{1}{a\log\lambda + z^{\pm} + \widetilde{O}_2(\lambda^{0+})} = \frac{b_1}{\log\lambda + z_2^{\pm}} + \frac{b_2}{(\log\lambda + z_2^{\pm})^2} + \widetilde{O}_2((\log\lambda)^{-3}). \end{aligned}$$

Where a_j, b_j are real-valued constants, the exact values are unimportant for our analysis. Since S_1 is a rank one projection, the operators K_j are all finite rank, and hence also absolutely bounded.

□

To attain better time-decay, we use

Lemma 3.5. *If $\mathcal{E}(\lambda) = \widetilde{O}_2((\log\lambda)^{-k})$, then for $k \geq 2$,*

$$\Big|\int_0^\infty e^{it\lambda^2}\lambda\chi(\lambda)\mathcal{E}(\lambda)\,d\lambda\Big| \lesssim \frac{1}{t(\log t)^{k-1}}, \qquad t > 2.$$

Proof. We first divide the integral into two pieces,

$$\int_0^\infty e^{it\lambda^2}\lambda\chi(\lambda)\mathcal{E}(\lambda)\,d\lambda = \int_0^{t^{-1/2}} e^{it\lambda^2}\lambda\chi(\lambda)\mathcal{E}(\lambda)\,d\lambda + \int_{t^{-1/2}}^\infty e^{it\lambda^2}\lambda\chi(\lambda)\mathcal{E}(\lambda)\,d\lambda$$

For the first integral one cannot utilize the oscillation of the Gaussian, instead we use

$$\begin{aligned}\Big|\int_0^{t^{-1/2}} e^{it\lambda^2}\lambda\chi(\lambda)\mathcal{E}(\lambda)\,d\lambda\Big| &\lesssim \int_0^{t^{-1/2}} \frac{\lambda}{(\log\lambda)^k}\,d\lambda \lesssim \int_0^{t^{-1/2}} \frac{\lambda^2}{\lambda(\log\lambda)^k}\,d\lambda \\ &\lesssim \frac{1}{t}\int_0^{t^{-1/2}} \frac{1}{\lambda(\log\lambda)^k}\,d\lambda \lesssim \frac{1}{t(\log t)^{k-1}}.\end{aligned}$$

For the second integral, we utilize the oscillation by integrating by parts twice to see

$$\begin{aligned}\Big|\int_{t^{-1/2}}^\infty e^{it\lambda^2}\lambda\chi(\lambda)\mathcal{E}(\lambda)\,d\lambda\Big| &= \Big|\frac{\mathcal{E}(t^{-1/2})}{2it} + \frac{1}{2it}\int_{t^{-1/2}}^\infty e^{it\lambda^2}\frac{d}{d\lambda}\big(\chi(\lambda)\mathcal{E}(\lambda)\big)\,d\lambda\Big| \\ &\lesssim \frac{|\mathcal{E}(t^{-1/2})|}{t} + \frac{|\mathcal{E}'(t^{-1/2})|}{t^{\frac{3}{2}}} + \frac{1}{t^2}\int_{t^{-1/2}}^\infty \Big|\partial_\lambda\Big(\frac{\mathcal{E}'(\lambda)}{\lambda}\Big)\Big|\,d\lambda \\ &\lesssim \frac{1}{t(\log t)^k} + \frac{1}{t(\log t)^{k+1}} + \frac{1}{t^2}\int_{t^{-1/2}}^{t^{-1/4}} \frac{1}{\lambda^3|\log t|^k}\,d\lambda + \frac{1}{t^2}\int_{t^{-\frac{1}{4}}}^{\frac{1}{2}} \frac{1}{\lambda^3}\,d\lambda \lesssim \frac{1}{t(\log t)^k}.\end{aligned}$$

Here we used that the integral converges on $[\frac{1}{2}, \infty)$.

□

One can similarly prove bounds with $(t\log(\log t))^{-1}$ if $k = 1$ and $1/t$ if $k = 0$.

Lemma 3.6. *In the case of a resonance of the first kind, dim$\{\psi \in L^{2,0+} : H\psi = 0\} = 1$. Furthermore, the integral kernel of the operator $P_r := G_0VG_0vS_1vG_0VG_0$ satisfies the identity*

$$P_r(x,y) = \psi(x)\psi(y), \qquad \textit{where } H\psi = 0, \quad \textit{and} \quad \langle v\psi, v\psi\rangle = 1.$$

Proof. Corollary 8.3 and the fact that $S_2 = 0$ in this case establishes the first claim. For the second, we first note that

$$S_1(U + vG_0v) = 0 \qquad \Rightarrow \qquad S_1 = -S_1vG_0w, \qquad \text{and} \qquad S_1 = -wG_0vS_1. \tag{45}$$

So that $G_0VG_0vS_1vG_0VG_0 = G_0vS_1vG_0$. Now, since S_1 is a one dimensional projection, we have $S_1f(x) = \phi(x)\langle f, \phi\rangle$ where we take $\phi \in S_1L^2(\mathbb{R}^4)$ such that $\|\phi\|_2 = 1$. Furthermore,

Lemma 8.2 gives us that $\phi = w\psi$ with $H\psi = 0$. Noting that $(-\Delta + V)\psi = 0$ is equivalent to $(I + G_0 V)\psi = 0$, we have

$$\begin{aligned} G_0 v S_1 v G_0(x,y) &= G_0 v(x,x_1)\phi(x_1)\phi(y_1) v G_0(y_1,y) = [G_0 v\omega\psi](x)[G_0 v\omega\psi](y) \\ &= [G_0 V\psi](x)[G_0 V\psi](y) = \psi(x)\psi(y) \end{aligned}$$

□

We note that since P_r is rank one, it is absolutely bounded. Noting that from the expansion in Lemma 2.2 and its proof, we have

$$R_0^{\pm}(\lambda^2)(x,y) = G_0(x,y) + g_1^{\pm}(\lambda) + \lambda^2 G_1(x,y) + \langle x\rangle^{\frac{1}{2}}\langle y\rangle^{\frac{1}{2}}\widetilde{O}_2(\lambda^{\frac{5}{2}}). \tag{46}$$

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. The proof follows from Lemma 2.11 and the expansions in Proposition 3.4. By (15), and the discussion following Lemma 2.11, we need now only bound the contribution of

$$[R_0^+ V R_0^+ v M^+(\lambda) v R_0^+ V R_0^+ - R_0^- V R_0^- v M^-(\lambda) v R_0^- V R_0^-](x,y) \tag{47}$$

to the Stone formula, (4). Using the algebraic fact,

$$\prod_{k=0}^{M} A_k^+ - \prod_{k=0}^{M} A_k^- = \sum_{\ell=0}^{M}\Big(\prod_{k=0}^{\ell-1} A_k^-\Big)(A_\ell^+ - A_\ell^-)\Big(\prod_{k=\ell+1}^{M} A_k^+\Big), \tag{48}$$

there are two cases. Either the '+/-' difference acts on a free resolvent, or on $M^{\pm}(\lambda)^{-1}$.

By (46) we have

$$R_0^+(\lambda^2)(x,y) - R_0^-(\lambda^2)(x,y) = c\lambda^2 + \langle x\rangle^{\frac{1}{2}}\langle y\rangle^{\frac{1}{2}}\widetilde{O}_2(\lambda^{\frac{5}{2}}). \tag{49}$$

When the '+/-' difference acts on a free resolvent, we can write

$$M^{\pm}(\lambda)^{-1} = f^{\pm}(\lambda) S_1 + \widetilde{O}_2(1).$$

We get, in this case, contributions of the form

$$\lambda^2 f^{\pm}(\lambda) C_1 + \langle x\rangle^{\frac{1}{2}}\langle y\rangle^{\frac{1}{2}}\widetilde{O}_2(\lambda^{\frac{1}{2}-}), \tag{50}$$

here $C_1 = 1VG_0vS_1vG_0VG_0 + G_0V1vS_1vG_0VG_0 + G_0VG_0vS_1v1VG_0 + G_0VG_0vS_1vG_0V1$ is a finite rank operator. By (44) and a simple integration by parts, noting that $\partial_\lambda(\log\lambda)^{-1}$

is integrable on $[0, \lambda_1]$, we have that contribution of C_1 to the Stone formula can be bounded by t^{-1}. The error term's contribution to (4) is of the form

$$\langle x\rangle^{\frac{1}{2}}\langle y\rangle^{\frac{1}{2}}\int_0^\infty e^{it\lambda}\chi(\lambda)\widetilde{O}_2(\lambda^{\frac{3}{2}-})\,d\lambda,$$

which can be bounded by $\langle x\rangle^{\frac{1}{2}}\langle y\rangle^{\frac{1}{2}}t^{-1-}$ using Lemma 8.7.

On the other hand, if the '+/-' difference acts on $M^{\pm}(\lambda)^{-1}$, we have

$$M^+(\lambda)^{-1} - M^-(\lambda)^{-1} = [f^+ - f^-](\lambda)S_1 + [f_1^+ - f_1^-](\lambda)K_1 + [f_2^+ - f_2^-](\lambda)K_2 + \widetilde{O}_2\left(\frac{1}{(\log\lambda)^3}\right),$$

There are now four subcases to consider. First, using the representation (46), if all the free resolvents contribute G_0, we have to bound the contribution of

$$[f^+ - f^-](\lambda)P_r + \widetilde{O}_2(1/\log\lambda)C_2 + \widetilde{O}_2\left(\frac{1}{(\log\lambda)^3}\right)$$

Here $C_2 = G_0VG_0vK_1vG_0VG_0 + G_0VG_0vK_2vG_0VG_0$ is a finite rank operator. By Lemma 3.2, the first term's contribution to the Stone formula is bounded by $1/\log t$. The second term is bounded by $1/t$ and the final term is bounded by $t^{-1}(\log t)^{-2}$ by Lemma 3.5. We note that the contribution of this error term, when all of its surronding free resolvents contribute G_0 defines the operator A_3 in Theorem 3.1. We can see from the expansion for $M^{\pm}(\lambda)^{-1}$, that we cannot expect this operator to be finite rank.

Another case to consider is when one free resolvent contributes $g_1(\lambda)$ while the other contribute G_0. Recall that $g_1(\lambda)$ comes with an operator whose integral kernel is a constant. In this case, we have to control

$$g_1^{\pm}(\lambda)[f^+ - f^-](\lambda)C_1 + \widetilde{O}_2(\lambda^{2-}).$$

Here C_1 the same operator encountered in (50). We note that $g_1^{\pm}(\lambda)[f^+ - f^-](\lambda) = \widetilde{O}_1(1/\log\lambda)$, thus the first term's contribution to the Stone formula is bounded by $1/t$ by Lemma 3.2, while the contribution of the second term is $O(t^{-1-})$ by Lemma 8.7.

Another case to consider is when one free resolvent contributes $\lambda^2 G_1$ while the other contribute G_0. In this case, we have to control

$$\lambda^2[f^+ - f^-](\lambda)C_3 + \widetilde{O}_2(\lambda^{2-}).$$

Here $C_3 = G_1VG_0vS_1vG_0VG_0 + G_0VG_1vS_1vG_0VG_0 + G_0VG_0vS_1vG_1VG_0 + G_0VG_0vS_1vG_0VG_1$ is a finite rank operator. By Lemma 3.5, the first term's contribution to the Stone formula is bounded by $1/(t\log t)$. Due to the presence of the operator

G_1, whose integral kernel is $G_1(x,y) = -\frac{1}{8\pi^2}\log|x-y|$, this bound is understood as mapping logarithmically weighted spaces, see the discussion around (52) below.

Finally, if the error term in (46) in any free resolvent is encountered, or if less than three G_0's are encountered, its contribution is bounded by $\langle x\rangle^{\frac{1}{2}}\langle y\rangle^{\frac{1}{2}}\widetilde{O}_2(\lambda^{\frac{1}{2}})$, which contributes $t^{-\frac{3}{2}}$ to the Stone formula as an operator from $L^{1,\frac{1}{2}}$ to $L^{\infty,-\frac{1}{2}}$.

To close the proof, we must establish that the spatial integrals converge. For the contribution of C_1, we note that due to the similarity of the four constitutent operators, and

$$|1VG_0|(x,z_2) = C\int_{\mathbb{R}^4}\frac{V(z_1)}{|z_1-x|^2}\,dz_1 \lesssim \int_{\mathbb{R}^4}\frac{\langle z_1\rangle^{-4-}}{|z_1-x|^2}\,dz_1 \lesssim 1$$

uniformly in x by Lemma 8.12. Similarly,

$$\begin{aligned}|G_0VG_0|(z_3,y) &= C\int_{\mathbb{R}^4}\frac{V(z_4)}{|z_3-z_4|^2|z_4-y|^2}\,dz_4\\ &\lesssim \int_{\mathbb{R}^4}\frac{\langle z_4\rangle^{-4-}}{|z_4-y|^2}\left(\frac{1}{|z_3-z_4|^{2+}}+\frac{1}{|z_3-z_4|^{2-}}\right)dz_4 \lesssim 1+\frac{1}{|z_3-y|^{0+}}.\end{aligned}$$

Under the assumptions on V, we have that $\sup_{y\in\mathbb{R}^4}\|v(\cdot)(1+|y-\cdot|^{-0-})\|_2 \lesssim 1$. Thus,

$$\sup_{x,y}|1VG_0vS_1vG_0VG_0(x,y)| \lesssim \sup_{x,y}\|1VG_0(x,\cdot)v\|_2\||S_1|\|_{2\to 2}\|vG_0VG_0(\cdot,y)\|_2 \lesssim 1. \tag{51}$$

Similarly, one can bound the contributions of P_r and C_2. For C_3 we must take some care to account for the operator G_1. Note that $G_1(x,z) = c\log|x-z|$, when G_1 is contributed by the leading or lagging free resolvent, we use that $\log|x-z| = \log^-|x-z| + \log^+|x-z|$. Since $\log^-|x-z| \lesssim |x-z|^{0-}$, we can control it as in the previous operators. For $\log^+$, we note that $\log^+$ is an increasing function and $|x-y| \le |x|+|y| \le 2\max(|x|,|y|)$. Then,

$$\begin{aligned}|G_1VG_0|(x,z_2) &= C\int_{\mathbb{R}^4}\frac{\log|x-z_1|V(z_1)}{|z_1-z_2|^2}\,dz_1\\ &\lesssim (1+\log^+|x|)\int_{\mathbb{R}^4}\frac{V(z_1)}{|z_1-z_2|^2}\left(1+\log^+|z_1|+\frac{1}{|z_1-x|^{0+}}\right)dz_1 \lesssim 1+\log^+|x|.\end{aligned} \tag{52}$$

Here we used the decay of the potential to control the $\log^+|z_1|$ growth and Lemma 8.12 to establish the boundedness of the resulting integrals. A similar analysis holds for the polynomially weighted error terms.

□

4. Resonance of the second kind

In this section we prove Theorem 1.2 in the case of a resonance of the second kind, when $S_1 \neq 0$, and $S_1 - S_2 = 0$. Recall that this means there is an eigenvalue at zero energy, but no resonance. In particular, we prove

Theorem 4.1. *Suppose that $|V(x)| \lesssim \langle x\rangle^{-8-}$. If there is a resonance of the second kind at zero, then*

$$e^{itH}\chi(H)P_{ac}(H) = O(1/t)A_1 + O(t^{-1-})A_3, \qquad t > 2.$$

$A_1 : L^1 \to L^\infty$, is a finite rank operator, and $A_3 : L^{1,\frac{1}{2}} \to L^{\infty,-\frac{1}{2}}$.

Despite the fact that the spectral measure is more singular as $\lambda \to 0$, the lack of resonances greatly simplifies our expansions for $M^\pm(\lambda)^{-1}$. Much of this simplification follows from the fact that $S_1 = S_2$, which by (102) shows that $PS_1 = 0$. This eliminates many of the terms containing powers of $\log\lambda$ in the expansion of the spectral measure as $\lambda \to 0$.

To understand the expansion for $M^\pm(\lambda)^{-1}$ in this case we need more terms in the expansion of $(M^\pm(\lambda) + S_1)^{-1}$ than was provided Lemma 2.9. From Lemma 2.2, specifically (33), we have by a Neumann series expansion

$$\begin{aligned}
&(M^\pm(\lambda) + S_1)^{-1} \\
&\qquad = D_0[\mathbb{1} + \widetilde{g}_1^\pm(\lambda)PD_0 + \lambda^2 vG_1vD_0 + g_2^\pm(\lambda)vG_2vD_0 + \lambda^4 vG_3vD_0 + M_2^\pm(\lambda)D_0]^{-1} \\
&\qquad = D_0 - \widetilde{g}_1^\pm(\lambda)D_0PD_0 - \lambda^2 D_0vG_1vD_0 + (\widetilde{g}_1^\pm(\lambda))^2 D_0PD_0PD_0 \\
&\qquad + \lambda^2\widetilde{g}_1^\pm(\lambda)[D_0PD_0vG_1vD_0 + D_0vG_1vD_0PD_0] - g_2^\pm(\lambda)D_0vG_2vD_0 \\
&\qquad - \lambda^4 D_0vG_3vD_0 + D_0E_2^\pm(\lambda)D_0
\end{aligned} \tag{53}$$

with $E_2^\pm(\lambda) = \widetilde{O}_1(\lambda^{4+})$.

