LIMITING ABSORPTION PRINCIPLE AND STRICHARTZ ESTIMATES FOR DIRAC OPERATORS IN TWO AND HIGHER DIMENSIONS

M. BURAK ERDOĞAN, MICHAEL GOLDBERG, WILLIAM R. GREEN

ABSTRACT. In this paper we consider Dirac operators in $\mathbb{R}^n$, $n \geq 2$, with a potential V. Under mild decay and continuity assumptions on V and some spectral assumptions on the operator, we prove a limiting absorption principle for the resolvent, which implies a family of Strichartz estimates for the linear Dirac equation. For large potentials the dynamical estimates are not an immediate corollary of the free case since the resolvent of the free Dirac operator does not decay in operator norm on weighted L^2 spaces as the frequency goes to infinity.

1. INTRODUCTION

In this paper we obtain limiting absorption principle bounds and Strichartz estimates for the linear Dirac equation in dimensions two and higher with potential:

$$i\partial_t \psi(x,t) = (D_m + V(x))\psi(x,t), \quad \psi(x,0) = \psi_0(x). \tag{1.1}$$

Here $x \in \mathbb{R}^n$ and $\psi(x,t) \in \mathbb{C}^{2^N}$ where $N = \lfloor \frac{n+1}{2} \rfloor$. The n-dimensional free Dirac operator D_m is defined by

$$D_m = -i\alpha \cdot \nabla + m\beta = -i\sum_{k=1}^{n} \alpha_k \partial_k + m\beta, \tag{1.2}$$

where $m \geq 0$ is a constant, and the $2^N \times 2^N$ Hermitian matrices α_j satisfy the anti-commutation relationships

$$\begin{cases} \alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} \mathbb{1}_{\mathbb{C}^{2^N}} & j,k \in \{1,2,\ldots,n\} \\ \alpha_j \beta + \beta \alpha_j = \mathbb{O}_{\mathbb{C}^{2^N}} \\ \beta^2 = \mathbb{1}_{\mathbb{C}^{2^N}} \end{cases} \tag{1.3}$$

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Physically, m represents the mass of the quantum particle. If $m = 0$ the particle is massless and if $m > 0$ the particle is massive. We note that dimensions $n = 2, 3$ are of particular physical interest. Following standard conventions, we define the free Dirac operator in dimension two with the Pauli spin matrices

$$\alpha_1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \ \alpha_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

In dimension three we use

$$\beta = \begin{bmatrix} I_{\mathbb{C}^2} & 0 \\ 0 & -I_{\mathbb{C}^2} \end{bmatrix}, \ \alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix},$$

$$\sigma_1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \ \sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

In higher dimensions $n > 3$, one can create a full set of anti-commuting matrices α_j iteratively, see [32] for an explicit construction.

The Dirac equation arose as an attempt to reconcile the theories of relativity and quantum mechanics and describe the behavior of subatomic particles at near luminal speeds. The relativistic relationship between energy, momentum and mass, $E^2 = c^2p^2 + m^2c^4$ can be combined with the quantum-mechanical notions of energy $E = i\hbar\partial_t$ and momentum $p = -i\hbar\nabla$ to obtain a Klein-Gordon equation

$$-\hbar^2\partial_t^2\psi(x,t) = -c^2\hbar^2\Delta\psi(x,t) + m^2c^4\psi(x,t).$$

Here $\hbar$ is Planck's constant and c is the speed of light. However, the Klein-Gordon does not preserve L^2 norm of the initial data and is incompatible with quantum mechanical interpretations of the wave function. By considering E directly, one arrives at the non-local equation

$$i\hbar\partial_t\psi(x,t) = \sqrt{-c^2\hbar^2\Delta + m^2c^4}\,\psi(x,t). \tag{1.4}$$

In our mathematical analysis, we rescale so that we may take the constants $\hbar$ and c to be one. Dirac's insight was to rewrite the right hand side in terms of the first order operator $D_m = -i\alpha\cdot\nabla + m\beta$. This leads to the free Dirac equation, (1.1) with $V = 0$, a system of coupled hyperbolic equations with α, β required to be matrices. Dirac's modification allows one to account for the spin of quantum particles, as well as providing a way to incorporate external electro-magnetic fields in a manner compatible with the relativistic theory where the Klein-Gordon and (1.4) cannot. In addition, we note that (1.4) has infinite speed of propagation, which is in contrast with the causality principle in relativity. In dimension $n = 3$, the Dirac equation models the evolution of spin 1/2 particles, while in dimension $n = 2$ the massless Dirac equation is of considerable interest due to its connection to graphene, see, e.g., [26].

Formally, the Dirac equation is a square root of a system of Klein-Gordon or wave equations when $m > 0$ and $m = 0$ respectively. One consequence is that the spectrum of the free Dirac operators is unbounded in both the positive and negative directions. In particular, the continuous spectrum of D_m is $(-\infty, -m] \cup [m, \infty)$. By Weyl's criterion, the continuous spectrum of the perturbed Dirac operator is also $(-\infty, -m] \cup [m, \infty)$ for a large class of potentials. The absence of embedded eigenvalues in the continuous spectrum in general dimensions was established in [10] for the class of potentials we are interested in by adapting the argument of [7] for three dimensions. This result was used to study linearizations about a solitary wave for a non-linear equation. For other results in this direction for small dimensions and specific classes of potentials see [40, 7, 43, 27]. Finally, there is no singular continuous spectrum, see [27]. For a further background on the Dirac equation see [42].

We denote the perturbed Dirac operator by $H := D_m + V$, then e^{-itH} is formally the solution operator to (1.1). For the class of potentials considered in Theorem 1.1, we note that H is self-adjoint by the Kato-Rellich theorem. We denote $a- := a - \varepsilon$ for a small, but fixed $\varepsilon > 0$. Further, we write $A \lesssim B$ to indicate there is a fixed absolute constant $C > 0$ so that $A \leq CB$.

Theorem 1.1. *Let V be a $2^N \times 2^N$ real Hermitian matrix for all $x \in \mathbb{R}^n$, $n \geq 2$, with continuous entries satisfying $|V_{ij}(x)| \lesssim \langle x \rangle^{-1-}$ when $m = 0$, and $|V_{ij}(x)| \lesssim \langle x \rangle^{-2-}$ when $m > 0$. Furthermore, assume that threshold energies are regular. Then, with P_c being the projection onto the continuous spectrum,*

$$\|\langle \nabla \rangle^{-\theta} e^{-itH} P_c f\|_{L^p_t(L^q_x)} \lesssim \|f\|_{L^2(\mathbb{R}^n)} \tag{1.5}$$

in the case $m > 0$, provided that

$$\theta \geq \frac{1}{2} + \frac{1}{p} - \frac{1}{q} \quad \text{and} \quad \frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \quad 2 \leq q < \frac{2n}{n-2}.$$

In the case $m = 0$, the bound is

$$\||\nabla|^{-\theta} e^{-itH} P_c f\|_{L^p_t(L^q_x)} \lesssim \|f\|_{L^2(\mathbb{R}^n)} \tag{1.6}$$

provided that

$$\theta = \frac{n}{2} - \frac{1}{p} - \frac{n}{q} \quad \text{and} \quad \frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2}, \quad p > 2, \ 2 \leq q < \infty.$$

The combinations of (p,q,θ) stated above are the same ones found in Strichartz estimates for the free massive ($m>0$) and massless ($m=0$) Dirac equation, respectively. Note that the range of admissible Strichartz exponents (p,q) match those for the Schrödinger equation in the massive case, and the derivative is not homogeneous. This reflects the fact that the low energy behavior of the Dirac system is comparable to the Schrödinger equation, while the high energy behavior is closer to the wave equation (which requires differentiability of initial data). See the Appendix of [18] for a derivation of Strichartz estimates for the free evolution e^{-itD_m}. The free massless Dirac system has the same scaling properties and admissible combinations as the free wave equation, which are proved in [33] for the wave equation.