In the case of a resonance of the second kind, we recall that $S_1 = S_2$. By Lemma 8.4 below the operator $S_1vG_1vS_1$ is invertible on S_1L^2 (which is S_2L^2 in this case). We define $D_2 = (S_1vG_1vS_1)^{-1}$ as an operator on $S_2L^2(\mathbb{R}^4)$. Noting that $D_2 = S_1D_2S_1$, the operator is finite rank and hence absolutely bounded.

Proposition 4.2. *If there is a resonance of the second kind at zero, then for small λ*

$$M^\pm(\lambda)^{-1} = -\frac{D_2}{\lambda^2} + \frac{g_2^\pm(\lambda)}{\lambda^4}K_1 + K_2 + \widetilde{O}_2(\lambda^{0+}) \tag{54}$$

where K_1, K_2 are λ independent, finite rank operators.

We note that the statement and proof of this Proposition are found in [14], see Proposition 4.2 with an error term of $\widetilde{O}_1(\lambda^{0+})$. Here, we need the extra derivative, but we note the same proof follows noting that the error term from Lemma 2.9 has two derivatives. In [14] only one derivative was needed to get the t^{-1} bound, we wish to gain more time decay from its contribution which necessitates the spatial weights. The fact that K_1, K_2 are finite rank operators follows from the fact that S_1 is finite rank.

Proof of Theorem 4.1. The proof follows by bounding the contribution of Proposition 4.2 to the Stone formula, (4). We to use cancellation between the '+' and '-' terms in

$$R_0^+ V R_0^+ v M^+(\lambda)^{-1} v R_0^+ V R_0^+ - R_0^- V R_0^- v M^-(\lambda)^{-1} v R_0^- V R_0^-.$$

As with resonances of the first kind, we use the algebraic fact (48). Two kinds of terms occur in this decomposition, one featuring the difference $M^+(\lambda)^{-1} - M^-(\lambda)^{-1}$ and those containing a difference of free resolvents. For the first case we use Proposition 4.2 and that $g_2^+(\lambda) - g_2^-(\lambda) = c\lambda^4$ to obtain

$$M^+(\lambda)^{-1} - M^-(\lambda)^{-1} = cK_1 + \widetilde{O}_2(\lambda^{0+}). \tag{55}$$

Recalling (46), we can write $R_0^\pm(\lambda^2) = G_0 + \langle x\rangle^{\frac{1}{2}}\langle y\rangle^{\frac{1}{2}}\widetilde{O}_2(\lambda^{2-})$ and consider the most singular terms this difference contributes, i.e.,

$$G_0 V G_0 v K_1 v G_0 V G_0 + \langle x\rangle^{\frac{1}{2}}\langle y\rangle^{\frac{1}{2}}\widetilde{O}_2(\lambda^{0+}).$$

The time decay of t^{-1} for the first term follows from Lemma 8.6 and the bound of $\langle x\rangle^{\frac{1}{2}}\langle y\rangle^{\frac{1}{2}} t^{-1-}$ for the error term follows from 8.7. An analysis of the spatial integrals as in Theorem 3.1 noting that K_1 is finite rank and hence absolutely bounded finishes the argument. For the terms of the second kind the difference of '+' and '-' terms in (48) acts on one of the resolvents. As usual, the most delicate case is of the form

$$(R_0^+(\lambda^2) - R_0^-(\lambda^2)) V R_0^+(\lambda^2) v[(54)] v R_0^+(\lambda^2) V R_0^+(\lambda^2). \tag{56}$$

Using (46), we have $[R_0^+ - R_0^-](\lambda^2)(x,y) = c\lambda^2 + \langle x\rangle^{\frac{1}{2}}\langle y\rangle^{\frac{1}{2}}\widetilde{O}_2(\lambda^{\frac{5}{2}})$. We then write $M^\pm(\lambda) = -D_2/\lambda^2 + \widetilde{O}_2(\lambda^{0-})$ to see

$$(56) = -cVG_0 v D_2 v G_0 V G_0 + \langle x\rangle^{\frac{1}{2}}\langle y\rangle^{\frac{1}{2}}\widetilde{O}_2(\lambda^{0+}).$$

Using Lemma 8.6, we see that the first term contributes t^{-1} to the Stone formula, while the second term is bounded by $\langle x\rangle^{\frac{1}{2}}\langle y\rangle^{\frac{1}{2}} t^{-1-}$ using Lemma 8.7. The remaining terms can be bounded similarly. The spatial integrals are controlled as in the case of a resonance of the first kind in Theorem 3.1.

□

5. Resonance of the third kind

In this section we prove Theorem 1.2 in the case of a resonance of the third kind, that is when $S_1 \neq 0$, $S_2 \neq 0$ and $S_1 - S_2 \neq 0$. Recall that this means there are both a zero energy resonance and a zero energy eigenvalue. In particular, we prove

Theorem 5.1. *Suppose that $|V(x)| \lesssim \langle x\rangle^{-8-}$. If there is a resonance of the third kind at zero, then for $t > 2$,*

$$e^{itH}\chi(H)P_{ac}(H) = \varphi(t)A_0 + O(1/t)A_1 + O((t\log t)^{-1})A_2 + O(t^{-1}(\log t)^{-2})A_3 + O(t^{-1-})A_4$$

with $A_0, A_1 : L^1 \to L^\infty$, $A_2 : L^1_w \to L^\infty_{w^{-1}}$ finite rank operators, $A_3 : L^1 \to L^\infty$ and $A_4 : L^{1,\frac{1}{2}} \to L^{\infty,-\frac{1}{2}}$.

The expansion in (53) remains valid, but in this section we do not have that $S_1P = 0$. Using (33) in Lemma 2.2, we have

$$\begin{aligned} B^\pm(\lambda) &= \widetilde{g}_1^\pm(\lambda)S_1PS_1 + \lambda^2 S_1vG_1vS_1 - (\widetilde{g}_1^\pm(\lambda))^2 S_1PD_0PS_1 \\ &\quad - \lambda^2\widetilde{g}_1^\pm(\lambda)[S_1PD_0vG_1vS_1 + S_1vG_1vD_0PS_1] + g_2^\pm(\lambda)S_1vG_2vS_1 \\ &\quad + \lambda^4 S_1vG_3vS_1 + \widetilde{O}_2(\lambda^{6-}) \\ &=: \widetilde{g}_1^\pm(\lambda)S_1PS_1 + \lambda^2 S_1vG_1vS_1 + (\widetilde{g}_1^\pm(\lambda))^2\Gamma_1 + \lambda^2\widetilde{g}_1^\pm(\lambda)\Gamma_2 + g_2^\pm(\lambda)\Gamma_3 \\ &\quad + \lambda^4\Gamma_4 + \widetilde{O}_2(\lambda^{6-}). \end{aligned} \tag{57}$$

Note that, since $S_2 \neq 0$ the kernel of S_1PS_1 is non-trivial. We, therefore, use Feshbach formula to invert $B^\pm(\lambda)$. To do that, we define the operator Γ by $S_1 = S_2 + \Gamma$ and express $B^\pm(\lambda)$ with respect to the decomposition $S_1L^2(\mathbb{R}^4) = S_2L^2(\mathbb{R}^4) \oplus \Gamma L^2(\mathbb{R}^4)$. We also define the finite rank operator S by

$$S := \begin{bmatrix} \Gamma & -\Gamma vG_1vD_2 \\ -D_2vG_1v\Gamma & D_2vG_1v\Gamma vG_1vD_2 \end{bmatrix}. \tag{58}$$

We note that the operator A_0 in the statement of Theorem 5.1 has rank at most two. This follows from the expansions detailed below and the fact that S has rank at most two. As in the previous cases, we give a refinement of the expansion in [14].

Lemma 5.2. *In the case of a resonance of the third kind we have for small λ*

$$B^\pm(\lambda)^{-1} = \tilde{f}^\pm(\lambda)S + \frac{D_2}{\lambda^2} + \frac{g_2^\pm(\lambda)}{\lambda^4}F_1 + F_2 + f_1(\lambda)F_3 + f_2(\lambda)F_4 + \widetilde{O}_2(\lambda^{0+}). \tag{59}$$

Here F_j are λ independent absolutely bounded operators, $\tilde{f}^+(\lambda) = (\lambda^2(a\log\lambda + z))^{-1}$ with $a \in \mathbb{R}\setminus\{0\}$ and $z \in \mathbb{C}\setminus\mathbb{R}$, with $\tilde{f}^-(\lambda) = \overline{\tilde{f}^+(\lambda)}$, and $f_j(\lambda) = \widetilde{O}_2((\log\lambda)^{-j})$.

Proof. Here we use the fact that $S_2P = PS_2 = 0$ to see that the two smallest terms of $B^\pm(\lambda)$ with respect to λ, see (57), may be written in the block form

$$A^\pm(\lambda) := \lambda^2 \begin{bmatrix} \frac{\widetilde{g}_1^\pm(\lambda)}{\lambda^2}\Gamma P\Gamma + \Gamma vG_1v\Gamma & \Gamma vG_1vS_2 \\ S_2vG_1v\Gamma & S_2vG_1vS_2 \end{bmatrix}. \tag{60}$$

Then, by the Feshbach formula we have

$$A^{\pm}(\lambda)^{-1} = \frac{1}{\lambda^2 h^{\pm}(\lambda)} \begin{bmatrix} \Gamma & -\Gamma v G_1 v D_2 \\ -D_2 v G_1 v \Gamma & D_2 v G_1 v \Gamma v G_1 v D_2 \end{bmatrix} + \frac{D_2}{\lambda^2} =: \tilde{f}^{\pm}(\lambda) S + \frac{D_2}{\lambda^2}. \tag{61}$$

Here $\tilde{f}^{\pm} := (\lambda^2[a \log \lambda + z])^{-1}$ for some $a \in \mathbb{R} \setminus \{0\}$ and $z \in \mathbb{C} \setminus \mathbb{R}$. One can see [14] for the details of this inversion.

By a Neumann series expansion, we obtain

$$\begin{aligned} B^{\pm}(\lambda)^{-1} &= A^{\pm}(\lambda)^{-1}[\mathbb{1} + (B^{\pm}(\lambda) - A^{\pm}(\lambda))A^{\pm}(\lambda)^{-1}]^{-1} \\ &= A^{\pm}(\lambda)^{-1} - A^{\pm}(\lambda)^{-1}[B^{\pm}(\lambda) - A^{\pm}(\lambda)]A^{\pm}(\lambda)^{-1} + \widetilde{O}_2(\lambda^{0+}). \end{aligned}$$

We note that $D_2 S_1 P = D_2 S_2 P = 0$. Therefore, the operators Γ_j in the expansion of $B^{\pm}(\lambda)$ in (57) satisfy

$$\Gamma_1 D_2 = D_2 \Gamma_1 = D_2 \Gamma_2 D_2 = 0.$$

Further recalling (21) for $\widetilde{g_1}^{\pm}(\lambda)$ and $g_2^{\pm}(\lambda)$, for small λ we have

$$\begin{aligned} \tilde{f}^{\pm}(\lambda)\widetilde{g_1}^{\pm}(\lambda) &= \frac{\widetilde{g_1}^{\pm}(\lambda)}{\lambda^2(a \log \lambda + z)} = c_1 + \frac{z_1}{a \log \lambda + z}, \\ \frac{\tilde{f}^{\pm}(\lambda)}{\lambda^2} g_2^{\pm}(\lambda) &= \frac{g_2^{\pm}(\lambda)}{\lambda^4(a \log \lambda + z)} = c_2 + \frac{z_2}{a \log \lambda + z}, \\ \widetilde{f}^{\pm}(\lambda)\lambda^2 &= \frac{1}{a \log \lambda + z} \\ [\tilde{f}^{\pm}(\lambda)]^2 g_2^{\pm}(\lambda) &= \frac{c_3}{a \log \lambda + z} + \frac{z_3}{(a \log \lambda + z)^2}, \end{aligned} \tag{62}$$

for some (unimportant) constants c_j, z_j, establishes the claim. □

Using Lemma 2.8 and (59), we have

Proposition 5.3. *If there is a resonance of the third kind at zero, then for small λ*

$$M^{\pm}(\lambda)^{-1} = \tilde{f}^{\pm}(\lambda) S_1 S S_1 + \frac{D_2}{\lambda^2} + \frac{g_2^{\pm}(\lambda)}{\lambda^4} D_2 \Gamma_3 D_2 + K_1 + f_1(\lambda) K_2 + f_2(\lambda) K_3 + \widetilde{O}_2(\lambda^{0+}),$$

where the operators are λ independent and finite rank except for the error term.

Proof of Theorem 5.1. We note that the expansion of $M^{\pm}(\lambda)^{-1}$ is a sum of terms similar to the ones encountered in Propositions 3.4 and 4.2. Accordingly, the methods used in the proofs of Theorems 3.1 and 4.1 apply with only minor adjustments.

□

6. Eigenvalue only and $P_eVx = 0$

We consider the evolution when there is a resonance of the third kind, that is an eigenvalue by not resonance at zero energy, and extra cancellation between the eigenfunctions and the potential. In particular, we show that the evolution satisfies the same $|t|^{-2}$ dispersive bound as an operator from L^1 and L^∞ as the free evolution. This bound is motivated by the work in [19, 20] which proved such bounds in higher dimensions $n \geq 5$. We note that the techniques of [19, 20] are not sufficient to obtain the $|t|^{-2}$ bound when $n = 4$. In dimensions $n > 4$, one has the expansion $R_0^\pm(\lambda^2) = G_0 + O(\lambda^2)$, in dimension $n = 4$ we instead have $R_0^\pm(\lambda^2) = G_0 + O(\lambda^2(1 + \log(\lambda|x - y|)))$. This small difference introduces many technical challenges which we overcome in this section.

For the purpose of obtaining this bound, one needs much longer expansion for $M^\pm(\lambda)^{-1}$.

Lemma 6.1. *Suppose that zero is not a regular point of the spectrum of $H = -\Delta + V$, and let S_1 be the corresponding Riesz projection. Then for sufficiently small $\lambda_1 > 0$, the operators $M^\pm(\lambda) + S_1$ are invertible for all $0 < \lambda < \lambda_1$ as bounded operators on $L^2(\mathbb{R}^4)$. Further, one has*

$$
\begin{aligned}
(M^\pm(\lambda) + S_1)^{-1} &= D_0 - \widetilde{g}_1^\pm(\lambda) D_0 P D_0 - \lambda^2 D_0 v G_1 v D_0 - g_2^\pm(\lambda) D_0 v G_2 v D_0 \\
&- \lambda^4 D_0 v G_3 v D_0 - g_3^\pm(\lambda) D_0 v G_4 v D_0 - \lambda^6 D_0 v G_5 v D_0 \\
&+ (\widetilde{g}_1^\pm(\lambda))^2 D_0 P D_0 P D_0 + \lambda^2 \widetilde{g}_1^\pm(\lambda)[D_0 P D_0 v G_1 v D_0 + D_0 v G_1 v D_0 P D_0] \\
&+ \lambda^4 \widetilde{g}_1^\pm(\lambda)[D_0 P D_0 v G_3 v D_0 + D_0 v G_3 v D_0 P D_0] \\
&+ \widetilde{g}_1^\pm(\lambda) g_2^\pm(\lambda)[D_0 P D_0 v G_2 v D_0 + D_0 v G_2 v D_0 P D_0] + \lambda^4 D_0 v G_1 v D_0 v G_1 v D_0 \\
&- (\widetilde{g}_1^\pm(\lambda))^3 D_0 P D_0 P D_0 P D_0 + \lambda^2 g_2^\pm(\lambda)[D_0 v G_1 v D_0 v G_2 v D_0 \\
&+ D_0 v G_2 v D_0 v G_1 v D_0] - \lambda^2 (\widetilde{g}_1^\pm(\lambda))^2 [D_0 P D_0 P D_0 v G_1 v D_0 \\
&+ D_0 P D_0 v G_1 v D_0 P D_0 + D_0 v G_1 v D_0 P D_0 P D_0] \\
&- \lambda^4 \widetilde{g}_1^\pm(\lambda)[D_0 P D_0 v G_1 v D_0 v G_1 v D_0 + D_0 v G_1 v D_0 v G_1 v D_0 P D_0 \\
&+ D_0 v G_1 v D_0 P D_0 v G_1 v D_0] - \lambda^6 D_0 v G_1 v D_0 v G_1 v D_0 v G_1 v D_0 + \widetilde{O}_2(\lambda^{6+})
\end{aligned} \tag{63}
$$

as an absolutely bounded operator on $L^2(\mathbb{R}^4)$ provided $v(x) \lesssim \langle x \rangle^{-6-}$.