These families of perturbed Strichartz estimates are a consequence of the uniform resolvent estimates that we prove. Much of the paper is devoted to proving the following resolvent bounds, which hold for any subset of the continuous spectrum of H.

Theorem 1.2. *Let V be a $2^N \times 2^N$ real Hermitian matrix for all $x \in \mathbb{R}^n$, $n \geq 2$, with continuous entries satisfying $|V_{ij}(x)| \lesssim \langle x\rangle^{-1-}$. Then for $\sigma > \frac{1}{2}$ there exists $\lambda_1 < \infty$ so that*

$$\sup_{|\lambda|>\lambda_1} \|\langle x\rangle^{-\sigma}(H-(\lambda+i0))^{-1}\langle x\rangle^{-\sigma}\|_{2\to 2} \lesssim 1. \tag{1.7}$$

Under the assumption that the threshold energies are regular, and if $m>0$ the stronger decay condition $|V_{ij}(x)| \lesssim \langle x\rangle^{-2-}$, this bound can be extended as follows

$$\sup_{|\lambda|>m} \|\langle x\rangle^{-\sigma}(H-(\lambda+i0))^{-1}\langle x\rangle^{-\sigma}\|_{2\to 2} \lesssim 1,$$

provided that $\sigma > \frac{1}{2}$ when $m=0$, and $\sigma > 1$ when $m>0$.

We note that the proof of the high energy limiting absorption principle (1.7) does not require V to be real or Hermitian. Since V is assumed to be bounded and the free Dirac operator D_m is self-adjoint, H has the same domain as D_m and for unit functions η in the domain the quadratic form $\langle H\eta,\eta\rangle$ is confined to a strip of finite width around the real axis.

The immediate consequence of (1.7) is that there cannot be any embedded eigenvalues or resonances on $(-\infty,-\lambda_1)\cup(\lambda_1,\infty)$ for $\lambda_1=\lambda_1(V)$ sufficiently large. A perturbation argument shows that the eigenvalue-free zone extends to a sector of the complex plane.

Corollary 1.3. *Under the hypotheses of Theorem 1.2, there exist* $\lambda_1 < \infty$ *and* $\delta > 0$ *depending on* V*,* m*, and* $\sigma > \frac{1}{2}$ *so that*

$$\sup_{\substack{|\lambda|>\lambda_1 \\ 0<|\gamma|<\delta|\lambda|}} \|\langle x\rangle^{-\sigma}(H-(\lambda+i\gamma))^{-1}\langle x\rangle^{-\sigma}\|_{2\to 2} \lesssim 1. \tag{1.8}$$

As a result, there is a compact subset of the complex plane outside of which the spectrum of H *is confined to the real axis.*

Our results apply to a broad class of electric potentials $V(x)$ and require no implicit smallness condition, only that V is bounded, continuous and satisfies a mild polynomial decay at infinity. The potentials need not be small, radial, or smooth. Our results apply for the potentials that naturally arise when linearizing about soliton solutions for the non-linear Dirac equation.

There is a rich history of results on limiting absorption principles and mapping estimates of dispersive equations. Much of this history is focused on the analysis of the Schrödinger, wave or Klein-Gordon equation. We refer the reader to [29, 38, 30, 28, 18, 41, 21, 8, 36, 22, 25, 20, 39], for example. There are far fewer results in the case of the Dirac system, due to its more complicated mathematical structure.

It is known that the Dirac resolvent does not decay in the spectral parameter, [44]. That is, the bound (1.7) does not decay as $\lambda \to \infty$. This is a stark contrast to the Schrödinger resolvent in which one obtains a decay in the spectral parameter λ as $\lambda \to \infty$. The bootstrapping argument of Agmon, [2], produces uniform bounds on the resolvent operators only on compact subsets of the purely absolutely continuous spectrum. Limiting absorption principles have been studied to establish the limiting behavior of resolvents as one approaches the spectrum, see [43, 4, 7]. The work of Georgescu and Mantoiu provides resolvent bounds on compact subsets of the spectrum, [27]. Other limiting absorption principles have been established, often in service of providing dispersive, smoothing or Strichartz estimates, [9, 19, 13]. Very recently, [15], established a limiting absorption principle for the free massless Dirac operator in dimensions $n \geq 2$.

One consequence of the resolvent bounds in Theorem 1.2 is the family of Strichartz estimates given in Theorem 1.1. Strichartz estimates have been used to study non-linear Dirac equations, [35, 17, 5, 6, 10, 11]. These are often adapted to the problem by localizing in frequency or considering specialized potentials. Strichartz estimates may be obtained by establishing a virial identity see, for example, [12, 14], which consider magnetic potentials with a certain smallness condition. The first and third author proved a class of Strichartz estimates for the two-dimensional Dirac equation, [23], by first establishing dispersive estimates of the two-dimensional Dirac propagator.

The paper is organized as follows: In Section 2 we show how the Strichartz estimates in Theorem 1.1 follows from the resolvent bounds in Theorem 1.2. The bulk of the paper is then devoted to proving Theorem 1.2.

In Section 3 we present the basic properties of the free resolvents of Dirac and Schrödinger operators. The small energy case of Theorem 1.2 is then treated in Section 4. In Section 5, we treat the case of large energies by adapting an intricate argument originally devised in [21, 22] for Schrödinger operators in dimensions $n \geq 3$ with a non-smooth magnetic potential. A brief argument in Section 6 derives Corollary 1.3 from the main high-energy bounds.

2. The basic setup

The Strichartz estimates stated in Theorem 1.1 will be proved using Proposition 2.1 below, which is essentially Theorem 4.1 in [38]. It is based on Kato's notion of smoothing operators, see [34]. We recall that for a self-adjoint operator H, an operator Γ is called H-smooth in the sense of Kato if for any $f \in \mathcal{D}(H)$

$$\|\Gamma e^{-itH} f\|_{L^2_t L^2_x} \leq C_\Gamma(H) \|f\|_{L^2_x}. \tag{2.1}$$

Let $\Omega \subset \mathbb{R}$ and let P_Ω be a spectral projection of H associated with a set Ω. We say that Γ is H-smooth on Ω if ΓP_Ω is H-smooth. It is not difficult to show (see e.g. [37, Theorems XIII.25 and XIII.30]) that, Γ is H-smooth on Ω if

$$\sup_{\lambda \in \Omega} \|\Gamma [R_H^+(\lambda) - R_H^-(\lambda)] \Gamma^*\|_{L^2 \to L^2} \leq C_\Gamma(H, \Omega). \tag{2.2}$$

Given the known Strichartz bounds for the free Dirac equation, the following proposition and Theorem 1.2 imply Theorem 1.1. For brevity we state only the $m > 0$ case.

Proposition 2.1. *Let $H_0 = D_m$, $m > 0$, and $H = H_0 + V$, where $|V(x)| \lesssim \langle x \rangle^{-2\sigma}$. Assume that $w(x) := \langle x \rangle^{-\sigma}$ is H_0-smooth and H-smooth on Ω for some $\Omega \subset \mathbb{R}$. Assume also that the unitary semigroup e^{-itH_0} satisfies the estimate*

$$\left\| \langle \nabla \rangle^{-\theta} e^{-itH_0} \right\|_{L^2 \to L^q_t L^r_x} < \infty \tag{2.3}$$

for some $q \in (2, \infty]$, $r \in [1, \infty]$, and $\theta \in \mathbb{R}$. Then the semigroup e^{-itH} associated with $H = H_0 + V$, restricted to the spectral set Ω, also verifies the estimate (2.3)*, i.e.,*

$$\left\| \langle \nabla \rangle^{-\theta} e^{-itH} P_\Omega \right\|_{L^2 \to L^q_t L^r_x} < \infty. \tag{2.4}$$

Proof. For completeness we supply the proof following [38]. We have

$$e^{-itH}P_\Omega f = e^{-itH_0}P_\Omega f - i\int_0^t e^{-i(t-s)H_0}Ve^{-isH}P_\Omega f ds.$$

By Christ-Kiselev Lemma [16], it suffices to prove that

$$\Big\|\langle\nabla\rangle^{-\theta}\int_0^\infty e^{-i(t-s)H_0}Ve^{-isH}P_\Omega f ds\Big\|_{L^q_tL^r_x} \lesssim \|f\|_{L^2}.$$

Using (2.3), we bound the left hand side by

$$\Big\|\int_0^\infty e^{isH_0}Vw^{-1}we^{-isH}P_\Omega f ds\Big\|_{L^2}. \tag{2.5}$$

Since w is H_0 smooth and H-smooth on Ω, and $|Vw^{-1}| \lesssim w$, we have

$$\|we^{-itH}P_\Omega f\|_{L^2_tL^2_x} \lesssim \|f\|_{L^2_x}, \quad \|Vw^{-1}e^{-itH_0}f\|_{L^2_tL^2_x} \lesssim \|f\|_{L^2_x},$$

and its dual

$$\Big\|\int_0^\infty e^{isH_0}Vw^{-1}g(s,x)ds\Big\|_{L^2} \lesssim \|g\|_{L^2_sL^2_x}.$$

Composing these two inequalities suffices to bound (2.5) by $\|f\|_{L^2}$. □

3. Properties of the Free Resolvent

The following identity,[1] which follows from (1.3),

$$(D_m-\lambda\mathbb{1})(D_m+\lambda\mathbb{1}) = (-i\alpha\cdot\nabla+m\beta-\lambda\mathbb{1})(-i\alpha\cdot\nabla+m\beta+\lambda\mathbb{1}) = (-\Delta+m^2-\lambda^2) \tag{3.1}$$

allows us to formally define the free Dirac resolvent operator $\mathcal{R}_0(\lambda) = (D_m-\lambda)^{-1}$ in terms of the free resolvent $R_0(\lambda) = (-\Delta-\lambda)^{-1}$ of the Schrödinger operator for λ in the resolvent set:

$$\mathcal{R}_0(\lambda) = (D_m+\lambda)R_0(\lambda^2-m^2). \tag{3.2}$$

We first discuss the properties of Schrödinger resolvent R_0. There are two possible continuations to the positive halfline, namely

$$R_0^\pm(\lambda^2) = \lim_{\varepsilon\to0^+} R_0(\lambda^2 \pm i\varepsilon), \quad \lambda>0,$$

where the limit is in the operator norm from L^2_σ to $L^2_{-\sigma}$, $\sigma > \frac{1}{2}$. Here L^2_σ denotes the weighted L^2 space with norm

$$\|f\|_{L^2_\sigma} := \|\langle\cdot\rangle^\sigma f\|_{L^2}.$$

[1]Here and throughout the paper, scalar operators such as $-\Delta+m^2-\lambda^2$ are understood as $(-\Delta+m^2-\lambda^2)\mathbb{1}_{\mathbb{C}^{2N}}$.

Existence of the limits $R_0^{\pm}(\lambda^2)$ is known as the limiting absorption principle. In fact $R_0^{\pm}(\lambda^2)$ varies continuously in λ over the interval $(0,\infty)$. In dimensions $n \geq 3$ the continuity extends to $\lambda \in [0,\infty)$ with a uniform bound

$$\|R_0^{\pm}(\lambda^2)\|_{L^2_\sigma \to L^2_{-\sigma}} \leq C_{\sigma,n}, \quad \lambda \geq 0 \tag{3.3}$$

provided $\sigma > 1$. In two dimensions the free Schrödinger operator has a threshold resonance and consequently $R_0^{\pm}(\lambda^2)$ is unbounded as λ approaches zero. However there is still a useful uniform estimate,

$$\|\nabla R_0^{\pm}(\lambda^2)\|_{L^2_\sigma \to L^2_{-\sigma}} + \lambda \|R_0^{\pm}(\lambda^2)\|_{L^2_\sigma \to L^2_{-\sigma}} \leq C_{\sigma,n}, \quad \lambda > 0, \ \sigma > \frac{1}{2}, \tag{3.4}$$

which is true in all dimensions $n \geq 2$. This bound for large λ is largely due to scaling considerations. The bound for small λ will be proved in the next section.

Using the limiting absorption bounds (3.4) for Schrödinger and (3.2), we obtain for $n \geq 2$

$$\|\mathcal{R}_0(\lambda)\|_{L^2_\sigma \to L^2_{-\sigma}} \leq C_{\sigma,\lambda_0,n}, \quad |\lambda| > \lambda_0 > m, \ \sigma > \frac{1}{2}. \tag{3.5}$$

An analogous uniform bound holds on the entire interval $|\lambda| > m$ if $n \geq 3$ and $\sigma > 1$. In the case $m = 0$ we have the following stronger uniform bound for $n \geq 2$

$$\|\mathcal{R}_0(\lambda)\|_{L^2_\sigma \to L^2_{-\sigma}} \leq C_{\sigma,n}, \quad |\lambda| > 0, \ \sigma > \frac{1}{2}. \tag{3.6}$$

In particular, two dimensional massless free Dirac operator does not have a threshold resonance.