Proof. The proof uses the expansion (34) for $M^\pm(\lambda)$ up to terms of size λ^6 with error term $M_3^\pm(\lambda)$, along with a Neumann series expansion that considers up to the 'x^3' term. □

Remark 6.2. *When there is an eigenvalue only, we take advantage of the facts that* $S_1 = S_2$, $S_1 D_0 = D_0 S_1 = S_1$ *and* $S_1 P = P S_1 = 0$. *The effect of this is that the leading terms in* $S_1(M + S_1)^{-1} S_1$ *containing only* $\widetilde{g}_1^{\pm}(\lambda)$, $(\widetilde{g}_1^{\pm}(\lambda))^2$, $(\widetilde{g}_1^{\pm}(\lambda))^3$, $\lambda^2 \widetilde{g}_1^{\pm}(\lambda)$, $\widetilde{g}_1^{\pm}(\lambda) g_2^{\pm}(\lambda)$ *and the* $\lambda^4 \widetilde{g}_1^{\pm}(\lambda)[D_0 P D_0 v G_3 v D_0 + D_0 v G_3 v D_0 P D_0]$ *all vanish.*

This observation allows us to prove the following.

Lemma 6.3. *Suppose there is a resonance of the second kind at zero and* $P_e V x = 0$. *If* $v(x) \lesssim \langle x \rangle^{-6}$ *then we have the following expansion.*

$$B^{\pm}(\lambda)^{-1} = \frac{D_2}{\lambda^2} + B_1 + \frac{g_3^{\pm}(\lambda)}{\lambda^4} B_2 + \frac{g_2^{\pm}(\lambda)}{\lambda^2} B_3 + \widetilde{g}_1^{\pm}(\lambda) B_4 + \lambda^2 B_5 + \widetilde{O}_2(\lambda^{2+}) \tag{64}$$

where B_i *are absolutely bounded operators with real-valued kernels.*

Proof. Note that by the identities $S_1 D_0 = D_0 S_1 = S_1$ and $S_1 P = P S_1 = 0$ in Remark 6.2 many terms in (63) cancels and recalling that by Lemma 8.4 the operator $D_2 = (S_1 v G_1 v S_1)^{-1}$ is bounded when $S_1 = S_2$, we obtain

$$\begin{aligned} B^{\pm}(\lambda)^{-1} &= [-\lambda^2 S_1 v G_1 v S_1 - g_2^{\pm}(\lambda) S_1 v G_2 v S_1 \\ &\qquad + \lambda^4 C_1 + g_3^{\pm}(\lambda) C_2 + \lambda^2 g_2^{\pm}(\lambda) C_3 + \lambda^4 \widetilde{g}_1^{\pm}(\lambda) C_4 + \lambda^6 C_5 + \widetilde{O}_2(\lambda^{6+})]^{-1} \\ &= -\lambda^{-2} D_2 [\mathbb{1} + \lambda^{-2} g_2^{\pm}(\lambda) S_1 v G_2 v S_1 D_2 + \lambda^2 C_1 D_2 \\ &\qquad + \lambda^{-2} g_3^{\pm}(\lambda) C_2 D_2 + g_2^{\pm}(\lambda) C_3 D_2 + \lambda^2 \widetilde{g}_1^{\pm}(\lambda) C_4 D_2 + \lambda^4 C_5 D_2 + \widetilde{O}_2(\lambda^{4+})]^{-1}. \end{aligned}$$

Here the operators C_i's can be written explicitly, however, in our analysis it is enough to know that the decay assumption on $v(x)$ ensures the boundedness of their Hilbert-Schmidt norms.

To effectively invert the above expression in a Neumann series, we first recall $D_2 = S_1 D_2 S_1$ and $S_1 = -w G_0 v S_1$. Also, by Lemma 8.5 we have $P_e = G_0 v S_2 D_2 S_2 v G_0$.

Using these and noting that in this section $S_1 = S_2$, we have

$$D_2 = S_1 D_2 S_1 = w G_0 v S_1 D_2 S_1 v G_0 w = w P_e w. \tag{65}$$

Further assuming $P_e V x = 0$, recalling that $G_2(x, y) = |x - y|^2 = (x - y) \cdot (x - y)$, we have

$$\begin{aligned} D_2 v G_2 v D_2 &= w P_e V [x^2 - 2x \cdot y + y^2] V P_e w \\ &= w P_e V x^2 1 V P_e w - 2 w P_e V x \cdot y V P_e w + w P_e V 1 y^2 V P_e w = 0 \end{aligned} \tag{66}$$

Note that using (66) the term $\lambda^{-2} g_2^{\pm}(\lambda) D_1 S_1 v G_2 v S_1$ is zero. Hence, we obtain (64). □

Proposition 6.4. *Assume $P_eVx = 0$. If there is a resonance of the second kind at zero, then for small λ we have*

$$M^{\pm}(\lambda)^{-1} = -\frac{D_1}{\lambda^2} + M_1 + \frac{g_3^{\pm}(\lambda)}{\lambda^4}M_2 + \frac{g_2^{\pm}(\lambda)}{\lambda^2}M_3 + \widetilde{g}_1^{\pm}(\lambda)M_4 + \lambda^2 M_5 + \widetilde{O}_2(\lambda^{2+}) \tag{67}$$

where M_i are λ independent and finite rank operators.

Proof. Using the expansion (63) together with the fact that $PS_1 = S_1P = 0$ we have

$$\begin{aligned}(M^{\pm}(\lambda) + S_1)S_1 &= S_1 - \lambda^2 D_0 v G_1 v S_1 - g_2^{\pm}(\lambda) D_0 v G_2 v S_1 - \lambda^4 D_0 v G_3 v S_1 \\ &\quad + \lambda^2 g_1^{\pm}(\lambda) D_0 P D_0 v G_1 v S_1 + \lambda^4 D_0 v G_1 D_0 v G_1 S_1 + \widetilde{O}_2(\lambda^{4+}),\end{aligned}$$

$$\begin{aligned}S_1(M^{\pm}(\lambda) + S_1) &= S_1 - \lambda^2 S_1 v G_1 v D_0 - g_2^{\pm}(\lambda) S_1 v G_2 v D_0 - \lambda^4 S_1 v G_3 v D_0 \\ &\quad + \lambda^2 g_1^{\pm}(\lambda) S_1 P D_0 v G_1 v D_0 + \lambda^4 S_1 v G_1 D_0 v G_1 D_0 + \widetilde{O}_2(\lambda^{4+}).\end{aligned}$$

The assertion follows by applying Lemma 2.8.

□

To prove the main Theorem, we need a few lemmas. The following variation of stationary phase from [41] will be useful in the analysis.

Lemma 6.5. *Let $\phi'(\lambda_0) = 0$ and $1 \le \phi'' \le C$. Then,*

$$\begin{aligned}\Big|\int_{-\infty}^{\infty} e^{it\phi(\lambda)} a(\lambda)\, d\lambda\Big| &\lesssim \int_{|\lambda-\lambda_0|<|t|^{-\frac{1}{2}}} |a(\lambda)|\, d\lambda \\ &\quad + |t|^{-1}\int_{|\lambda-\lambda_0|>|t|^{-\frac{1}{2}}} \left(\frac{|a(\lambda)|}{|\lambda-\lambda_0|^2} + \frac{|a'(\lambda)|}{|\lambda-\lambda_0|}\right) d\lambda.\end{aligned}$$

Define $\mathcal{G}_n(\pm\lambda, |x-y|)$ to be kernel of n-dimensional free resolvent operator $R_0^{\pm}(\lambda^2)$. Then we recall Lemma 2.1 in [12],

Lemma 6.6. *For $n \ge 2$, the following recurrence relation holds.*

$$\Big(\frac{1}{\lambda}\frac{d}{d\lambda}\Big)\mathcal{G}_n(\pm\lambda, |x-y|) = \frac{1}{2\pi}\mathcal{G}_{n-2}(\pm\lambda, |x-y|).$$

To prove Theorem 1.3, we need to bound the contribution of

$$R_0^+ V R_0^+ v M^+(\lambda)^{-1} v R_0^+ V R_0^+ - R_0^- V R_0^- v M^-(\lambda)^{-1} v R_0^- V R_0^-$$

to the Stone formula, (4). There are a number of terms that arise when considering the difference with (48), which we bound in a series of Lemmas.

Lemma 6.7. *Under the assumptions of Theorem 1.3, we have the bound*

$$\sup_{x,y\in\mathbb{R}^4}\Bigg|\int_{\mathbb{R}^{16}}\int_0^\infty e^{it\lambda^2}\lambda\chi(\lambda)\frac{[R_0^+-R_0^-](\lambda^2)(x,z_1)}{\lambda^2}V(z_1)G_0(z_1,z_2)v(z_2)D_2(z_2,z_3)v(z_3)$$
$$[R_0^+(\lambda^2)-G_0](z_3,z_4)V(z_4)R_0^+(\lambda^2)(z_4,y)\,d\lambda\,dz_1\,dz_2\,dz_3\,dz_4\Bigg|\lesssim|t|^{-2}.$$

Proof. The proof follows from a delicate case analysis. The integrand can be seen to be of small enough in the spectral variable λ to allow for one integration by parts without a boundary term. The λ integral is now equal to

$$\frac{1}{2it}\int_0^\infty e^{it\lambda^2}\chi(\lambda)\Bigg\{\Bigg[\frac{\partial_\lambda(R_0^+-R_0^-)(\lambda^2)}{\lambda^2}VG_0VP_eV[R_0^+(\lambda^2)-G_0]VR_0^+(\lambda^2)\Bigg] \tag{68}$$
$$+\Bigg[\frac{(R_0^+-R_0^-)(\lambda^2)}{\lambda^2}VG_0VP_eV\partial_\lambda[R_0^+(\lambda^2)-G_0]VR_0^+(\lambda^2)\Bigg] \tag{69}$$
$$+\Bigg[\frac{(R_0^+-R_0^-)(\lambda^2)}{\lambda^2}VG_0VP_eV[R_0^+(\lambda^2)-G_0]V\partial_\lambda R_0^+(\lambda^2)\Bigg] \tag{70}$$
$$-2\Bigg[\frac{(R_0^+-R_0^-)(\lambda^2)}{\lambda^3}VG_0VP_eV[R_0^+(\lambda^2)-G_0]VR_0^+(\lambda^2)\Bigg]\Bigg\}\,d\lambda. \tag{71}$$

When the derivative acts on $\chi(\lambda)$, it can be controlled as in (71). We first consider (68), using $R_0^+(\lambda^2)-G_0=g_1^+(\lambda)+\lambda^2G_1+\langle x\rangle^{\frac12}\langle y\rangle^{\frac12}\widetilde{O}_2(\lambda^{\frac52})$ on the inner resolvent, and Lemma 6.6 yields $\partial_\lambda[R_0^+-R_0^-](\lambda^2)(x,z_1)=c\lambda J_0(\lambda|x-z_1|)$, we need only bound

$$\frac{1}{t}\int_0^\infty e^{it\lambda^2}\chi(\lambda)\Bigg[\frac{\lambda J_0(\lambda|x-z_1|)}{\lambda^2}VG_0VP_eV[g_1^+(\lambda)+\lambda^2G_1+\widetilde{O}(\lambda^{\frac52})]VR_0^+(\lambda^2)\Bigg].$$

Here J_0 is the Bessel function of order zero. Since $P_eV1=0$, the term containing $g_1^+(\lambda)$ is immediately zero. Writing $R_0^\pm(\lambda^2)(x,y)=G_0+(1+\log^-|x-y|)\widetilde{O}_1(\lambda^{2-})$ for the lagging free resolvent, we then need only bound

$$\frac{1}{t}\int_0^\infty e^{it\lambda^2}\chi(\lambda)\lambda J_0(\lambda|x-z_1|)VG_0VP_eVG_1VG_0+(1+\log^-|z_4-y|)\widetilde{O}_1(\lambda^{1+})\,d\lambda$$

The first term can be bounded by $|t|^{-2}$ uniformly in x,y by Lemma 12 of [41]. The second term can be bounded by Lemma 8.7 using the observation $J_0(\lambda|x-z_1|)=\widetilde{O}_1(1)$.

For (69), we use that $\partial_\lambda[R_0^+(\lambda^2)-G_0]=c\lambda(\log\lambda+1)+2\lambda G_1+\langle x\rangle^{\frac12}\langle y\rangle^{\frac12}\widetilde{O}(\lambda^{\frac32})$. The first term can be safely ignored due to $P_eV1=0$ to consider

$$\frac{1}{t}\int_0^\infty e^{it\lambda^2}\chi(\lambda)\frac{(R_0^+-R_0^-)(\lambda^2)}{\lambda^2}VG_0VP_eV$$
$$[\lambda G_1+\langle z_3\rangle^{\frac12}\langle z_4\rangle^{\frac12}\widetilde{O}(\lambda^{\frac32})]V[G_0+(1+\log^-|z_4-y|)\widetilde{O}_1(\lambda^{2-})]\,d\lambda.$$

By writing

$$[R_0^+ - R_0^-](\lambda^2)(x,z_1) = \begin{cases} c_1\lambda^2 + \widetilde{O}_1(\lambda^4|x-z_1|^2) & \lambda|x-z_1| \ll 1 \\ \frac{\lambda}{|x-z_1|}e^{\pm i\lambda|x-z_1|}\widetilde{O}((1+\lambda|x-z_1|)^{-\frac{1}{2}}) & \lambda|x-z_1| \gtrsim 1 \end{cases},$$

one can employ an approach as in the Born series. First, for $\lambda|x-z_1| \ll 1$, the λ integral is of the form

$$\frac{1}{t}\int_0^\infty e^{it\lambda^2}\lambda\chi(\lambda)[1+\widetilde{O}_1(\lambda^{0+})+\widetilde{O}_1(\lambda^2|x-z_1|^2)]\chi(\lambda|x-z_1|)\,d\lambda.$$

The first two terms are easily seen to be bounded by $|t|^{-2}$ by integration by parts. For the last term, we note that the error term is supported on the set $\lambda \lesssim |x-z_1|^{-1}$ to integrate by parts and bound by

$$\frac{|x-z_1|^2}{|t|^2}\int_0^{|x-z_1|^{-1}} \lambda\,d\lambda \lesssim \frac{1}{|t|^2}.$$

For $\lambda|x-z_1| \gtrsim 1$, we note that the bound follows as in Lemmas 3.6 and 3.8 of [20] by using Lemma 6.5. The bound follows as long as we can write the resulting integral in the form

$$\int_0^\infty e^{it\lambda^2 \pm i\lambda|x-z_1|}a(\lambda)\,d\lambda \qquad \text{with} \qquad |a(\lambda)| \lesssim \frac{\lambda^{\frac{1}{2}}}{|x-z_1|^{\frac{1}{2}}}, \qquad |a'(\lambda)| \lesssim \frac{1}{\lambda^{\frac{1}{2}}|x-z_1|^{\frac{1}{2}}}.$$

Using $|x-z_1|^{-1} \lesssim \lambda$, these can be established from the bounds above.

For (70), we use Lemma 6.6 and $P_eV1 = 0$ to bound

$$\frac{1}{t}\int_0^\infty e^{it\lambda^2}\lambda\chi(\lambda)\frac{(R_0^+ - R_0^-)(\lambda^2)}{\lambda^2}VG_0VP_eV$$
$$[\lambda^2 G_1 + \langle z_3\rangle^{\frac{1}{2}}\langle z_4\rangle^{\frac{1}{2}}\widetilde{O}_2(\lambda^{\frac{5}{2}})]VR_2(\lambda^2)(z_4,y)\,d\lambda$$

where R_2 denotes the two-dimensional Schrödinger free resolvent. Since

$$\frac{(R_0^+ - R_0^-)(\lambda^2)}{\lambda^2} = \widetilde{O}_1(1),$$

we are left to bound

$$\frac{1}{t}\int_0^\infty e^{it\lambda^2}\lambda\chi(\lambda)\widetilde{O}_1(\lambda^3)R_2(\lambda^2)(z_4,y)\,d\lambda.$$

When $\lambda|z_4-y| \gtrsim 1$, the analysis of Lemma 12 in [41] yields the desired bound. When $\lambda|z_4-y| \ll 1$, we note that

$$|[J_0+iY_0](\lambda|z_4-y|)| \lesssim 1 + |\log\lambda| + \log^-|z_4-y|, \qquad |\partial_\lambda[J_0+iY_0](\lambda|z_4-y|)| \lesssim \frac{1}{\lambda},$$

which allows us to integrate by parts a second time without growth in x or y.