The kernel of the free resolvent $R_0^+(\lambda^2)$ in $\mathbb{R}^n$ is given by[2]

$$R_0^+(\lambda^2)(x,y) = C_n \frac{\lambda^{\frac{n-2}{2}}}{|x-y|^{\frac{n-2}{2}}} H^+_{\frac{n-2}{2}}(\lambda|x-y|)$$

where H_ν^+ is a Hankel function. There is the scaling relation

$$R_0^+(\lambda^2)(x,y) = \lambda^{n-2} R_0^+(1)(\lambda x, \lambda y) \quad \forall\, \lambda > 0 \tag{3.7}$$

and the representation, see the asymptotics of H_ν^+ in [1],

$$R_0^+(1)(x,y) = \frac{e^{i|x-y|}}{|x-y|^{\frac{n-1}{2}}} a(|x-y|) + \frac{b(|x-y|)}{|x-y|^{n-2}} \tag{3.8}$$

provided $n \geq 2$. Here

$$|a^{(k)}(r)| \lesssim r^{-k} \quad \forall\, k \geq 0, \quad a(r) = 0 \quad \forall\, 0 < r < \frac{1}{2} \tag{3.9}$$

[2]Constants C_n are allowed to change from line to line.

and $b(r) = 0$ for all $r > \frac{3}{4}$, with

$$|b^{(k)}(r)| \lesssim 1 \quad \forall\, k \geq 0, \quad n \;\; \text{odd} \tag{3.10}$$

$$\left.\begin{array}{lll} |b^{(k)}(r)| & \lesssim 1 & \forall\, 0 \leq k < n-2 \\ |b^{(k)}(r)| & \lesssim r^{n-k-2}|\log r| & \forall\, k \geq n-2 \end{array}\right\} \; n \geq 2 \;\; \text{even} \tag{3.11}$$

for all $r > 0$. In dimension $n = 2$ we will need the following more detailed expansion of b

$$b(r) = \left(-\frac{1}{2\pi}\big(\log(r/2) + \gamma\big) + \frac{i}{4} \right) + \mathcal{E}(r), \tag{3.12}$$

where

$$\big|\mathcal{E}^{(k)}(r)\big| \lesssim \left| \frac{d^k}{dr^k}(r^2 \log r) \right| \qquad k = 0, 1, 2.$$

In order to gain sharp control over the scaling behavior as $\lambda \to \infty$ we discuss the $\sigma = \frac{1}{2}$ endpoint of the limiting absorption principle. As in Chapter XIV of [31] define

$$\|f\|_B := \sum_{j=0}^{\infty} 2^{\frac{j}{2}} \|f\|_{L^2(D_j)}, \quad \|f\|_{B^*} := \sup_{j \geq 0} 2^{-\frac{j}{2}} \|f\|_{L^2(D_j)},$$

where $D_j = \{x \;:\; |x| \sim 2^j\}$ for $j \geq 1$ and $D_0 = \{|x| \leq 1\}$. For each $\sigma > \frac{1}{2}$, there are containment relations $L^2_\sigma \subset B$ and $B^* \subset L^2_{-\sigma}$. It is known that $\|R_0(1)\|_{B \to B^*} < \infty$.

Note that

$$\|Vf\|_B \lesssim \|f\|_{B^*} \sum_{j=0}^{\infty} 2^j \|V\|_{L^\infty(D_j)} \lesssim \|\langle x \rangle^{1+} V\|_{L^\infty} \|f\|_{B^*}. \tag{3.13}$$

Also recall that, by Lemma 3.1 in [22], we have the following scaling relations for any $\lambda \geq 1$

$$\|f(\lambda^{-1}\cdot)\|_B \lesssim \lambda^{\frac{n+1}{2}} \|f\|_B, \quad \|g(\lambda\cdot)\|_{B^*} \lesssim \lambda^{-\frac{n-1}{2}} \|g\|_{B^*},$$

provided the right-hand sides are finite. This and (3.7) immediately imply the following statement. In what follows, R_0 stands for either of $R_0^{\pm}$.

Proposition 3.1. *For all $\lambda \geq 1$, we have*

$$\|R_0(\lambda^2)\|_{B \to B^*} \lesssim \lambda^{-1} \|R_0(1)\|_{B \to B^*}.$$

Proof. First, from (3.7)

$$(R_0^+(\lambda^2) f)(x) = \lambda^{-2} [R_0^+(1) f(\cdot \lambda^{-1})](\lambda x)$$

Hence, by the previous lemma,

$$\|R_0^+(\lambda^2) f\|_{B^*} \lesssim \lambda^{-2} \lambda^{-\frac{n-1}{2}} \|R_0^+(1) f(\cdot \lambda^{-1})\|_{B^*} \lesssim \lambda^{-1} \|R_0^+(1) f\|_{B^*}$$

as claimed. □

4. Energies close to $\{-m, m\}$

In this section, assuming the regularity of threshold energies, we prove Theorem 1.2 when the spectral parameter λ is sufficiently close to the threshold energy $\lambda = m$, respectively $\lambda = -m$. We consider the positive portion of the spectrum $[m, \infty)$, the negative part can be controlled similarly. That is, for sufficiently small λ_0

$$\sup_{m<\lambda<\lambda_0} \|w(\mathcal{R}_V^+(\lambda) - \mathcal{R}_V^-(\lambda))w\|_{2\to 2} < \infty, \tag{4.1}$$

where $w = \langle x \rangle^{-\sigma}$ for $\sigma > 1$. In fact, we prove that

$$\sup_{m<\lambda<\lambda_0} \|w\mathcal{R}_V^\pm(\lambda)w\|_{2\to 2} < \infty, \tag{4.2}$$

provided that $\sigma > 1$ when $m > 0$, and provided $\sigma > \frac{1}{2}$ when $m = 0$. A similar statement holds for negative energies.

We refer to reader to [23] for the case $n = 2$ and $m > 0$, as this argument is substantially different from the other cases. In all the remaining cases we have

$$\begin{aligned}\|w\mathcal{R}_V^\pm(\lambda)w\|_{2\to 2} &= \|w(1 + \mathcal{R}_0^\pm(\lambda)V)^{-1}w^{-1}w\mathcal{R}_0^\pm(\lambda)w\|_{2\to 2} \\ &\leq \|w(1 + \mathcal{R}_0^\pm(\lambda)V)^{-1}w^{-1}\|_{2\to 2}\|w\mathcal{R}_0^\pm(\lambda)w\|_{2\to 2},\end{aligned}$$

so it suffices to show that

$$\sup_{m<\lambda<\lambda_0} \|w(1 + \mathcal{R}_0^\pm(\lambda)V)^{-1}w^{-1}\|_{2\to 2} < \infty, \tag{4.3}$$

$$\sup_{m<\lambda<\lambda_0} \|w\mathcal{R}_0^\pm(\lambda)w\|_{2\to 2} < \infty. \tag{4.4}$$

In dimensions $n \geq 3$, (4.4) is an immediate consequence of the fact that the free Dirac operator is regular at the threshold, provided $\sigma > 1$. We will show below that (4.4) is also true when $m = 0$ and $n \geq 2$ with $\sigma > \frac{1}{2}$. The $n = 2$ case is somewhat surprising because the threshold is not regular for the free Schrödinger operator, nor for the free Dirac operator with $m > 0$.

In dimensions $n \geq 3$, let $G = \mathcal{R}_0^\pm(m) = (D_m + m)R_0(0)$. In the case $n = 2$, $m = 0$, define $G = D_0 G_0$, where

$$G_0 f(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x - y| f(y)\, dy. \tag{4.5}$$

Let $B_\lambda^\pm = \mathcal{R}_0^\pm(\lambda) - G$. We assume that the threshold m is a regular point of the spectrum, namely the boundedness of the operators

$$w(I + GV)^{-1}w^{-1} = (I + wGVw^{-1})^{-1} : L^2 \to L^2. \tag{4.6}$$

By a standard Fredholm alternative argument, (4.6) is equivalent to the absence of resonances and eigenfunctions at m. We now prove that under suitable conditions

$$\|wB_\lambda^\pm Vw^{-1}\|_{2\to 2} \lesssim \|wB_\lambda^\pm w\|_{2\to 2} \to 0 \qquad \text{as } \lambda \to m^+. \tag{4.7}$$

This and (4.6) imply (4.3) by summing the Neumann series directly, and it implies (4.4) since wGw is L^2-bounded for $\sigma > 1$ if $m > 0$ and $\sigma > \frac{1}{2}$ if $m = 0$.

To prove the bound (4.7) recall the properties of the kernel of $B_\lambda^\pm$: with $\lambda = \sqrt{m^2+z^2}$, we have

$$\begin{aligned} B_\lambda^\pm &= \mathcal{R}_0^\pm(\lambda) - G = \big(D_m + \sqrt{m^2+z^2}\big)R_0(z^2) - (D_m+m)R_0(0) \\ &= \big(\sqrt{m^2+z^2} - m\big)R_0(z^2) + m(\beta+I)\big[R_0(z^2) - R_0(0)\big] - i\alpha\cdot\nabla\big[R_0(z^2)-R_0(0)\big] \end{aligned} \tag{4.8}$$

in dimensions $n \geq 3$. When $m = 0$ we have

$$B_\lambda^\pm = zR_0(z^2) - i\alpha\cdot\nabla\big[R_0(z^2) - R_0(0)\big] \tag{4.9}$$

and this holds for $n = 2$ by replacing $R_0(0)$ with G_0.

By the limiting absorption principle for the free Schrödinger operator, the second summand of (4.8) goes to zero as $z \to 0$ as operators from L^2_σ to $L^2_{-\sigma}$, provided that $\sigma > 1$, see (3.3). This can also be proved using the limiting absorption bound (3.4) at frequency 1 and scaling, similar to the remaining cases that we discuss below. The remaining terms are identical to those in (4.9) or better, since $0 \leq \sqrt{m^2+z^2} - m \leq z$. We will prove that both terms of (4.9) go to zero for $\sigma > \frac{1}{2}$ for dimensions $n \geq 2$.

For the first term, using the scaling relation (3.7) and the representation (3.8) we have

$$zR_0(z^2) = zR_0(z^2)(x,y)\widetilde{\chi}(z|x-y|) + z\frac{b(z|x-y|)}{|x-y|^{n-2}},$$

where $\widetilde{\chi}$ is a smooth cutoff for the complement of the unit ball. Using (3.10) and (3.11), the low energy term can be bounded as follows

$$\Big|z\frac{b(z|x-y|)}{|x-y|^{n-2}}\Big| \lesssim z\frac{(z|x-y|)^{0-}}{|x-y|^{n-2}}\chi(z|x-y|) \lesssim z\frac{(z|x-y|)^{-1+}}{|x-y|^{n-2}} = \frac{z^{0+}}{|x-y|^{n-1-}}.$$

By the weighted version of the Schur's test, this operator is $O(z^{0+})$ as $z \to 0$ as an operator from L^2_σ to $L^2_{-\sigma}$, provided that $\sigma > 1/2$. We can rewrite the high energy term using the scaling relation (3.7):

$$zR_0(z^2)(x,y)\widetilde{\chi}(z|x-y|) = z^{n-1}\big[R_0(1)\widetilde{\chi}\big](zx,zy).$$

Therefore, with χ be a smooth cutoff for 1/10 neighborhood of the origin,

$$\begin{aligned}\Big\|\langle x\rangle^{-\sigma} z\Big[R_0(z^2)(x,y)\widetilde{\chi}(z|x-y|)\Big]\langle y\rangle^{-\sigma}\Big\|_{L^2\to L^2}&\\ \lesssim z^{-1}\big\|\langle x/z\rangle^{-\sigma}\big[R_0(1)\widetilde{\chi}\big](x,y)\langle y/z\rangle^{-\sigma}\big\|_{L^2\to L^2}&\\ \lesssim z^{2\sigma-1}\big\|(z+|x|)^{-\sigma}\big[R_0(1)\widetilde{\chi}\big](x,y)(|y|+z)^{-\sigma}\big\|_{L^2\to L^2}&\\ \lesssim z^{2\sigma-1}\big\|\langle x\rangle^{-\sigma}\big[R_0(1)\widetilde{\chi}\big](x,y)\langle y\rangle^{-\sigma}\big\|_{L^2\to L^2}&\\ + z^{2\sigma-1}\big\|(z+|x|)^{-\sigma}\chi(|x|)\big[R_0(1)\widetilde{\chi}\big](x,y)\langle y\rangle^{-\sigma}\big\|_{L^2\to L^2}&\\ + z^{2\sigma-1}\big\|\langle x\rangle^{-\sigma}\big[R_0(1)\widetilde{\chi}\big](x,y)(z+|y|)^{-\sigma}\chi(|y|)\big\|_{L^2\to L^2}.&\end{aligned}$$

The first summand converges to zero provided that $\sigma > \frac{1}{2}$ by the limiting absorption bound for the free Schrödinger operator for $\lambda = 1$. Using the representation (3.8) and the bound (3.9), and considering the Hilbert Schmidt norms, the second and third summands can be bounded by the square root of

$$z^{4\sigma-2}\int \langle x\rangle^{-2\sigma}\frac{1}{\langle x-y\rangle^{n-1}}(z+|y|)^{-2\sigma}\chi(|y|)dxdy \lesssim z^{0+}\to 0,$$

provided that $\sigma > \frac{1}{2}$.

We now consider the second summand in (4.9). In dimensions $n \geq 3$ we may use the scaling relation (3.7) and the representation (3.8) to write

$$\nabla\big[R_0(z^2)-R_0(0)\big] = \nabla\Big[R_0(z^2)(x,y)\widetilde{\chi}(z|x-y|) + \frac{b(z|x-y|)-b(0)}{|x-y|^{n-2}}\Big],$$

where $\widetilde{\chi}$ is a smooth cutoff for the complement of the unit ball. Using (3.10) and (3.11), the low energy term can be bounded as follows

$$\Big|\nabla\frac{b(z|x-y|)-b(0)}{|x-y|^{n-2}}\Big| \lesssim \frac{z^{0+}}{|x-y|^{n-1-}},$$

which goes to zero as $z\to 0$ as an operator from L^2_σ to $L^2_{-\sigma}$, provided that $\sigma > 1/2$. If $n = 2$ we use (3.12) to claim an analogous bound

$$\big|\nabla\big[b(z|x-y|)-G_0\big]\big| \lesssim \frac{z^{0+}}{|x-y|^{1-}}.$$

Turning our attention to the high energy term, we use the scaling relation (3.7) to write

$$\nabla\Big[R_0(z^2)(x,y)\widetilde{\chi}(z|x-y|)\Big] = z^{n-1}\nabla\big[R_0(1)\widetilde{\chi}\big](zx,zy).$$