For the final term, (71), we again need to consider cases based on the size of $\lambda|x-z_1|$ and consider

$$\frac{1}{t}\int_0^\infty e^{it\lambda^2}\chi(\lambda)\Big[(R_0^+-R_0^-)(\lambda^2)VG_0VP_eVG_1VG_0\widetilde{O}_1(\lambda)+(1+\log^-|z_4-y|)\widetilde{O}_1(\lambda^{1+})\Big]\,d\lambda$$

The bound for the first term follows as in the bound for (69), while the bound for the second term follows from Lemma 8.7.

□

The following bound is proved similarly.

Lemma 6.8. *Under the assumptions of Theorem 1.3, we have the bound*

$$\sup_{x,y\in\mathbb{R}^4}\Bigg|\int_{\mathbb{R}^{16}}\int_0^\infty e^{it\lambda^2}\lambda\chi(\lambda)\frac{[R_0^+-R_0^-](\lambda^2)(x,z_1)}{\lambda^2}V(z_1)[R_0^+(\lambda^2)-G_0](z_1,z_2)$$
$$v(z_2)D_2(z_2,z_3)v(z_3)G_0(z_3,z_4)V(z_4)R_0^+(\lambda^2)(z_4,y)\,d\lambda\,dz_1\,dz_2\,dz_3\,dz_4\Bigg|\lesssim|t|^{-2}.$$

Then, we have

Proposition 6.9. *Under the assumptions of Theorem 1.3, we have the bound*

$$\sup_{x,y\in\mathbb{R}^4}\Bigg|\int_{\mathbb{R}^{16}}\int_0^\infty e^{it\lambda^2}\lambda\chi(\lambda)\frac{[R_0^+-R_0^-](\lambda^2)(x,z_1)}{\lambda^2}V(z_1)G_0(z_1,z_2)v(z_2)D_2(z_2,z_3)v(z_3)$$
$$G_0(z_3,z_4)V(z_4)R_0^+(\lambda^2)(z_4,y)\,d\lambda\,dz_1\,dz_2\,dz_3\,dz_4\Bigg|\lesssim|t|^{-2}.$$

To prove this proposition, we write

$$\frac{R_0^+-R_0^-}{\lambda^2}VG_0vD_2vG_0VR_0^+$$
$$=\frac{R_0^+-R_0^-}{\lambda^2}VG_0vD_2vG_0VG_0+\frac{R_0^+-R_0^-}{\lambda^2}VG_0vD_2vG_0V[R_0^+-G_0].$$

The first term is handled by the machinery set up in [20], specifically Lemma 4.5. The second term requires more care. We introduce some ideas and techniques inspired by the two-dimenstional treatment from [41, 13]. We first note that

$$\frac{R_0^+-R_0^-}{\lambda^2}VG_0vD_2vG_0V[R_0^+-G_0]=\frac{R_0^+-R_0^-}{\lambda^2}VP_eV[R_0^+-G_0]$$

and when $P_eV1=0$, we have

$$\int_{\mathbb{R}^8}\frac{[R_0^+-R_0^-](\lambda^2)(x,z_1)}{\lambda^2}VP_eV[R_0^+-G_0](z_4,y)\,dz_1\,dz_4 \tag{72}$$

$$= \int_{\mathbb{R}^8} \frac{\big\{[R_0^+ - R_0^-](\lambda^2)(x,z_1) - f(\lambda,x)\big\}}{\lambda^2} V P_e V \big\{[R_0^+ - G_0](z_4,y) - g(\lambda,y)\big\}\, dz_1\, dz_4$$

for any funtions f, g that are independent of z_1 and z_4 respectively. To utilize this cancellation, we must consider the different behavior for the resolvents for small and large arguments. We begin by noting (7) and (8) to see

$$[R_0^+ - R_0^-](\lambda^2)(x,y) = c\frac{\lambda}{|x-y|} J_1(\lambda|x-y|) = c\lambda^2 \frac{J_1(\lambda|x-y|)}{\lambda|x-y|} := \lambda^2 A(\lambda|x-y|) \tag{73}$$

In particular, we use (72) to subtract off $\lambda^2 A(\lambda(1+|x|))$. We define

$$G(\lambda,p,q) := A(\lambda p)\chi(\lambda p) - A(\lambda q)\chi(\lambda q). \tag{74}$$

To control the contribution of $R_0^+ - G_0$, recalling (8), we defined $G(\lambda,p,q)$ to control the contribution of J_1, we now turn to the contribution of Y_1. We have

$$[R_0^+(\lambda^2)(x,y) - G_0(x,y)] = \frac{i\lambda}{8\pi|x-y|} J_1(\lambda|x-y|) - \frac{\lambda^2}{8\pi}\left[\frac{Y_1(\lambda|x-y|)}{\lambda|x-y|} - \frac{2}{\pi\lambda^2|x-y|^2}\right].$$

This leads us to define the function

$$F(\lambda,p,q) = \chi(\lambda p)\left[\frac{Y_1(\lambda p)}{\lambda p} + \frac{2}{\pi\lambda^2 p^2}\right] - \chi(\lambda q)\left[\frac{Y_1(\lambda q)}{\lambda q} + \frac{2}{\pi\lambda^2 q^2}\right]. \tag{75}$$

In addition, we define $k(x,y) := 1 + \log^+|y| + \log^-|x-y|$.

Lemma 6.10. *Let $p := |x-y|$ and $q = 1+|x|$, then for $0 < \lambda < 2\lambda_1 \ll 1$,*

$$|\partial_\lambda^k G(\lambda,p,q)| \lesssim \lambda^{1-k}|p-q| \lesssim \lambda^{1-k}\langle y\rangle, \qquad k = 0,1,2,$$
$$|\partial_\lambda^k F(\lambda,p,q)| \lesssim \lambda^{-k} k(x,y), \qquad k = 0,1,2.$$

Proof. We begin with the bounds for $G(\lambda,p,q)$. We note that by the asymptotic expansion in (9), we have $g(\lambda p) = A(\lambda p)\chi(\lambda p) = 1 + (\lambda p)^2 + \widetilde{O}_4((\lambda p)^2)$ Thus, $g'(z) = O(1)$, and by the mean value theorem, we see

$$|G(\lambda,p,q)| = |g(\lambda p) - g(\lambda q)| \lesssim \lambda|p-q||g'(c)| \lesssim \lambda|p-q|.$$

For the derivatives, we note that for $k = 1,2$,

$$|\partial_\lambda^k G(\lambda,p,q)| = |p^k g^{(k)}(\lambda p) - q^k g^{(k)}(\lambda q)| = \frac{|(\lambda p)^k g^{(k)}(\lambda p) - (\lambda q)^k g^{(k)}(\lambda q)|}{\lambda^k}$$

Again, by (9), we have $|\partial_z[z^k g^{(k)}](z)| = O(1)$ and then the mean value theorem we have

$$|(\lambda p)^k g^{(k)}(\lambda p) - (\lambda q)^k g^{(k)}(\lambda q)| \lesssim \lambda\big|p-q\big|\big|\partial_z[z^k g^{(k)}(z)]\big| \lesssim \lambda|p-q|$$

We now turn to the bounds for $F(\lambda,p,q)$. The bounds follow by the expansion (10). Note that adding the term $2(\pi\lambda^2p^2)^{-1}$ exactly cancels out the singular term. We thus have

$$\chi(\lambda p)\left[\frac{Y_1(\lambda p)}{\lambda p}+\frac{2}{\pi\lambda^2p^2}\right]=\frac{1}{\pi}\log(\lambda p/2)+b_1\lambda p-\frac{1}{8\pi}(\lambda p)^2\log(\lambda p)+b_2(\lambda p)^2 + \widetilde{O}((\lambda p)^4\log(\lambda p)):=b(\lambda p).$$

From this expansion, we can see that

$$F(0+,p,q)=\log\left(\frac{|x-y|}{1+|x|}\right)+c\lesssim k(x,y). \tag{76}$$

Similar to the two dimensional case considered in [13], we have

$$|\partial_\lambda F(\lambda,p,q)|=|p\chi'(\lambda p)b(\lambda p)-q\chi'(\lambda q)b(\lambda q)+\partial_\lambda b(\lambda p)\chi(\lambda p)-\partial_\lambda b(\lambda q)\chi(\lambda q)|.$$

The bound of $\lambda^{-1}k(x,y)$ can be seen by using $\chi'(z)$ is supported on $z\approx 1$ and

$$|p\chi'(\lambda p)b(\lambda p)|\lesssim\frac{1}{\lambda}|z\chi'(z)(1+\log z)|\lesssim\frac{1}{\lambda}.$$

Further, one has $|b'(\lambda p)|\lesssim\frac{1}{\lambda}$. To bound F, we note (76) allows us to bound

$$\int_0^{2\lambda_1}|\partial_\lambda F(\lambda,p,q)|\,d\lambda\lesssim\int_0^{2\lambda_1}|p\chi'(\lambda p)b(\lambda p)|+|q\chi'(\lambda q)b(\lambda q)|\,d\lambda \tag{77}$$

$$+\int_0^{2\lambda_1}|[\partial_\lambda b(\lambda p)]\chi(\lambda p)-[\partial_\lambda b(\lambda q)]\chi(\lambda q)|\,d\lambda. \tag{78}$$

The first line is seen to be bounded by the previous discussion. For the second line, we note that

$$\partial_\lambda b(\lambda p)=\frac{1}{\pi\lambda}+b_1p+O(p(\lambda p)^{1-})=\frac{1}{\pi\lambda}+O\left(\frac{p^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}}\right).$$

Thus, we can bound (78) by

$$\int_0^{2\lambda_1}\frac{1}{\lambda}[\chi(\lambda p)-\chi(\lambda q)]+\chi(\lambda p)\frac{p^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}}+\chi(\lambda q)\frac{q^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}}\,d\lambda\lesssim k(x,y).$$

The first integrand is bounded since $\chi(\lambda p)-\chi(\lambda q)$ is supported on the set $[\frac{\lambda_1}{2p},2\frac{\lambda_1}{q}]$, while the remaining pieces follow by integration using that $\lambda\lesssim p^{-1}$ on the support of $\chi(\lambda p)$.

For the second derivative, we note that

$$|\partial_\lambda^2F(\lambda,p,q)|=|p^2\chi''(\lambda p)b(\lambda p)-q^2\chi''(\lambda q)b(\lambda q)+p[\partial_\lambda b(\lambda p)]\chi'(\lambda p)-q[\partial_\lambda b(\lambda q)]\chi'(\lambda q) + [\partial_\lambda^2b(\lambda p)]\chi(\lambda p)-[\partial_\lambda^2b(\lambda p)]\chi(\lambda q)|$$

By multiplying and dividing by λ, and using that $|z^2 b(z)|, |zb(z)| \lesssim 1$ for the terms with two derivatives on b, we note that since $\lambda p \leq 2\lambda_1 \ll 1$, we have

$$|\partial_\lambda^2 b(\lambda p)| \lesssim p^2 + p^2 |\log(\lambda p)| \lesssim \frac{1}{\lambda^2} + \frac{\lambda^2 p^2 |\log(\lambda p)|}{\lambda^2} \lesssim \frac{1}{\lambda^2}.$$

□

Lemma 6.11. *Under the assumptions of Theorem 1.3, we have the bound*

$$\sup_{x,y\in\mathbb{R}^4} \Bigg| \int_{\mathbb{R}^{16}} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \frac{R_0^+ - R_0^-}{\lambda^2}(\lambda^2)(x, z_1)\chi(\lambda|x - z_1|)$$
$$VP_eV[R_0^+(\lambda^2) - G_0](z_4, y)\chi(\lambda|z_4 - y|)\, d\lambda\, dz_1\, dz_2\, dz_3\, dz_4 \Bigg| \lesssim |t|^{-2}.$$

Proof. Considering the λ integral, and (72), we can replace $[R_0^+ - R_0^-](\lambda^2)(x, z_1)\chi(\lambda|x - z_1|)$ with $\lambda^2 G(\lambda, p_1, q_1)$ with $p_1 = |x - z_1|$ and $q_1 = 1 + |x|$. We can also replace $[R_0^+(\lambda^2) - G_0](z_4, y)\chi(\lambda|z_4 - y|)$ with $\lambda^2 G(\lambda, p_2, q_2) + \lambda^2 F(\lambda, p_2, q_2)$ with $p_2 = |z_4 - y|$ and $q_2 = 1 + |y|$. Thus, we are lead to bound the integral

$$\int_0^\infty e^{it\lambda^2} \lambda^3 \chi(\lambda) G(\lambda, p_1, q_1)[F(\lambda, p_2, q_2) + G(\lambda, p_2, q_2)]\, d\lambda$$

By the bounds in Lemma 6.10, we can express this integral as

$$[k(z_4, y) + \langle z_4\rangle]\langle z_1\rangle \int_0^\infty e^{it\lambda^2} \chi(\lambda) \widetilde{O}_2(\lambda^4)\, d\lambda \lesssim \frac{[k(z_4, y) + \langle z_4\rangle]\langle z_1\rangle}{|t|^2}.$$

The λ smallness allows us to integrate by parts twice without boundary terms to gain the $|t|^{-2}$ time decay.

We close the argument by bounding the spatial integrals by

$$\sup_{x,y\in\mathbb{R}^4} \|\langle\cdot\rangle V\|_{L^2} \||P_e|\|_{L^2\to L^2} \|\langle\cdot\rangle k(\cdot, y)V\|_{L^2} \lesssim 1.$$

□

For when the Bessel functions are supported on a large argument, we recall that asymptotics (12) and define the functions

$$\widetilde{G}^\pm(\lambda, p, q) = \widetilde{\chi}(\lambda p)\tilde{w}_\pm(\lambda p) - e^{\pm i\lambda(p-q)}\widetilde{\chi}(\lambda q)\tilde{w}_\pm(\lambda q), \qquad \tilde{w}_\pm(z) = \widetilde{O}(z^{-\frac{3}{2}}). \tag{79}$$

Here we have absorbed the $\lambda/|x - y|$ from (7) to the asymptotic expansion. This allows us to write

$$[R_0^+ - R_0^-](\lambda^2)(p)\widetilde{\chi}(\lambda p) - [R_0^+ - R_0^-](\lambda^2)(q)\widetilde{\chi}(\lambda q) = \lambda^2 \Big[e^{i\lambda p}\widetilde{G}^+(\lambda, p, q) + e^{-i\lambda p}\widetilde{G}^-(\lambda, p, q) \Big].$$

Lemma 6.12. *For any $0 \le \tau \le 1$, we have the bounds*

$$|\widetilde{G}^{\pm}(\lambda,p,q)| \lesssim (\lambda|p-q|)^{\tau}\Big(\frac{\widetilde{\chi}(\lambda p)}{|\lambda p|^{\frac{3-\tau}{2}}}+\frac{\widetilde{\chi}(\lambda q)}{|\lambda q|^{\frac{3-\tau}{2}}}\Big),$$

$$|\partial_\lambda \widetilde{G}^{\pm}(\lambda,p,q)| \lesssim (|p-q|+\lambda^{-1})\Big(\frac{\widetilde{\chi}(\lambda p)}{|\lambda p|^{\frac{3}{2}}}+\frac{\widetilde{\chi}(\lambda q)}{|\lambda q|^{\frac{3}{2}}}\Big),$$

$$|\partial_\lambda^2 \widetilde{G}^{\pm}(\lambda,p,q)| \lesssim (|p-q|+\lambda^{-1})^2\Big(\frac{\widetilde{\chi}(\lambda p)}{|\lambda p|^{\frac{3}{2}}}+\frac{\widetilde{\chi}(\lambda q)}{|\lambda q|^{\frac{3}{2}}}\Big).$$

Proof. We prove the case for $\widetilde{G}^{+}$ and ignore the superscript. The bound

$$|\widetilde{G}(\lambda,p,q)| \lesssim \Big(\frac{\widetilde{\chi}(\lambda p)}{|\lambda p|^{\frac{3}{2}}}+\frac{\widetilde{\chi}(\lambda q)}{|\lambda q|^{\frac{3}{2}}}\Big) \tag{80}$$

is clear. To gain λ smallness, we define the function $c(s) = \widetilde{\chi}(s)w(s)$ which satisfies $|c^{(k)}(s)| \lesssim \widetilde{\chi}(s)|s|^{-\frac{3}{2}-k}$ to write

$$\widetilde{G}(\lambda,p,q) = c(\lambda p) - c(\lambda q) + \big(1-e^{i\lambda(p-q)}\big)c(\lambda q). \tag{81}$$

The second summand is easily seen to be bounded by

$$\lambda|p-q||c(\lambda q)| \lesssim \lambda|p-q|\frac{\widetilde{\chi}(\lambda q)}{|\lambda q|^{\frac{3}{2}}}.$$

For the first term, without loss of generality we assume that $p > q$ to see

$$|c(\lambda p)-c(\lambda q)| = \Big|\int_{\lambda q}^{\lambda p} c'(s)\,ds\Big| \lesssim \int_{\lambda q}^{\lambda p} \widetilde{\chi}(s)|s|^{-\frac{5}{2}}\,ds.$$

In the case that $1 < \lambda q < \lambda p$, this integral is bounded by

$$\lambda|p-q|\frac{\widetilde{\chi}(\lambda q)}{|\lambda q|^{\frac{3}{2}}}.$$

In the case that $\lambda q < 1 < \lambda p$, we bound as

$$\int_{\lambda q}^{\lambda p} \widetilde{\chi}(s)|s|^{-\frac{5}{2}}\,ds \lesssim \widetilde{\chi}(\lambda p)\int_1^{\lambda p} s^{-\frac{5}{2}}\,ds \lesssim \widetilde{\chi}(\lambda p)\frac{(\lambda p)^{\frac{3}{2}}-1}{|\lambda p|^{\frac{3}{2}}} \lesssim \widetilde{\chi}(\lambda p)\frac{(\lambda p)^{\frac{3}{2}}-(\lambda q)^{\frac{3}{2}}}{|\lambda p|^{\frac{3}{2}}}$$
$$\lesssim \widetilde{\chi}(\lambda p)\frac{\lambda|p-q|}{|\lambda p|},$$

where the last bound follows by

$$|b^{\frac{3}{2}}-a^{\frac{3}{2}}| \lesssim \int_a^b \sqrt{s}\,ds \lesssim |b-a|\sqrt{b}.$$

Since $\lambda p, \lambda q \gtrsim 1$, the dominant term is the bound

$$\lambda|p-q|\Big(\frac{\widetilde{\chi}(\lambda p)}{|\lambda p|}+\frac{\widetilde{\chi}(\lambda q)}{|\lambda q|}\Big).$$

Interpolating between this and the trivial bound (80), one obtains the desired bound.