Therefore

$$\begin{aligned}\Big\|\langle x\rangle^{-\sigma}\nabla\Big[R_0(z^2)(x,y)\widetilde{\chi}(z|x-y|)\Big]\langle y\rangle^{-\sigma}\Big\|_{L^2\to L^2}&\\ \lesssim z^{-1}\big\|\langle x/z\rangle^{-\sigma}\nabla\big[R_0(1)\widetilde{\chi}\big](x,y)\langle y/z\rangle^{-\sigma}\big\|_{L^2\to L^2}&\end{aligned}$$

$$\begin{aligned}
&\lesssim z^{2\sigma-1}\big\|(z+|x|)^{-\sigma}\nabla\big[R_0(1)\widetilde{\chi}\big](x,y)(|y|+z)^{-\sigma}\big\|_{L^2\to L^2}\\
&\lesssim z^{2\sigma-1}\big\|\langle x\rangle^{-\sigma}\nabla\big[R_0(1)\widetilde{\chi}\big](x,y)\langle y\rangle^{-\sigma}\big\|_{L^2\to L^2}\\
&\qquad+z^{2\sigma-1}\big\|(z+|x|)^{-\sigma}\chi(|x|)\nabla\big[R_0(1)\widetilde{\chi}\big](x,y)\langle y\rangle^{-\sigma}\big\|_{L^2\to L^2}\\
&\qquad\quad+z^{2\sigma-1}\big\|\langle x\rangle^{-\sigma}\nabla\big[R_0(1)\widetilde{\chi}\big](x,y)(z+|y|)^{-\sigma}\chi(|y|)\big\|_{L^2\to L^2},
\end{aligned}$$

where χ is a smooth cutoff for 1/10 neighborhood of the origin. The first summand converges to zero provided that $\sigma > \frac{1}{2}$ by the limiting absorption bound for the free Schrödinger operator for $\lambda = 1$. Using the representation (3.8) and the bound (3.9), and considering the Hilbert Schmidt norms, the second and third summands can be bounded by the square root of

$$z^{4\sigma-2}\int \langle x\rangle^{-2\sigma}\frac{1}{\langle x-y\rangle^{n-1}}(z+|y|)^{-2\sigma}\chi(|y|)dxdy \lesssim z^{0+}\to 0,$$

provided that $\sigma > \frac{1}{2}$.

5. The high energies limiting absorption principle

Let us briefly consider intermediate energies, i.e., $\lambda \in I := [\lambda_0, \lambda_1] \subset (-\infty, -m) \cup (m, \infty)$. It was shown in [27], see Theorem 1.6, that the resolvent of H satisfies the limiting absorption principle uniformly in λ:

$$\sup_{\lambda\in I}\|\langle x\rangle^{-\sigma}\mathcal{R}_V^{\pm}(\lambda)\langle x\rangle^{-\sigma}\|_{L^2\to L^2} \le C_I,$$

provided that there are no embedded eigenvalues, $\sigma > \frac{1}{2}$, and $|V(x)| \lesssim \langle x\rangle^{-1-}$.

In this section we complete the proof of Theorem 1.2 by considering energies sufficiently far from threshold, in the non-compact interval $|\lambda| \in (\lambda_1, \infty)$. In other words, we establish a limiting absorption principle for the perturbed Dirac resolvent $\mathcal{R}_V^{\pm}(\lambda)$ at high energies:

$$\sup_{|\lambda|>\lambda_0}\|\mathcal{R}_V^{\pm}(\lambda)\|_{L^{2,\sigma}\to L^{2,-\sigma}} \lesssim 1, \qquad \sigma > \frac{1}{2}. \tag{5.1}$$

In fact we control the slightly stronger operator norm from B to B^*, and show that embedded eigenvalues are absent in this part of the spectrum.

Recall that (with $z = \sqrt{\lambda^2 - m^2}$) we have

$$\mathcal{R}_V^{\pm}(\lambda) = \mathcal{R}_0^{\pm}(\lambda)\big[I + V\mathcal{R}_0^{\pm}(\lambda)\big]^{-1} = \mathcal{R}_0^{\pm}(\lambda)\big[I + L_z R_0^{\pm}(z^2)\big]^{-1}, \tag{5.2}$$

where

$$L_z = V(D_m + \lambda).$$

Scaling arguments nearly identical to Proposition 3.1 show that $\|\mathcal{R}_0^\pm(\lambda)\|_{B\to B^*} \lesssim 1$ for all $\lambda \geq \lambda_0 > m$.

Since multiplication by V maps B^* to B (see (3.13)), the limiting absorption principle for R_0 implies a bound $\|L_z R_0^\pm(z^2)\|_{B\to B} < C\|\langle x\rangle^{1+}V\|_{L^\infty}$ uniformly in $z \geq z_0 > 0$. If V is sufficiently small then the operator norm of $L_z R_0^\pm(z^2)$ is less than 1 for all $z \geq z_0$. Then one can conclude

$$\sup_{z\geq z_0} \big\| \big[I + L_z R_0^\pm(z^2)\big]^{-1} \big\|_{B\to B} \lesssim 1.$$

The main goal of this section is to show that the same bound on $[I + L_z R_0^\pm(z^2)]^{-1}$ holds even when V is not small. This cannot be proved directly from the size of $L_z R_0^\pm(z^2)$, which need not become small as $z \to \infty$. Instead, the following crucial lemma shows that the Neumann series

$$\big[I + L_z R_0^\pm(z^2)\big]^{-1} = \sum_{\ell=0}^{\infty} (-1)^\ell \big(L_z R_0^\pm(z^2)\big)^\ell \tag{5.3}$$

is absolutely convergent for large z due to the behavior of later terms in the series.

Lemma 5.1. *Assume the entries of V are continuous and satisfy $|V_{ij}(x)| \lesssim \langle x\rangle^{-1-}$. There exist sufficiently large $M = M(V)$ and $z_1 = z_1(V)$ such that*

$$\sup_{z>z_1} \|(L_z R_0^\pm(z^2))^M\|_{B\to B} \leq \frac{1}{2}. \tag{5.4}$$

Assuming for the moment that (5.4) holds, the operator inverse in (5.2) is bounded uniformly in $z > z_1$, thus we conclude (5.1) for $\lambda_0 = \lambda_0(V)$ sufficiently large. The remainder of this section is devoted to the proof of Lemma 5.1. The method will be similar to the one in [22].

5.1. **The directed free resolvent.** The first step is to decompose the free Schrödinger resolvent into a large number of pieces according to the size of $|x-y|$ and where $\frac{x-y}{|x-y|}$ lies on the unit sphere. This section presents a limiting absorption estimate for these truncated free resolvent kernels and for their first-order derivatives. The constants will not depend on the parameters of truncation, which gives us the freedom to choose those values later on. Similar estimates were obtained in [22] in dimensions $n \geq 3$, with derivatives of up to second order. We emphasize here the steps where $n \geq 2$ and the number of derivatives are most prominent, and refer the reader to [22] for technical details that are shared by both arguments.

For any $\delta \in (0, 1)$, let Φ_δ be a smooth cut-off function to a δ-neighborhood of the north pole in S^{n-1}. Also, for any $d \in (0, \infty)$, $\eta_d(x) = \eta(|x|/d)$ denotes a smooth cut-off to the set $|x| > d$. In what follows, we shall use the notation

$$R_{d,\delta}(\lambda^2)(x,y) = [R_0(\lambda^2)\eta_d\Phi_\delta](x,y) = R_0(\lambda^2)(x,y)\eta_d(|x-y|)\Phi_\delta\Big(\frac{x-y}{|x-y|}\Big).$$

Note that this operator obeys the same scaling as R_0, see (3.7). More precisely,

$$R_{d,\delta}(\lambda^2)(x,y) = \lambda^{n-2}R_{d\lambda,\delta}(1)(\lambda x, \lambda y).$$

Thus, Proposition 3.1 applies to $R_{d,\delta}(\lambda^2)$ in the form

$$\|R_{d,\delta}(\lambda^2)\|_{B\to B^*} \lesssim \lambda^{-1}\|R_{d\lambda,\delta}(1)\|_{B\to B^*} \tag{5.5}$$

for all $\lambda \geq 1$ or, more generally,

$$\|D^\alpha R_{d,\delta}(\lambda^2)\|_{B\to B^*} \lesssim \lambda^{-1+|\alpha|}\|D^\alpha R_{d\lambda,\delta}(1)\|_{B\to B^*} \tag{5.6}$$

for all multi-indices α and $\lambda \geq 1$.