We now turn to derivatives, rather than rewriting $\widetilde{G}$ we use (79) directly to see

$$\begin{aligned}|\partial_\lambda \widetilde{G}(\lambda,p,q)| &= |pc'(\lambda p) - i(p-q)e^{i\lambda(p-q)}c(\lambda q) - e^{i\lambda(p-q)}qc'(\lambda q)| \\ &\lesssim \frac{c_1(\lambda p)+c_1(\lambda q)}{\lambda} + |p-q|c(\lambda q)\end{aligned}$$

Here $c_1(s) := sc'(s)$ satisfies the same bounds as $c(s)$. This suffices to prove the desired bound for the first derivative. For the second derivative, we again use (79) to see

$$\begin{aligned}|\partial_\lambda^2 \widetilde{G}(\lambda,p,q)| &\lesssim |p^2c''(\lambda p) + (p-q)^2c(\lambda q) + (p-q)qc'(\lambda q) + q^2c'(\lambda q)| \\ &\lesssim \frac{c_2(\lambda p)+c_2(\lambda q)}{\lambda^2} + |p-q|^2c(\lambda q) + \frac{|p-q|}{\lambda}c_1(\lambda q).\end{aligned}$$

With $c_2(s) := s^2c''(s)$ satisfies the same bounds as $c(s)$. This establishes the desired bound. □

Lemma 6.13. *Under the assumptions of Theorem 1.3, we have the bound*

$$\sup_{x,y\in\mathbb{R}^4}\Bigg|\int_{\mathbb{R}^{16}}\int_0^\infty e^{it\lambda^2}\lambda\chi(\lambda)\frac{R_0^+-R_0^-}{\lambda^2}(\lambda^2)(x,z_1)\widetilde{\chi}(\lambda|x-z_1|) \\ VP_eV[R_0^+(\lambda^2)-G_0](z_4,y)\chi(\lambda|z_4-y|)\,d\lambda\,dz_1\,dz_2\,dz_3\,dz_4\Bigg| \lesssim |t|^{-2}.$$

Proof. As in the proof of Lemma 6.11, we employ the functions F, G and $\widetilde{G}$ as needed. We will prove the bound for F in place of $[R_0^+(\lambda^2)-G_0]$ as this is larger in λ. We define $p_1 := \max(|x-z_1|, 1+|x|)$ and $p_2 := \min(|x-z_1|, 1+|x|)$. Accordingly, we see to bound

$$\int_0^\infty e^{it\lambda^2}\lambda^3\chi(\lambda)e^{\pm i\lambda p_2}\widetilde{G}(\lambda,p_1,p_2)F(\lambda,q_1,q_2)\,d\lambda.$$

The λ smallness of the integrand allows us to integrate by parts once without boundary terms to bound with

$$\frac{1}{t}\int_0^\infty e^{it\lambda^2}\partial_\lambda\Big[\lambda^2\chi(\lambda)e^{\pm i\lambda p_2}\widetilde{G}(\lambda,p_1,p_2)F(\lambda,q_1,q_2)\Big]\,d\lambda = \frac{1}{t}\int_0^\infty e^{it\phi_\pm(\lambda)}a(\lambda)\,d\lambda.$$

Here $\phi_\pm(\lambda) = \lambda^2 \pm \lambda p_2 t^{-1}$. In Lemma 3.8 of [13], using Lemma 6.5 it is proven that

$$\Big|\int_0^\infty e^{it\phi_\pm(\lambda)}a(\lambda)\,d\lambda\Big| \lesssim \frac{1}{|t|},$$

provided

$$|a(\lambda)| \lesssim \chi(\lambda)\lambda^{\frac12}\Big(\frac{\widetilde{\chi}(\lambda p_1)}{p_1^{\frac12}} + \frac{\widetilde{\chi}(\lambda p_2)}{p_2^{\frac12}}\Big), \qquad |a'(\lambda)| \lesssim \chi(\lambda)\lambda^{-\frac12}\Big(\frac{\widetilde{\chi}(\lambda p_1)}{p_1^{\frac12}} + \frac{\widetilde{\chi}(\lambda p_2)}{p_2^{\frac12}}\Big).$$

Since we have

$$a(\lambda) = e^{\mp i\lambda p_2}\partial_\lambda\Big[\lambda^2\chi(\lambda)e^{\pm i\lambda p_2}\widetilde{G}(\lambda,p_1,p_2)F(\lambda,q_1,q_2)\Big].$$

The bounds of Lemma 6.10 and 6.12 give us

$$a(\lambda) = p_2\widetilde{O}_1(\lambda^2)\widetilde{G}(\lambda,p_2,p_1) + \widetilde{O}_1(\lambda)\widetilde{G}(\lambda,p_1,p_2) + \widetilde{O}(\lambda^2)\partial_\lambda\widetilde{G}(\lambda,p_2,p_1)$$

satisfies the desired bounds.

□

Lemma 6.14. *Under the assumptions of Theorem 1.3, we have the bound*

$$\sup_{x,y\in\mathbb{R}^4}\Big|\int_{\mathbb{R}^{16}}\int_0^\infty e^{it\lambda^2}\lambda\chi(\lambda)\frac{R_0^+-R_0^-}{\lambda^2}(\lambda^2)(x,z_1)\widetilde{\chi}(\lambda|x-z_1|)$$
$$VP_eV[R_0^+(\lambda^2)-G_0](z_4,y)\widetilde{\chi}(\lambda|z_4-y|)\,d\lambda\,dz_1\,dz_2\,dz_3\,dz_4\Big| \lesssim |t|^{-2}.$$

Proof. We note that the contribution of $G_0(z_4,y)\widetilde{\chi}(\lambda|z_4-y|) = \widetilde{O}_2(\lambda^2)$. Thus, it's contribution may be bounded by

$$\int_0^\infty e^{it\lambda^2}\lambda\chi(\lambda)\frac{R_0^+-R_0^-}{\lambda^2}(\lambda^2)(x,z_1)\widetilde{\chi}(\lambda|x-z_1|)\widetilde{O}_2(\lambda^2)\,d\lambda.$$

This can be bounded by $|t|^{-2}$ as in the proof of Lemma 6.13 when the auxiliary function $F(\lambda,p,q)$ is used.

Assume that $t>0$ and recall (7) and (12)

$$\begin{aligned}R_0^\pm(\lambda^2)(x,y)\chi(\lambda|x-y|) &= \pm\frac{i}{4}\frac{\lambda}{2\pi|x-y|}H_1^\pm(\lambda|x-y|)\chi(\lambda|x-y|)\\ &= \frac{\lambda}{|x-y|}e^{\pm i\lambda|x-y|}\omega_\pm(\lambda|x-y|).\end{aligned}$$

While for the difference of resolvents we have both the '+' and '-' phases,

$$[R_0^+-R_0^-](\lambda^2)(x,y)\chi(\lambda|x-y|) = \frac{\lambda}{|x-y|}\Big[e^{i\lambda|x-y|}\omega_+(\lambda|x-y|)+e^{-i\lambda|x-y|}\omega_-(\lambda|x-y|)\Big].$$

To employ the auxiliary functions $\widetilde{G}$ we denote $p_1=\max(|x-z_1|,1+|x|)$, $p_2=\min(|x-z_1|,1+|x|)$, $q_1=\max(|y-z_4|,1+|y|)$ and $q_2=\min(|y-z_4|,1+|y|)$. Assuming $p_1,p_2,q_1,q_2>0$ we can exchange $[R_0^+-R_0^-](\lambda^2)(x,z_1)$ by sum of two terms $\lambda^2e^{\pm i\lambda p_2}\widetilde{G}^\pm(\lambda,p_1,p_2)$ and $[R_0^+(\lambda^2)-G_0](z_4,y)$ by $\lambda^2e^{i\lambda q_2}\widetilde{G}^+(\lambda,q_1,q_2)+\widetilde{O}_2(\lambda^2)$. As a result, we need to establish

$$\int_{\mathbb{R}^{16}}\int_0^\infty e^{it\phi_\pm(\lambda)}\lambda^3\chi(\lambda)\widetilde{G}^\pm(\lambda,p_1,p_2)VP_eV\widetilde{G}^+(\lambda,q_1,q_2)\,d\lambda\,d\vec{z}\lesssim t^{-2},$$

where

$$\phi_\pm(\lambda) = \lambda^2 + \lambda \frac{q_2 \pm p_2}{t}.$$

We consider first when the phase is ϕ_+. Note that the powers of λ allow us to integrate by parts once without boundary terms to obtain

$$\frac{1}{t}\int_0^\infty e^{it\lambda^2}\partial_\lambda\Big[\lambda^2\chi(\lambda)e^{i\lambda(q_2+p_2)}\widetilde{G}^\pm(\lambda,p_1,p_2)\widetilde{G}^+(\lambda,q_1,q_2)\Big]\,d\lambda = \frac{1}{t}\int_0^\infty e^{it\phi_\pm(\lambda)}a(\lambda)\,d\lambda.$$

It is now enough to show

$$\Big|\int_0^\infty e^{it\phi_\pm(\lambda)}a(\lambda)\,d\lambda\Big| \lesssim \frac{1}{t}. \tag{82}$$

To do that we need to determine the upper bounds for $|a(\lambda)|$ and $|a'(\lambda)|$. We have

$$\begin{aligned} |a(\lambda)| &\lesssim \partial_\lambda\big[\lambda^2\chi(\lambda)e^{i\lambda(p_2+q_2)}\widetilde{G}^+(\lambda,p_1,p_2)\widetilde{G}^+(\lambda,q_1,q_2)\big] \\ &\lesssim \chi(\lambda)\langle z_1\rangle\langle z_4\rangle\Big(\frac{\widetilde{\chi}(\lambda p_1)}{p_1^{\frac12}}+\frac{\widetilde{\chi}(\lambda q_2)}{q_2^{\frac12}}\Big)\Big(\frac{\widetilde{\chi}(\lambda q_1)}{q_1^{\frac12}}+\frac{\widetilde{\chi}(\lambda q_2)}{q_2^{\frac12}}\Big). \end{aligned} \tag{83}$$

If the derivative acts on one of the $\widetilde{G}(\lambda,p_1,p_2)$, using the bounds of Lemma 6.12 we have

$$\begin{aligned} &\chi(\lambda)\lambda^2(|p_1-p_2|+\lambda^{-1})\Big(\frac{\widetilde{\chi}(\lambda p_1)}{|\lambda p_1|^{\frac32}}+\frac{\widetilde{\chi}(\lambda q_2)}{|\lambda p_2|^{\frac32}}\Big)\Big(\frac{\widetilde{\chi}(\lambda q_1)}{|\lambda q_1|^{\frac32}}+\frac{\widetilde{\chi}(\lambda q_2)}{|\lambda q_2|^{\frac32}}\Big) \\ &\lesssim \chi(\lambda)\lambda\langle z_1\rangle\Big(\frac{\widetilde{\chi}(\lambda p_1)}{|\lambda p_1|^{\frac12}}+\frac{\widetilde{\chi}(\lambda q_2)}{|\lambda p_2|^{\frac12}}\Big)\Big(\frac{\widetilde{\chi}(\lambda q_1)}{|\lambda q_1|^{\frac12}}+\frac{\widetilde{\chi}(\lambda q_2)}{|\lambda q_2|^{\frac12}}\Big), \end{aligned}$$

where we used that $\lambda p_j, \lambda q_j \gtrsim 1$ and $|p_1-p_2| \lesssim \langle z_1\rangle$. The desired bound then follows by cancelling the λ in the numerator with $\lambda^{\frac12}$ in the denominator of each factor. The argument is identical for $\widetilde{G}(\lambda,q_1,q_2)$. Similar bounds hold if the derivative acts on λ^2 or the cut-off function $\chi(\lambda)$, noting that $|\chi'(\lambda)| \approx \lambda^{-1}$ on the range of λ under consideration.

On the other hand, if the derivative acts on $e^{i\lambda(p_2+q_2)}$ in (83) we use

$$\begin{aligned} \lambda^2 p_2\,\widetilde{G}(\lambda,p_1,p_2)\widetilde{G}(\lambda,q_1,q_2) &= \lambda(\lambda p_2)\Big(\frac{\widetilde{\chi}(\lambda p_1)}{|\lambda p_1|^{\frac32}}+\frac{\widetilde{\chi}(\lambda p_2)}{|\lambda p_2|^{\frac32}}\Big)\Big(\frac{\widetilde{\chi}(\lambda q_1)}{|\lambda q_1|^{\frac32}}+\frac{\widetilde{\chi}(\lambda q_2)}{|\lambda q_2|^{\frac32}}\Big) \\ &\lesssim \lambda\Big(\frac{\widetilde{\chi}(\lambda p_1)}{|\lambda p_1|^{\frac12}}+\frac{\widetilde{\chi}(\lambda p_2)}{|\lambda p_2|^{\frac12}}\Big)\Big(\frac{\widetilde{\chi}(\lambda q_1)}{|\lambda q_1|^{\frac12}}+\frac{\widetilde{\chi}(\lambda q_2)}{|\lambda q_2|^{\frac12}}\Big) \\ &\lesssim \Big(\frac{\widetilde{\chi}(\lambda p_1)}{p_1^{\frac12}}+\frac{\widetilde{\chi}(\lambda p_2)}{p_2^{\frac12}}\Big)\Big(\frac{\widetilde{\chi}(\lambda q_1)}{q_1^{\frac12}}+\frac{\widetilde{\chi}(\lambda q_2)}{q_2^{\frac12}}\Big). \end{aligned}$$

Here we used $p_2 \le p_1$. An identical argument holds for q_2. Similarly, one can obtain the first derivative of $a(\lambda)$ as

$$|a'(\lambda)| \lesssim \langle z_1\rangle^2\langle z_4\rangle^2\chi(\lambda)\lambda^{-1}\Big(\frac{\widetilde{\chi}(\lambda p_1)}{p_1^{\frac12}}+\frac{\widetilde{\chi}(\lambda q_2)}{q_2^{\frac12}}\Big)\Big(\frac{\widetilde{\chi}(\lambda q_1)}{q_1^{\frac12}}+\frac{\widetilde{\chi}(\lambda q_2)}{q_2^{\frac12}}\Big). \tag{84}$$

This can be seen by noting that the bound in Lemma 6.12 show that if $0 < \lambda \ll 1$,

$$|\partial_\lambda^k \widetilde{G}^{\pm}(\lambda, p_1, p_2)| \lesssim \left(\frac{\langle p_1 - p_2\rangle}{\lambda}\right)^k \left(\frac{\widetilde{\chi}(\lambda p_1)}{|\lambda p_1|^{\frac{3}{2}}} + \frac{\widetilde{\chi}(\lambda p_2)}{|\lambda p_2|^{\frac{3}{2}}}\right), \qquad k = 0, 1, 2.$$

The desired bound, (82), follows as in Lemma 3.10 of [13].