We sketch a proof of a limiting absorption bound for $R_{d,\delta}$ and its derivatives of order at most one uniformly in the parameters $d, \delta \in (0,1)$, see Proposition 5.3 below. This will be based on the oscillatory integral estimate in Lemma 5.2, which was proved in [22] for $n \geq 3$ and $\frac{n-1}{2} \leq p \leq \frac{n+3}{2}$. The extension here to dimension $n = 2$ with a smaller range of p can be obtained from the proof of Lemma 3.4 in [22] by minor changes in the case analysis. More specifically, there is a step where one can replace the inequality $3(\frac{n-1}{2}) \geq \frac{n+3}{2}$ (which is true only if $n \geq 3$) with the slightly weaker bound $3(\frac{n-1}{2}) \geq \frac{n+1}{2}$.

Lemma 5.2. *Let χ denote a smooth cut-off function to the region $1 < |x| < 2$. With $a(r)$ as in (3.9), define*

$$(T_{\delta,p,R_1,R_2}f)(x) = \int \chi\big(\frac{x}{R_1}\big)\frac{e^{i|x-y|}}{|x-y|^p}a(|x-y|)\Phi_\delta\Big(\frac{x-y}{|x-y|}\Big)\chi\big(\frac{y}{R_2}\big)f(y)\,dy. \tag{5.7}$$

Then, for any $n \geq 2$, and $\frac{n-1}{2} \leq p \leq \frac{n+1}{2}$,

$$\|T_{\delta,p,R_1,R_2}f\|_2 \leq C_n\,\delta^{p-\frac{n-1}{2}}\sqrt{R_1R_2}\,\|f\|_2 \tag{5.8}$$

for all $R_1, R_2 \geq 1$, $\delta \in (0,1)$. The constant C_n only depends on $n \geq 2$.

Proposition 5.3. *Let $n \geq 2$. Then for any $d \in (0,\infty)$, $\delta \in (0,1)$, and $\lambda \geq 1$ there is the bound*

$$\|D^\alpha R_{d,\delta}(\lambda^2)f\|_{B^*} \leq C_n\lambda^{-1+|\alpha|}\|f\|_B \tag{5.9}$$

for any $0 \leq |\alpha| \leq 1$. The constant C_n depends only on the dimension $n \geq 2$.

Proof. In view of (5.5) and (5.6) it suffices to prove this estimate for $\lambda = 1$. We need to prove that for any $0 \le |\alpha| \le 1$

$$\|\chi(\cdot/R_1)D^\alpha R_{d,\delta}(1)\,\chi(\cdot/R_2)f\|_2 \le C_n \sqrt{R_1R_2}\,\|f\|_2 \tag{5.10}$$

where $R_1, R_2 \ge 1$ are arbitrary. We write

$$R_{d,\delta}(1) = R_0^+(1)\eta_d\Phi_\delta = T_0 + T_1 \tag{5.11}$$

where the kernels of T_0, T_1 are

$$\begin{aligned} T_0(x,y) &= \frac{b(|x-y|)}{|x-y|^{n-2}}\eta_d(|x-y|)\Phi_\delta(x,y), \\ T_1(x,y) &= \frac{e^{i|x-y|}}{|x-y|^{\frac{n-1}{2}}}\eta_d(|x-y|)a(|x-y|)\Phi_\delta(x,y), \end{aligned} \tag{5.12}$$

respectively, see (3.8). The modified function $\eta_d(r)a(r)$ satisfies all decay estimates in (3.9) with constants independent of the choice of d.

We begin by showing that $\widehat{T_0f} = m_0\hat{f}$ where $|m_0(\xi)| \lesssim \langle\xi\rangle^{-1}$. This will imply (5.10) for T_0. By definition

$$m_0(\xi) = \int_0^\infty \int_{S^{n-1}} rb(r)\eta_d(r)e^{-ir\omega\cdot\xi}\Phi_\delta(\omega)\,\sigma(d\omega)\,dr.$$

Since $b(r) = 0$ if $r > \frac{3}{4}$, $|m_0(\xi)| \lesssim 1$. Hence we may assume that $|\xi| \ge 1$. If $|\xi_n| \ge |\xi|/10$, then $|\omega\cdot\xi| \gtrsim |\xi|$ and

$$|m_0(\xi)| \lesssim \int_{S^{n-1}} \Phi_\delta(\omega)\langle\omega\cdot\xi\rangle^{-2}\,\sigma(d\omega) \lesssim \delta^{n-1}|\xi|^{-2},$$

where we have used that

$$\Big|\int_0^\infty e^{-ir\rho}rb(r)\eta_d(r)\chi(r)\,dr\Big| \lesssim \langle\rho\rangle^{-1}.$$

This follows from (3.10) and (3.11) after an integration by parts. Now suppose that $|\xi_n| \le |\xi|/10$. Set $\xi = |\xi|\hat{\xi}$ and change integration variables as follows:

$$\begin{aligned} &\int_{S^{n-1}}\int_0^\infty rb(r)\eta_d(r)\chi(r)e^{-ir|\xi|\omega\cdot\hat{\xi}}\,dr\,\Phi_\delta(\omega)\,\sigma(d\omega) \\ &= \int_{\mathbb{R}^{n-1}}\int_0^\infty rb(r)\eta_d(r)\chi(r)e^{-ir|\xi|u_1}\,dr\,\widetilde{\Phi}_\delta(u_1,\dots,u_{n-1})\,du_1du_2\dots du_{n-1} \\ &= \delta^{n-2}\int_0^\infty\int_{\mathbb{R}} rb(r)\eta_d(r)\chi(r)e^{-ir|\xi|u_1}\Psi_\delta(u_1)\,du_1dr, \end{aligned}$$

where $(u_1, \dots, u_{n-1})$ is a parametrization of the support of Φ_δ, aligning u_1 with $\hat{\xi}$. The function Ψ_δ is a smooth cut-off supported on an interval of length $\sim \delta$ resulting from the integration of $\widetilde{\Phi}_\delta$. Thus,

$$|m_0(\xi)| \lesssim \delta^{n-2} \int_0^1 |\widehat{\Psi_\delta}(r|\xi|)|\, dr \lesssim \delta^{n-2}|\xi|^{-1} \|\widehat{\Psi_\delta}(u)\|_{L^1_u} \lesssim \delta^{n-2}|\xi|^{-1}.$$

In conclusion, $|m_0(\xi)| \lesssim \langle \xi \rangle^{-1}$ as claimed.