We now turn to the case of $\phi_-(\lambda)$ which has opposing phases. We wish to reduce to previously considered cases as much as possible. We note that if $\frac{1}{2}q_2 > p_2$, we have $q_2 - p_2 \approx q_2$, this allows us effectively reduce to an integral of the form

$$\int_0^\infty e^{it\lambda^2 + i\lambda a_2} \lambda^3 \chi(\lambda) \widetilde{G}^-(\lambda, p_1, p_2) \widetilde{G}^+(\lambda, q_1, q_2)\, d\lambda.$$

This can be controlled as in the proof of Lemma 6.13 to get the desired bound since the constant (in λ) a_2 satisfies the same bounds as q_2. The bounds on $\widetilde{G}^-(\lambda, p_1, p_2)$ are bounded by those used in $F(\lambda, p_1, p_2)$ in this proof.

In the case that $q_2 < \frac{1}{2}p_2$, we have $q_2 - p_2 \approx -p_2$, this allows us effectively reduce to an integral of the form

$$\int_0^\infty e^{it\lambda^2 - i\lambda b_2} \lambda^3 \chi(\lambda) \widetilde{G}^-(\lambda, p_1, p_2) \widetilde{G}^+(\lambda, q_1, q_2)\, d\lambda.$$

This also can be controlled as in the proof of Lemma 6.13 to get the desired bound using that the constant b_2 satisfies the same bounds as p_2.

We now consider the final case in which $q_2 \approx p_2$, and we cannot effectively reduce to the previous cases. In this case, we need to bound an integral of the form

$$\int_0^\infty e^{it\lambda^2 + i\lambda(q_2 - p_2)} \lambda^3 \chi(\lambda) \widetilde{G}^-(\lambda, p_1, p_2) \widetilde{G}^+(\lambda, q_1, q_2)\, d\lambda.$$

In the previous cases, we do not integrate by parts twice to avoid spatial weights. In this case, we use the λ smallness to integrate by parts twice. We need to bound

$$\int_0^\infty e^{it\lambda^2} e^{i\lambda(q_2 - p_2)} b(\lambda)\, d\lambda. \tag{85}$$

Using the bounds in Lemma 6.12, $b(\lambda)$ is a function that is supported on $[0, 2\lambda_1)$ that satisfies

$$|\partial_\lambda^k b(\lambda)| \lesssim \lambda^{3-k} \left(\frac{\widetilde{\chi}(\lambda p_1)}{|\lambda p_1|^{\frac{3}{2}}} + \frac{\widetilde{\chi}(\lambda p_2)}{|\lambda p_2|^{\frac{3}{2}}}\right)\left(\frac{\widetilde{\chi}(\lambda q_1)}{|\lambda q_1|^{\frac{3}{2}}} + \frac{\widetilde{\chi}(\lambda q_2)}{|\lambda q_2|^{\frac{3}{2}}}\right), \qquad k = 0, 1, 2. \tag{86}$$

Thus, upon integrating by parts twice, we have

$$|(85)| \lesssim \frac{1}{t^2} + \frac{1}{t^2} \int_0^\infty \left| \left(\frac{\partial}{\partial\lambda} \frac{1}{\lambda}\right)^2 \left(e^{i\lambda(q_2 - p_2)} b(\lambda)\right) \right| d\lambda.$$

The boundary term occurs if the first derivative when integrating by parts acts on $b(\lambda)$, then to set up the second integration by parts there is an effective loss of three powers of λ.

We then note that (86) gives us that $|\lambda^{-2}b'(\lambda)| \lesssim 1$. We now move to control the integral. By absorbing the division by λ into $b(\lambda)$, we have to bound

$$\int_0^\infty \sum_{k=0}^{2} |p_2 - q_2|^k \lambda^{k-1} \chi(\lambda) \left(\frac{\widetilde{\chi}(\lambda p_1)}{|\lambda p_1|^{\frac{3}{2}}} + \frac{\widetilde{\chi}(\lambda p_2)}{|\lambda p_2|^{\frac{3}{2}}} \right) \left(\frac{\widetilde{\chi}(\lambda q_1)}{|\lambda q_1|^{\frac{3}{2}}} + \frac{\widetilde{\chi}(\lambda q_2)}{|\lambda q_2|^{\frac{3}{2}}} \right) d\lambda. \tag{87}$$

When $k = 0$, the integral is seen to be bounded by

$$\int_0^\infty \lambda^{-1} \chi(\lambda) \left(\frac{\widetilde{\chi}(\lambda p_1)}{|\lambda p_1|^{\frac{3}{2}}} + \frac{\widetilde{\chi}(\lambda p_2)}{|\lambda p_2|^{\frac{3}{2}}} \right) \left(\frac{\widetilde{\chi}(\lambda q_1)}{|\lambda q_1|^{\frac{3}{2}}} + \frac{\widetilde{\chi}(\lambda q_2)}{|\lambda q_2|^{\frac{3}{2}}} \right) d\lambda \lesssim \int_{\mathbb{R}} \frac{\widetilde{\chi}(\lambda p_2)}{\lambda^{\frac{5}{2}} p_2^{\frac{3}{2}}} \, d\lambda \lesssim 1.$$

Here we used the crude bound of a constant for all the terms involving q_1, q_2, and using that $p_2 \leq p_1$. If $k = 1$, we recall that $p_2 \leq p_1$, $q_2 \leq q_1$ and $p_2 \approx q_2$. We use the bound $|p_2 - q_2| \lesssim p_2$, and bound the terms containing q_1, q_2 with a constant, to see

$$\begin{aligned} &\int_0^\infty |p_2 - q_2| \chi(\lambda) \left(\frac{\widetilde{\chi}(\lambda p_1)}{|\lambda p_1|^{\frac{3}{2}}} + \frac{\widetilde{\chi}(\lambda p_2)}{|\lambda p_2|^{\frac{3}{2}}} \right) \left(\frac{\widetilde{\chi}(\lambda q_1)}{|\lambda q_1|^{\frac{3}{2}}} + \frac{\widetilde{\chi}(\lambda q_2)}{|\lambda q_2|^{\frac{3}{2}}} \right) d\lambda \\ &\qquad \lesssim \int_0^\infty \frac{p_2 \widetilde{\chi}(\lambda p_1)}{|\lambda p_1|^{\frac{3}{2}}} + \frac{\widetilde{\chi}(\lambda p_2)}{\lambda^{\frac{3}{2}} |p_2|^{\frac{1}{2}}} \, d\lambda \lesssim \int_0^\infty \frac{\widetilde{\chi}(\lambda p_2)}{\lambda^{\frac{3}{2}} |p_2|^{\frac{1}{2}}} \, d\lambda \lesssim 1. \end{aligned}$$

Finally, if $k = 2$, we seek to bound

$$\int_0^\infty |p_2 - q_2|^2 \lambda \chi(\lambda) \left(\frac{\widetilde{\chi}(\lambda p_1)}{|\lambda p_1|^{\frac{3}{2}}} + \frac{\widetilde{\chi}(\lambda p_2)}{|\lambda p_2|^{\frac{3}{2}}} \right) \left(\frac{\widetilde{\chi}(\lambda q_1)}{|\lambda q_1|^{\frac{3}{2}}} + \frac{\widetilde{\chi}(\lambda q_2)}{|\lambda q_2|^{\frac{3}{2}}} \right) d\lambda.$$

This time, we may not ignore any terms with a crude bound of a constant. Instead, we use the dominating terms and $p_2 \approx q_2$ to bound with

$$\int_0^\infty p_2 q_2 \lambda \left(\frac{\widetilde{\chi}(\lambda p_2)}{|\lambda p_2|^{\frac{3}{2}}} \frac{\widetilde{\chi}(\lambda q_2)}{|\lambda q_2|^{\frac{3}{2}}} \right) d\lambda \lesssim \int_0^\infty \frac{1}{\lambda^2} \left(\frac{\widetilde{\chi}(\lambda p_2)}{p_2^{\frac{1}{2}}} \frac{\widetilde{\chi}(\lambda q_2)}{q_2^{\frac{1}{2}}} \right) d\lambda \lesssim \int_{\mathbb{R}} \frac{\widetilde{\chi}(\lambda p_2)}{\lambda^2 p_2} \, d\lambda \lesssim 1.$$

□

It is now a simple matter to prove

Lemma 6.15. *Under the assumptions of Theorem 1.3, we have the bound*

$$\begin{aligned} \sup_{x,y \in \mathbb{R}^4} \Bigg| \int_{\mathbb{R}^{16}} \int_0^\infty e^{it\lambda^2} \lambda \chi(\lambda) \frac{R_0^+ - R_0^-}{\lambda^2} (\lambda^2)(x, z_1) \chi(\lambda |x - z_1|) \\ V P_e V [R_0^+(\lambda^2) - G_0](z_4, y) \widetilde{\chi}(\lambda |z_4 - y|) \, d\lambda \, dz_1 \, dz_2 \, dz_3 \, dz_4 \Bigg| \lesssim |t|^{-2}. \end{aligned}$$

We can now prove Proposition 6.9.

Proof of Proposition 6.9. The bound follows from the bounds in Lemmas 6.11, 6.15, 6.13 and 6.14. The ample decay of V more than suffices to ensure that the spatial integrals are bounded as in Theorem 3.1.

□

We now proceed to the proof of Theorem 1.3. We note that the assumption $P_eV1 = 0$ is already satisfied when there is a resonance of the second kind at zero.

Proof of Theorem 1.3. As in the previous sections, we need to understand the contribution of (67) to the Stone formula. Thanks to the algebraic fact (48) we have three cases to consider; the case when the '+/-' difference acts on $M^{\pm}(\lambda)^{-1}$, the case when the difference acts on an inner resolvent, and the case when the difference acts on the leading/lagging resolvent.

We first consider when the '+/-' difference acts on $M^{\pm}(\lambda)^{-1}$. By Proposition 6.4, we have

$$M^+(\lambda)^{-1} - M^-(\lambda)^{-1} = \lambda^2 M_6 + \widetilde{O}_2(\lambda^{2+})$$

for an absolutely bounded operator M_6. The λ smallness this brings to

$$R_0^+ V R_0^+ v M^+(\lambda)^{-1} v R_0^+ V R_0^+ - R_0^- V R_0^- v M^-(\lambda)^{-1} v R_0^- V R_0^-$$

allows us to consider this under the framework of the Born Series. The analysis in Lemma 3.8 of [20] can be applied to show that the contribution of this difference to the Stone formula can be bounded by $|t|^{-2}$ uniformly in x, y.

When the '+/-' difference acts on an inner resolvent, with some work, we can again reduce this to integrals bounded in the analysis of the Born series. Consider, the following as a representative term

$$R_0^-(\lambda^2) V [R_0^+ - R_0^-](\lambda^2) v M^+(\lambda)^{-1} v R_0^+(\lambda) V R_0^+(\lambda^2) \tag{88}$$

Here we note that we can express $[R_0^+(\lambda^2) - R_0^-(\lambda^2)] = c\lambda^2 + \widetilde{O}_2(\lambda^4 |z_j - z_{j+1}|^2)$, by using the small argument expansion of the Bessel function (9), while if $\lambda |z_j - z_{j+1}| \gtrsim 1$, one employs (35) with $\alpha = \frac{7}{2} - k$ for $k = 0, 1, 2$. Then, writing $M^+(\lambda)^{-1} = -D_2/\lambda^2 + \widetilde{O}(1)$, we have

$$\begin{aligned}
[R_0^+ - R_0^-](\lambda^2) v M^+(\lambda)^{-1} &= (c\lambda^2 + \widetilde{O}_2(\lambda^4 |z_j - z_{j+1}|^2)) v \big[-\frac{D_2}{\lambda^2} + \widetilde{O}(1) \big] \\
&= -c 1 v D_2 + \langle z_j \rangle^2 \langle z_{j+1} \rangle^2 \widetilde{O}_2(\lambda^2) = \langle z_j \rangle^2 \langle z_{j+1} \rangle^2 \widetilde{O}_2(\lambda^2).
\end{aligned}$$

The last equality holds due to the identity $vD_2 = vS_1 D_2 S_1 = VP_e w$ we have $1vD_2 = 1VP_e w = 0$. The growth in the inner spatial variables z_j, z_{j+1} can be absorbed by the decay of the potential functions V and v respectively. This again allows us to use the analysis of the Born series in Lemma 3.8 of [20] to bound its contribution to the Stone formula by $|t|^{-2}$ uniformly in x, y.

Finally, we consider when the '+/-' difference acts on a leading free resolvent. By symmetry, the calculations are identical if the difference acts on the lagging free resolvent. We first note that by Proposition 6.4, we have $M^+(\lambda)^{-1} = -D_2/\lambda^2 + \widetilde{O}_2(1)$. When the error term $\widetilde{O}_2(1)$ is substituted into

$$[R_0^+ - R_0^-]VR_0^+ vM^+(\lambda)^{-1} vR_0^+ VR_0^+ \tag{89}$$

the desired bound again falls under the framework of the analysis of the Born series, this time using Lemma 3.6 of [20]. We now consider only the contribution of $-D_2/\lambda^2$.

We need only consider the contribution of

$$\frac{R_0^+ - R_0^-}{\lambda^2} VR_0^+ vD_2 vR_0^+ VR_0^+$$

$$= \frac{R_0^+ - R_0^-}{\lambda^2} V[R_0^+ - G_0]vD_2 v[R_0^+ - G_0]VR_0^+ \tag{90}$$

$$+ \frac{R_0^+ - R_0^-}{\lambda^2} V[R_0^+ - G_0]vD_2 vG_0 VR_0^+ \tag{91}$$

$$+ \frac{R_0^+ - R_0^-}{\lambda^2} VG_0 vD_2 v[R_0^+ - G_0]VR_0^+ \tag{92}$$

$$+ \frac{R_0^+ - R_0^-}{\lambda^2} VG_0 vD_2 vG_0 VR_0^+ \tag{93}$$

The smallness of $R_0^+ - G_0$ occuring at 'inner resolvents' in (90) allows us to bound this term as in the Born series. The remaining terms are bounded by Lemmas 6.7, 6.8 and Proposition 6.9.

□

7. The Klein-Gordon Equation with a Potential

In this section we prove the bounds in Theorem 1.4 that control the solution of a perturbed Klein-Gordon equation. We note that much of our analysis is greatly simplified due to the expansions and analysis performed in previous sections in the context of the Schrödinger operator. Much of the analysis of the oscillatory integrals proceeds similarly with the multipliers $\frac{\sin(t\sqrt{\lambda^2+m^2})}{\sqrt{\lambda^2+m^2}}\,\lambda$ and $\cos(t\sqrt{\lambda^2+m^2})\,\lambda$ in place of the multiplier $e^{it\lambda^2}\lambda$.

We employ the following consequence of the classical Van der Corput lemma, see for example [42].

Lemma 7.1. *If $\phi : [0,1] \to \mathbb{R}$ obeys the bound $|\phi^{(2)}(\lambda)| \geq ct > 0$ for all $\lambda \in [0,1]$, and if $\psi : [0,1] \to \mathbb{C}$ is such that $\psi' \in L^1([0,1])$, then*

$$\Big| \int_0^1 e^{i\phi(\lambda)}\psi(\lambda)\, d\lambda \Big| \lesssim t^{-\frac{1}{2}} \Big\{ |\psi(1)| + \int_0^1 |\psi'(\lambda)|\, d\lambda \Big\}.$$

In particular, we employ this lemma with the phase $\phi(\lambda) = \pm t\sqrt{\lambda^2+m^2} + \lambda\nu$ for some $\nu \in \mathbb{R}$. In this case, we have

$$|\phi''(\lambda)| = t\frac{m^2}{(m^2+\lambda^2)^{\frac{3}{2}}} \geq \frac{t}{m}, \qquad m > 0.$$

Accordingly, this lemma is quite useful when analyzing the Klein-Gordon with non-zero mass m^2, but an alternative approach is required for the wave equation, when $m^2 = 0$.

Recalling (15), we need separate analysis for the contribution of the finite Born series, (16), and the singular portion of the expansion which is sensitive to the existence of zero-energy resonances and eigenvalues, (17).

To control the first term in the Born series, we note

Lemma 7.2. *One has the bound*

$$\left|\int_0^\infty \frac{\sin(t\sqrt{\lambda^2+m^2})}{\sqrt{\lambda^2+m^2}}\lambda\chi(\lambda)[R_0^+ - R_0^-](\lambda^2)(x,y)\,d\lambda\right| \lesssim |t|^{-\frac{3}{2}},$$
$$\left|\int_0^\infty \cos(t\sqrt{\lambda^2+m^2})\lambda\chi(\lambda)[R_0^+ - R_0^-](\lambda^2)(x,y)\,d\lambda\right| \lesssim |t|^{-\frac{3}{2}},$$

uniformly in x, y.

This reflects the natural dispersive decay rate of $|t|^{-\frac{3}{2}}$ for a wave-like equation in $\mathbb{R}^4$.

Proof. The bound is established by integrating by parts once, then using Lemma 7.1. We need to consider two cases, based on the size of $\lambda|x-y|$. We consider the first integral by writing $\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$, the second integral follows similarly using that $\sqrt{\lambda^2+m^2} = \widetilde{O}(1)$ on the support of $\chi(\lambda)$.

In the first case, if $\lambda|x-y| \ll 1$, we have from (46),

$$R_0^+ - R_0^-(\lambda^2)(x,y) = c\lambda^2 + \widetilde{O}_2(\lambda^2(\lambda|x-y|)^\epsilon), \qquad 0 \leq \epsilon < 2. \tag{94}$$

Using $\partial_\lambda e^{\pm it\sqrt{\lambda^2+m^2}} = e^{\pm it\sqrt{\lambda^2+m^2}}\frac{\pm it\lambda}{\sqrt{\lambda^2+m^2}}$, we need to control

$$\left|\int_0^\infty \frac{e^{\pm it\sqrt{\lambda^2+m^2}}}{\sqrt{\lambda^2+m^2}}\lambda\chi(\lambda)[R_0^+ - R_0^-](\lambda^2)(x,y)\chi(\lambda|x-y|)\,d\lambda\right|$$
$$= \frac{1}{t}\left|\int_0^\infty e^{\pm it\sqrt{\lambda^2+m^2}}\partial_\lambda\big(\chi(\lambda)[R_0^+ - R_0^-](\lambda^2)(x,y)\big)\,d\lambda\right|.$$

From the expansion (94), there are no boundary terms when integrating by parts. Using (94), we note that differentiation in λ is comparable to division by λ we can apply Lemma 7.1 with $\psi(\lambda) = \chi(\lambda)[2c\lambda + \widetilde{O}_1(\lambda(\lambda|x-y|)^\epsilon)] + \chi'(\lambda)[c\lambda^2 + \widetilde{O}_2(\lambda^2(\lambda|x-y|)^\epsilon)]$, then $\psi(\lambda) = \chi(\lambda)\widetilde{O}_1(\lambda)$, and $\psi(1) = 0$, $\psi' \in L^1([0,1])$.

When $\lambda|x-y| \gtrsim 1$, we do not employ any of the cancellation between R_0^+ and R_0^-, instead we use the representation (12) to bound

$$\Big| \int_0^\infty \frac{e^{\pm it\sqrt{\lambda^2+m^2}}}{\sqrt{\lambda^2+m^2}} \lambda\chi(\lambda) \frac{\lambda e^{i\lambda|x-y|}\omega_+(\lambda|x-y|)}{|x-y|}\, d\lambda \Big|$$

Here we consider the '+' phase in (12), the '-' phase is handled identically. As before, we can integrate by parts once without boundary terms and bound

$$\frac{1}{t}\int_0^\infty e^{\pm it\sqrt{\lambda^2+m^2}+i\lambda|x-y|} a(\lambda)\, d\lambda, \qquad a(\lambda) = e^{-i\lambda|x-y|}\partial_\lambda\Big(\chi(\lambda)\frac{\lambda e^{i\lambda|x-y|}\omega_+(\lambda|x-y|)}{|x-y|}\Big)$$

One can see that $|a(\lambda)| \lesssim \lambda^{\frac12}|x-y|^{-\frac12}$ and $|a'(\lambda)| \lesssim (\lambda|x-y|)^{-\frac12}$. Thus, using Lemma 7.1, we can bound with

$$\frac{1}{|t|^{\frac32}}\int_0^\infty |a'(\lambda)|\, d\lambda \lesssim \frac{1}{|t|^{\frac32}}\int_0^1 \frac{\lambda^{-\frac12}}{|x-y|^{\frac12}}\, d\lambda \lesssim \frac{1}{|t|^{\frac32}|x-y|^{\frac12}}$$

Noting that, on the support of $\chi(\lambda)$ if $\lambda|x-y| \gtrsim 1$, then $|x-y| \gtrsim 1$.

□

For the remaining terms of the Born series, we have the following bound.

Proposition 7.3. *If $|V(x)| \lesssim \langle x\rangle^{-\frac52-}$, then for any $\ell \in \mathbb{N}\cup\{0\}$, we have*

$$\sup_{x,y\in\mathbb{R}^4}\Big| \int_0^\infty \Big(\cos(t\sqrt{\lambda^2+m^2}) + \frac{\sin(t\sqrt{\lambda^2+m^2})}{\sqrt{\lambda^2+m^2}}\Big) \lambda\chi(\lambda)\Big[\sum_{k=0}^{2\ell+1}(-1)^k\big\{R_0^+(VR_0^+)^k - R_0^-(VR_0^-)^k\big\}\Big](\lambda^2)(x,y)\, d\lambda\Big| \lesssim |t|^{-\frac32}.$$

Proof. We note that the case of $k=0$ was handled separately in Lemma 7.2. To handle $k\geq 1$, we recall (8) and the asymptotic expansions, (10), (11) and (12) to write

$$R_0^\pm(\lambda^2)(x,y) = \begin{cases} \frac{c}{|x-y|^2} + \widetilde{O}_2(\lambda^{\frac32}|x-y|^{-\frac12}) & \lambda|x-y| \ll 1 \\ \frac{\lambda}{|x-y|}(e^{i\lambda|x-y|}\omega_+(\lambda|x-y|) + e^{-i\lambda|x-y|}\omega_-(\lambda|x-y|)) & \lambda|x-y| \gtrsim 1 \end{cases}$$

When we encounter $[R_0^+ - R_0^-](\lambda^2)(x,y)$, there is cancellation when $\lambda|x-y| \ll 1$, but no useful cancellation can be found when $\lambda|x-y| \gtrsim 1$. Thus when $\lambda|x-y| \gtrsim 1$, the same asymptotics apply, with slightly different functions $\omega_\pm$ that satisfy the same bounds.

Recall that we have

$$\begin{aligned}[R_0^+ - R_0^-](\lambda^2)(x,y) &= c\lambda^2 + \widetilde{O}_2(\lambda^2(\lambda|x-y|)^\epsilon) \qquad 0\leq\epsilon<2, \qquad \lambda|x-y|\ll 1 \\ &= \widetilde{O}_2(\lambda^{\frac32}|x-y|^{-\frac12}) \end{aligned} \tag{95}$$

Let $J \cup J^* = \{1, 2 \ldots, k+1\}$ be a partition. Omitting the potentials for the moment, we need to control the contribution of

$$(R_0^{\pm})^{k+1} = \prod_{j \in J} \frac{\lambda}{r_j} \left(e^{i\lambda r_j} \omega_+(\lambda r_j) + e^{-i\lambda r_j} \omega_-(\lambda r_j) \right) \prod_{i \in J^*} \left(\frac{1}{r_i^2} + \widetilde{O}_2(\lambda^{\frac{3}{2}} r_i^{-\frac{1}{2}}) \right)$$

where $r_j = |z_{j-1} - z_j|$ are the differences of the inner spatial variables with $z_0 = x$ and $z_{k+1} = y$. We note that for $j \in J$ we have the support condition that $\lambda r_j \gtrsim 1$ and for $i \in J^*$, we have $\lambda r_i \ll 1$. As the different phases for the large λr_j contributions do not matter for our analysis, we will abuse notation slightly and write $e^{\pm i \prod_J \lambda r_j}$ to indicate a sum over all possible combinations of positive and negative phases in the product.

We note that we need only use the difference of the '+' and '-' phases when $J = \{\emptyset\}$. For the remaining cases, we can estimate each term separately without relying on any cancellation. We first consider when $J \neq \{\emptyset\}$. We wish to bound

$$\int_0^\infty e^{\pm it\sqrt{\lambda^2+m^2}} \left(\lambda + \frac{\lambda}{\sqrt{\lambda^2+m^2}} \right) \chi(\lambda) \prod_{j \in J} \frac{\lambda}{r_j} \left(e^{\pm i\lambda r_j} \omega_\pm(\lambda r_j) \right) \prod_{i \in J^*} \left(\frac{1}{r_i^2} + \widetilde{O}_2(\lambda^{\frac{3}{2}} r_i^{-\frac{1}{2}}) \right) d\lambda.$$

Since $J \neq \{\emptyset\}$, we can integrate by parts once without boundary terms to bound

$$\frac{1}{t} \int_0^\infty e^{\pm it\sqrt{\lambda^2+m^2}} \partial_\lambda \Big\{ \big(1 + \sqrt{\lambda^2+m^2}\big) \chi(\lambda) \prod_{j \in J} \frac{\lambda}{r_j} \left(e^{\pm i\lambda r_j} \omega_\pm(\lambda r_j) \right) \prod_{i \in J^*} \left(\frac{1}{r_i^2} + \widetilde{O}_2(\lambda^{\frac{3}{2}} r_i^{-\frac{1}{2}}) \right) \Big\} d\lambda.$$

Thus, we wish to control integrals of the form

$$\frac{1}{t} \int_0^\infty e^{\pm it\sqrt{\lambda^2+m^2} \pm i\lambda \prod_J r_j} \psi(\lambda)\, d\lambda. \tag{96}$$

To control ψ, we note the following bounds:

$$\chi(\lambda), \sqrt{\lambda^2+m^2} = \widetilde{O}(1), \qquad \left| \frac{e^{\pm i\lambda r_j} \lambda}{r_j} \omega_\pm(\lambda r_j) \right| \lesssim \frac{\lambda^{\frac{1}{2}}}{r_j^{\frac{3}{2}}}, \qquad \left| \frac{1}{r_i^2} + \widetilde{O}_2(\lambda^{\frac{3}{2}} r_i^{-\frac{1}{2}}) \right| \lesssim \frac{1}{r_i^2},$$

$$\left| \partial_\lambda \left(\frac{e^{\pm i\lambda r_j} \lambda}{r_j} \omega_\pm(\lambda r_j) \right) \right| \lesssim \frac{\lambda^{\frac{1}{2}}}{r_j^{\frac{1}{2}}}, \qquad \left| \partial_\lambda \left(\frac{1}{r_i^2} + \widetilde{O}_2(\lambda^{\frac{3}{2}} r_i^{-\frac{1}{2}}) \right) \right| = \widetilde{O}_1 \left(\frac{\lambda^{\frac{1}{2}}}{r_i^{\frac{1}{2}}} \right).$$

Where we used that $O_2(\lambda^{\frac{3}{2}} r_i^{-\frac{1}{2}}) \lesssim r_i^{-2}$ when $\lambda r_i \ll 1$. So that

$$|\psi(\lambda)| \lesssim \sum_{\ell=1}^{k+1} \frac{\lambda^{\frac{1}{2}}}{r_\ell^{\frac{1}{2}}} \Big[\prod_{\ell \neq j \in J} \frac{\lambda^{\frac{1}{2}}}{r_j^{\frac{3}{2}}} \prod_{i \in J^*} \frac{1}{r_i^2} + \prod_{j \in J} \frac{\lambda^{\frac{1}{2}}}{r_j^{\frac{3}{2}}} \prod_{\ell \neq i \in J^*} \frac{1}{r_i^2} \Big]. \tag{97}$$

Since we have factored out the high energy phases $e^{\pm i\lambda r_j}$, differentiation of ψ is comparable to division by λ, so

$$|\partial_\lambda \psi(\lambda)| \lesssim \sum_{\ell=1}^{k+1} \frac{\lambda^{-\frac{1}{2}}}{r_\ell^{\frac{1}{2}}} \Big[\prod_{\ell\neq j\in J} \frac{\lambda^{\frac{1}{2}}}{r_j^{\frac{3}{2}}} \prod_{i\in J^*} \frac{1}{r_i^2} + \prod_{j\in J} \frac{\lambda^{\frac{1}{2}}}{r_j^{\frac{3}{2}}} \prod_{\ell\neq i\in J^*} \frac{1}{r_i^2} \Big]. \tag{98}$$

Lemma 7.1 shows that

$$(96) \lesssim \frac{1}{|t|^{\frac{3}{2}}} \sum_{\ell=1}^{k+1} \frac{1}{r_\ell^{\frac{1}{2}}} \Big[\prod_{\ell\neq j\in J} \frac{1}{r_j^{\frac{3}{2}}} \prod_{i\in J^*} \frac{1}{r_i^2} + \prod_{j\in J} \frac{1}{r_j^{\frac{3}{2}}} \prod_{\ell\neq i\in J^*} \frac{1}{r_i^2} \Big]$$

when $J \neq \{\emptyset\}$.

On the other hand, when $J = \{\emptyset\}$ we need to use the difference of + and - resolvents to be able to integrate by parts without boundary terms. In this case, we need to control integrals of the form

$$\frac{1}{t}\int_0^\infty e^{\pm it\sqrt{\lambda^2+m^2}} \partial_\lambda \Big\{ \big(1+\sqrt{\lambda^2+m^2}\big)\chi(\lambda) \sum_{\ell=1}^{k+1} \widetilde{O}_2(\lambda^{\frac{3}{2}} r_\ell^{-\frac{1}{2}}) \prod_{\ell\neq i} \Big(\frac{1}{r_i^2} + \widetilde{O}_2(\lambda^{\frac{3}{2}} r_i^{-\frac{1}{2}}) \Big) \Big\} \, d\lambda$$

Again, we gain the extra $|t|^{-\frac{1}{2}}$ decay by using Lemma 7.1. In this case, again using that $\widetilde{O}_2(\lambda^{\frac{3}{2}} r_i^{-\frac{1}{2}}) \lesssim r_i^{-2}$ when $\lambda r_i \ll 1$, we have

$$|\psi(\lambda)| \lesssim \sum_{\ell=1}^{k+1} \frac{\lambda^{\frac{1}{2}}}{r_\ell^{\frac{1}{2}}} \sum_{\ell\neq i} \frac{1}{r_i^2}, \qquad |\partial_\lambda \psi(\lambda)| \lesssim \sum_{\ell=1}^{k+1} \frac{\lambda^{-\frac{1}{2}}}{r_\ell^{\frac{1}{2}}} \sum_{\ell\neq i} \frac{1}{r_i^2}.$$

So that, in this case,

$$(96) \lesssim \frac{1}{|t|^{\frac{3}{2}}} \sum_{\ell=1}^{k+1} \frac{1}{r_\ell^{\frac{1}{2}}} \sum_{\ell\neq i} \frac{1}{r_i^2}.$$

To see the necessary decay assumptions on the potential, we have to ensure that

$$\Big| \int_{\mathbb{R}^{4k}} \sum_{\ell=1}^{k+1} \frac{1}{r_\ell^{\frac{1}{2}}} \Big[\prod_{\ell\neq j\in J} \frac{1}{r_j^{\frac{3}{2}}} \prod_{i\in J_*} \frac{1}{r_i^2} + \prod_{j\in J} \frac{1}{r_j^{\frac{3}{2}}} \prod_{\ell\neq i\in J_*} \frac{1}{r_i^2} \Big] \prod_{m=1}^{k} V(z_m)\, d\vec{z} \Big|, \tag{99}$$

with $d\vec{z} = dz_1\, dz_2 \ldots dz_k$ is bounded uniformly in x, y for every choice of partitions J and J^*. Using Lemma 8.12, we note that

$$\int_{\mathbb{R}^4} \frac{\langle z_\ell\rangle^{-\frac{5}{2}-}}{|z_{\ell-1}-z_\ell|^{\frac{1}{2}}} \Big(\frac{1}{|z_\ell - z_{\ell+1}|^{\frac{3}{2}}} + \frac{1}{|z_\ell - z_{\ell+1}|^2} \Big) dz_\ell \lesssim \langle z_{\ell-1} - z_{\ell+1}\rangle^{-\frac{1}{2}} \lesssim \frac{1}{|z_{\ell-1}-z_{\ell+1}|^{\frac{1}{2}}}. \tag{100}$$

So that, upon integrating in z_ℓ, we pass forward a decay of size $|z_{\ell-1}-z_{\ell+1}|^{-\frac{1}{2}}$, which allows us to iterate the bound in (100) until we have

$$\sup_{x,y\in\mathbb{R}^4}(99)\lesssim\sup_{x,y\in\mathbb{R}^4}\langle x-y\rangle^{-\frac{1}{2}}\lesssim 1,$$

as desired.

□

We note that, in our application, we need only establish the bound in Proposition 7.3 for $\ell=1$. That is, we need only bound the first three terms of the Born series. To accomplish this, one can lower the assumptions on the potential to $|V(x)|\lesssim\langle x\rangle^{-\frac{9}{4}-}$, which is slightly less restrictive than we assume in the statement of Proposition 7.3.

Proof of Theorem 1.4. We need only bound

$$\begin{aligned}(101)\quad \sup_{x,y\in\mathbb{R}^4}\Bigg|\int_0^\infty&\left(\cos(t\sqrt{\lambda^2+m^2})+\frac{\sin(t\sqrt{\lambda^2+m^2})}{\sqrt{\lambda^2+m^2}}\right)\lambda\chi(\lambda)\\ &[R_0^+(\lambda^2)VR_0^+(\lambda^2)vM^+(\lambda)^{-1}vR_0^+(\lambda^2)VR_0^+(\lambda^2)\\ &\qquad-R_0^-(\lambda^2)VR_0^-(\lambda^2)vM^-(\lambda)^{-1}vR_0^-(\lambda^2)VR_0^-(\lambda^2)]\,d\lambda\Bigg|.\end{aligned}$$

We proceed as in the proofs of Theorems 3.1, 4.1 and 5.1, using the oscillatory bounds in Lemmas 8.8, 8.9, 8.10 and 8.11 in place of Lemmas 3.2, 3.5, and 8.7 respectively.

Again, a sharper bound requires an interpolation with the results in [14]. In considering the wave equation, we have growth at a rate of $t/\log t$ for the sine operator due to the bound

$$\left|\int_0^\infty\sin(t\lambda)\chi(\lambda)\mathcal{E}(\lambda)\,d\lambda\right|\lesssim\frac{t}{\log t},\qquad t>2$$

when $\mathcal{E}(\lambda)=\widetilde{O}_1((\lambda\log\lambda)^{-2})$. This can be replaced with the bound of Lemma 8.8 for the desired bound of $1/(\log t)$ for the Klein-Gordon using the ideas and methods illustrated above.

□

For the wave equation, we need to bound integrals of the form

$$\int_0^\infty e^{\pm it\lambda}\lambda\left(1+\frac{1}{\lambda}\right)\chi(\lambda)\prod_{j\in J}\frac{\lambda}{r_j}\left(e^{\pm i\lambda r_j}\omega_\pm(\lambda r_j)\right)\prod_{i\in J^*}\left(\frac{1}{r_i^2}+\widetilde{O}_2(\lambda^{\frac{3}{2}}r_i^{-\frac{1}{2}})\right).\,d\lambda$$

Here, one cannot use the Van der Corput lemma, Lemma 7.1. Instead, one must use an integration by parts argument and case analysis based on the size of $t\pm r_j$ compared to t. This can be done as in the case when $n=2$ considered in Section 4 of [26]. The most

delicate case will be when 1 or $k+1 \in J$. In that case, we can safely integrate by parts once. Without loss of generality, we assume $t > 0$. Consider the worst case for the index $j = 1$, when the high energy contributes the '-' phase. The analysis for $j = k+1$ is identical. When $t - r_1 \leq t/2$, we have that $t \leq r_1$, and we need to bound

$$\frac{1}{t}\int_0^\infty \left|\chi(\lambda)\frac{\lambda^{\frac{1}{2}}}{r_1^{\frac{1}{2}}}\prod_{1\neq j\in J}\frac{\lambda^{\frac{1}{2}}}{r_j^{\frac{3}{2}}}\prod_{i\in J^*}\left(\frac{1}{r_i^2}+\widetilde{O}_2(\lambda^{\frac{3}{2}}r_i^{-\frac{1}{2}})\right)\right| d\lambda \lesssim \frac{1}{t^{\frac{3}{2}}}\prod_{1\neq j\in J}\frac{1}{r_j^{\frac{3}{2}}}\prod_{i\in J^*}\frac{1}{r_i^2}.$$

On the other hand, if $t - r_1 \geq t/2$, we can integrate by parts against the phase $e^{i\lambda(t-r_1)}$ to gain a time decay of $|t|^{-2}$. Interpolating between that bound and the bound of $|t|^{-1}$ one can get from the above integral gets the desired $|t|^{-\frac{3}{2}}$ decay rate. If $j = 1, k+1 \notin J$, one can integrate by parts twice without a case analysis to get a bound of $|t|^{-2}$.

This analysis will require that $|V(x)| \lesssim \langle x\rangle^{-3-}$. As in the analysis of the Born series for the Schrödinger evolution, this can be seen by examing the case when two derivatives act on a phase $e^{\pm i\lambda r_j}$ when integrating by parts, and we need to bound the integral

$$\int_{\mathbb{R}^4}\frac{\langle z_j\rangle^{\frac{1}{2}}V(z_j)}{|z_{j-1}-z_j|^{\frac{3}{2}}}\,dz_j.$$

This can be bounded by a constant, uniformly in z_{j-1} provided $|V(x)| \lesssim \langle x\rangle^{-3-}$. We leave the remaining details to the interested reader.

We conclude this section by remarking that the bounds proven for the Klein-Gordon allow us to conclude similar bounds for wave equation with the unfortunate growth in t for the sine operator as seen from the estimate above used in [14]. We do not investigate the high-energy dispersive bounds, as this requires a much different approach and requires smoothness on the potential and initial data, see [8]. We suspect that high energy bounds for the Klein-Gordon should follow from the bounds for the wave equation. Similar issues in, for example, Kato smoothing estimates are discussed in [10]. High energy weighted L^2 bounds for the Klein-Gordon were proven in [33] in two spatial dimensions, we believe a similar analysis can be performed in four spatial dimensions. Our low energy $L^1 \to L^\infty$ bounds imply the weighted L^2 estimates on $L^{2,\sigma} \to L^{2,-\sigma'}$ for any $\sigma, \sigma' > 2$.

8. Spectral Theory and Integral Estimates

We repeat the characterization of the spectral subspaces of $L^2(\mathbb{R}^4)$ and their relation to the invertibility of operators in our resolvent expansions performed in [14] for completeness. We omit the proofs. The results below are essentially Lemmas 5–7 of [15] modified to suit

four spatial dimensions. In addition, we give proofs of some integral estimates that are used in the preceding analysis.

Lemma 8.1. *Suppose* $|V(x)| \lesssim \langle x\rangle^{-4}$. *Then* $f \in S_1L^2 \setminus \{0\}$ *if and only if* $f = wg$ *for some* $g \in L^{2,0-} \setminus \{0\}$ *such that*

$$(-\Delta + V)g = 0$$

holds in the sense of distributions.

Recall that S_2 is the projection onto the kernel of S_1PS_1. Note that for $f \in S_2L^2$, since S_1, S_2 and P are projections and hence self-adjoint we have

$$0 = \langle S_1PS_1f, f\rangle = \langle Pf, Pf\rangle = \|Pf\|_2^2 \tag{102}$$

Thus $PS_2 = S_2P = 0$.

Lemma 8.2. *Suppose* $|V(x)| \lesssim \langle x\rangle^{-4-}$. *Then* $f \in S_2L^2 \setminus \{0\}$ *if and only if* $f = wg$ *for some* $g \in L^2 \setminus \{0\}$ *such that*

$$(-\Delta + V)g = 0$$

holds in the sense of distributions.

Corollary 8.3. *Suppose* $|V(x)| \lesssim \langle x\rangle^{-4-}$. *Then*

$$Rank(S_1) \leq \ Rank(S_2) + 1.$$

Lemma 8.4. *If* $|V(x)| \lesssim \langle x\rangle^{-5-}$, *then the kernel of* $S_2vG_1vS_2 = \{0\}$ *on* S_2L^2.

Lemma 8.5. *The projection onto the eigenspace at zero is* $G_0vS_2[S_2vG_1vS_2]^{-1}S_2vG_0$.

8.1. **Oscillatory Integral Estimates.** We have the following oscillatory integral bounds which prove useful in the preceding analysis. Some of these Lemmas along with their proofs appear in Section 6 of [19] or Section 5 of [20], accordingly we state them without proof.

Lemma 8.6. *If* $k \in \mathbb{N}_0$, *we have the bound*

$$\left| \int_0^\infty e^{it\lambda^2}\chi(\lambda)\lambda^k \, d\lambda \right| \lesssim |t|^{-\frac{k+1}{2}}.$$

Lemma 8.7. *For a fixed* $\alpha > -1$, *let* $f(\lambda) = \widetilde{O}_{k+1}(\lambda^\alpha)$ *be supported on the interval* $[0, \lambda_1]$ *for some* $0 < \lambda_1 \lesssim 1$. *Then, if* k *satisfies* $-1 < \alpha - 2k < 1$ *we have*

$$\left| \int_0^\infty e^{it\lambda^2} f(\lambda) \, d\lambda \right| \lesssim |t|^{-\frac{\alpha+1}{2}}.$$

In addition, we make use of the following oscillatory integral estimates that allow us to bound the Klein-Gordon and wave equations.

Lemma 8.8. *If $\mathcal{E}(\lambda) = \widetilde{O}_1((\lambda \log \lambda)^{-2})$, then for $m > 0$*

$$\left| \int_0^\infty \frac{\sin(t\sqrt{\lambda^2+m^2})}{\sqrt{\lambda^2+m^2}} \lambda \chi(\lambda) \mathcal{E}(\lambda)\, d\lambda \right| \lesssim \frac{1}{|\log t|}, \qquad t > 2.$$

Further,

$$\left| \int_0^\infty \cos(t\sqrt{\lambda^2+m^2}) \lambda \chi(\lambda) \mathcal{E}(\lambda)\, d\lambda \right| \lesssim \frac{1}{|\log t|}, \qquad t > 2.$$

Proof. We prove the first bound, the second bound follows similarly. We divide the integral into two pieces. First, on $0 < \lambda < t^{-1}$, we cannot use the oscillation. Instead, we use $0 < m \leq \sqrt{\lambda^2+m^2}$ to bound with

$$\left| \int_0^{t^{-1}} \frac{1}{\lambda(\log \lambda)^{-2}}\, d\lambda \right| \lesssim \frac{1}{|\log t|}.$$

On the remaining piece, where $\lambda \geq t^{-1}$, we write

$$\sin(t\sqrt{\lambda^2+m^2}) = -\frac{\sqrt{\lambda^2+m^2}}{\lambda t} \partial_\lambda \cos(t\sqrt{\lambda^2+m^2})$$

to facilitate an integration by parts. So we need to bound

$$\begin{aligned}
\int_{t^{-1}}^\infty \frac{\sin(t\sqrt{\lambda^2+m^2})}{\sqrt{\lambda^2+m^2}} \lambda \chi(\lambda) \mathcal{E}(\lambda)\, d\lambda &= \frac{1}{t} \int_{t^{-1}}^\infty -\partial_\lambda \cos(t\sqrt{\lambda^2+m^2}) \chi(\lambda) \mathcal{E}(\lambda)\, d\lambda \\
&= \frac{-\cos(t\sqrt{\lambda^2+m^2})\chi(\lambda)\mathcal{E}(\lambda)}{t}\Bigg|_{t^{-1}}^\infty + \int_{t^{-1}}^\infty \cos(t\sqrt{\lambda^2+m^2}) \chi(\lambda) \mathcal{E}(\lambda)\, d\lambda \\
&\lesssim \frac{1}{t} \frac{t}{(\log t)^2} + \int_{t^{-1}}^\infty \left|\partial_\lambda \mathcal{E}(\lambda)\right| d\lambda \lesssim \frac{1}{(\log t)^2}.
\end{aligned}$$

The final integral is bounded as in the proof of Lemma 3.2. For the cosine integral, one uses that $\sqrt{\lambda^2+m^2} \lesssim 1$ on the support of $\chi(\lambda)$.

□

Lemma 8.9. *If $\mathcal{E}(\lambda) = \widetilde{O}_2(\lambda^\alpha)$ for some $-1 < \alpha < 1$, then for $m > 0$*

$$\left| \int_0^\infty \frac{\sin(t\sqrt{\lambda^2+m^2})}{\sqrt{\lambda^2+m^2}} \lambda \chi(\lambda) \mathcal{E}(\lambda)\, d\lambda \right| \lesssim t^{-1-\frac{\alpha}{2}}, \qquad t > 2.$$

Further,

$$\left| \int_0^\infty \cos(t\sqrt{\lambda^2+m^2}) \lambda \chi(\lambda) \mathcal{E}(\lambda)\, d\lambda \right| \lesssim t^{-1-\frac{\alpha}{2}}, \qquad t > 2.$$

Proof. We prove the first bound, the second bound follows similarly. We divide the integral into two pieces. First, on $0 < \lambda < t^{-\frac{1}{2}}$, we cannot use the oscillation. Instead, we use $0 < m \leq \sqrt{\lambda^2 + m^2}$ to bound with

$$\left| \int_0^{t^{-\frac{1}{2}}} \lambda^{1+\alpha} \, d\lambda \right| \lesssim t^{-1-\frac{\alpha}{2}}.$$

On the remaining piece, where $\lambda \geq t^{-\frac{1}{2}}$, we need to integration by parts twice. So we need to bound

$$\frac{\mathcal{E}(t^{-\frac{1}{2}})}{t} + \frac{\partial_\lambda \mathcal{E}(t^{-\frac{1}{2}})}{t^{-\frac{3}{2}}} + \frac{1}{t^2} \int_{t^{-\frac{1}{2}}}^\infty \partial_\lambda \left(\frac{\sqrt{\lambda^2 + m^2}}{\lambda} \partial_\lambda \mathcal{E}(\lambda) \right) d\lambda$$
$$\lesssim t^{-1-\frac{\alpha}{2}} + \frac{1}{t^2} \int_{t^{-\frac{1}{2}}}^\infty \lambda^{\alpha-3} \, d\lambda \lesssim t^{-1-\frac{\alpha}{2}}$$

as desired.

□

Lemma 8.10. *If $\mathcal{E}(\lambda) = \widetilde{O}_2((\log \lambda)^{-k})$, then for $k \geq 2$,*

$$\left| \int_0^\infty \frac{\sin(t\sqrt{\lambda^2 + m^2})}{\sqrt{\lambda^2 + m^2}} \lambda \chi(\lambda) \mathcal{E}(\lambda) \, d\lambda \right| \lesssim \frac{1}{t(\log t)^{k-1}}, \qquad t > 2.$$

Further,

$$\left| \int_0^\infty \cos(t\sqrt{\lambda^2 + m^2}) \lambda \chi(\lambda) \mathcal{E}(\lambda) \, d\lambda \right| \lesssim \frac{1}{t(\log t)^{k-1}}, \qquad t > 2.$$

Proof. The proof follows as in the proof of Lemma 3.5 with the modifications made above in Lemmas 8.8 and 8.9.

□

For completeness, we include the following bound which follows from a simple integration by parts.

Lemma 8.11. *One has the bounds*

$$\left| \int_0^\infty \frac{\sin(t\sqrt{\lambda^2 + m^2})}{\sqrt{\lambda^2 + m^2}} \lambda \chi(\lambda) \, d\lambda \right|, \left| \int_0^\infty \cos(t\sqrt{\lambda^2 + m^2}) \lambda \chi(\lambda) \, d\lambda \right| \lesssim \frac{1}{t}.$$

8.2. **Spatial Integral Estimates.** The following bound is needed to show that certain operators are bounded. The proof is straight-forward and can be found in, for example [24].

Lemma 8.12. *Fix $u_1, u_2 \in \mathbb{R}^n$ and let $0 \leq k, \ell < n$, $\beta > 0$, $k + \ell + \beta \geq n$, $k + \ell \neq n$. We have*

$$\int_{\mathbb{R}^n} \frac{\langle z \rangle^{-\beta-}}{|z - u_1|^k |z - u_2|^\ell} \, dz \lesssim \begin{cases} \left(\frac{1}{|u_1 - u_2|}\right)^{\max(0, k+\ell-n)} & |u_1 - u_2| \leq 1 \\ \left(\frac{1}{|u_1 - u_2|}\right)^{\min(k, \ell, k+\ell+\beta-n)} & |u_1 - u_2| > 1 \end{cases}$$

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Department of Mathematics, Rose-Hulman Institute of Technology, Terre Haute, IN 47803, U.S.A.

E-mail address: green@rose-hulman.edu

Department of Mathematics, University of Illinois, Urbana, IL 61801, U.S.A.

E-mail address: toprak2@illinois.edu