Next, consider T_1. By the Leibniz rule,

$$\begin{aligned} D_x^\alpha T_1(x,y) &= \sum_{\beta \le \alpha} c_{\alpha,\beta}\, D_x^{\alpha-\beta} \Big[\frac{e^{i|x-y|}}{|x-y|^{\frac{n-1}{2}}} \eta_d(|x-y|) a(|x-y|) \Big] D_x^\beta \Phi_\delta(x,y) \\ &= \sum_{\beta \le \alpha} \delta^{-|\beta|}\, c_{\alpha,\beta}\, \frac{e^{i|x-y|}}{|x-y|^{\frac{n-1}{2}+|\beta|}} a_{\alpha,\beta,d}(|x-y|) \Phi_{\delta,\beta}(x,y), \end{aligned} \tag{5.13}$$

where $\Phi_{\delta,\beta} = \delta^{|\beta|} D^\beta \Phi_\delta$ is a modified angular cut-off and $a_{\alpha,\beta,d}$ satisfies the same bounds as a, see (3.9), with constants that do not depend on d. The estimate (5.10) for T_1 follows from Lemma 5.2 with $p = \frac{n-1}{2} + |\beta|$. □

A partition of unity $\{\Phi_i\}$ over S^{n-1} induces a directional decomposition of the free resolvent, namely

$$R_0(\lambda^2) = \sum_i R_i(\lambda^2) + R_d(\lambda^2) \tag{5.14}$$

where $R_i(\lambda^2) := R_{d,\delta}(\lambda^2)$ with Φ_i playing the role of Φ_δ from above. Moreover, $R_d(\lambda^2)(x) = (1 - \eta_d(|x|))R_0(\lambda^2)(|x|)$ is the "short range piece". We have the following L^2 bound for $R_d(\lambda^2)$:

Lemma 5.4. *With $R_d^+(\lambda^2)$ defined as above, the mapping estimate*

$$\|D^\alpha R_d(\lambda^2) f\|_2 \le C_n\, \lambda^{-2+|\alpha|} \langle d\lambda \rangle \|f\|_2 \tag{5.15}$$

holds uniformly for every choice of $d \in (0,\infty)$, $0 \le |\alpha| \le 1$, and $\lambda \ge 1$.

Proof. By the scaling relation (3.7), for any α,

$$\|D^\alpha R_0(\lambda^2) \chi_{[|x|<d]}\|_{2\to 2} = \lambda^{-2+|\alpha|} \|D^\alpha R_0(1) \chi_{[|x|<\lambda d]}\|_{2\to 2}$$

where $\chi_{[|x|<\rho]} = \chi(|x|/\rho)$ is a smooth cut-off to the set $|x| < \rho$ with $\rho > 0$ arbitrary. The notation is somewhat ambiguous here; we are seeking an estimate for the convolution operator with kernel $D^\alpha R_0(1)\chi_{[|x|<\lambda d]}$. The lemma is proved by showing that the Fourier transform of $R_0(1)\chi_{[|x|<\rho]}$ is bounded point-wise by $\langle \rho \rangle \langle \xi \rangle^{-1}$.

Consider first the case $\rho \leq 1$. The decomposition (3.8) implies that

$$\int_{\mathbb{R}^n} |R_0(1)(x)\chi(|x|/\rho)|\, dx \lesssim \rho^2 \log(\rho).$$

Furthermore, since $(\Delta + 1)R_0(1)$ is a point mass at the origin, the distribution $\Delta[R_0(1)\chi_{[|x|<\rho]}]$ consists of a point mass plus a function of L^1 norm $\lesssim |\log(\rho)|$. This implies that the Fourier transform of $R_0(1)\chi_{[|x|<\rho]}$ is bounded by $|\log(\rho)||\xi|^{-2}$. The desired Fourier transform estimate follows by interpolating these two bounds.

When $\rho > 1$, it is more convenient to estimate

$$\rho^n \Big| \int \big[\text{P.V.}\frac{1}{|\eta|^2 - 1} + i\sigma_{S^{n-1}}(d\eta)\big]\hat{\chi}((\xi - \eta)\rho)\, d\eta\Big|.$$

A standard calculation shows this to be less than $\rho\langle\rho(|\xi|^2 - 1)\rangle^{-1} < \rho\langle\xi\rangle^{-2}$. □

5.2. **Proof of Lemma 5.1.** Decomposing each free resolvent in the M-fold product $(L_z R_0(z^2))^M$ as in (5.14) yields the identity

$$(L_z R_0(z^2))^M = \sum_{i_1 \ldots i_M} \prod_{k=1}^{M} \big(L_z R_{i_k}(z^2)\big). \tag{5.16}$$

The indices i_k may take numerical values corresponding to the partition of unity $\{\Phi_i\}$, or else the letter d to indicate a short-range resolvent. There are two main types of products represented here, namely:

- *Directed Products*, where the support of functions Φ_{i_k} and $\Phi_{i_{k+1}}$ are separated by less than 10δ for each k. A product is also considered to be directed if it has this property once all instances of $i_k = d$ are removed. The term $(L_z R_d(z^2))^M$ is a vacuous example of a directed product.
- All other terms not meeting the above criteria are *Undirected Products*. An undirected product must contain two adjacent numerical indices (i.e., after discarding all instances where $i_k = d$) for which the corresponding functions Φ_i have disjoint support with distance at least 10δ between them.

Lemma 5.5. *For any $\delta > 0$, there exists a partition of unity $\{\Phi_i\}$ with approximately δ^{1-n} elements, having* $\operatorname{diam}\operatorname{supp}(\Phi_i) < \delta$ *for each i and admitting no more than $\delta^{1-n}(C_n)^M$ directed products of length M in* (5.16)*.*

Proof. The first claim is a standard fact from differential geometry. For the second claim note that there are $\lesssim \delta^{1-n}$ choices for the first element in a directed product, but only C_n choices at each subsequent step. □

The iterated resolvent $(L_z R_0(z^2))^M$ is an oscillatory integral operator with phase $e^{iz\sum_{k=1}^M |x_k - x_{k-1}|}$, where $x_0 = y$ and $x_M = x$. Loosely speaking, there is a region of stationary phase where $\sum_{k=1}^M |x_k - x_{k-1}| \approx |x-y|$. The integral kernel of a directed product is supported here, hence one cannot gain any benefit from oscillation as $z \to \infty$ beyond the bounds for individual resolvents in Proposition 5.3 and Lemma 5.4. Those bounds do not decrease to zero in the limit of large z. It appears that the operator norm of a directed product does not decrease to zero either.

Never the less, one can show that long directed products have an operator norm that is small enough for our purposes. The geometric idea is relatively simple: If $\delta < \frac{1}{20M}$, then all the angular cutoffs Φ_{i_k} have support within a single hemisphere. The convolution operators $R_{i_k}(z^2)$ are therefore biased consistently to one side. This introduces a gain from the product $\prod_{k=1}^M V(x_k)$, as only a handful of x_k can be located near the origin, and $V(x_k)$ is small everywhere else.

The following lemma is adapted from Lemma 4.8 of [22]. There is a small but significant difference in the structure of the perturbation. Here $L_z = V(D_m + \lambda)$ has the property that $L_z R_0(z^2)$ is a bounded operator on the space B. The symmetrized version $(L_z + L_z^*)R_0(z^2)$ does not map B to itself unless $V(x)$ is assumed to be differentiable. This explains the use of more elaborate function spaces in [22] and the need for bounds on the second derivative of the truncated free resolvent (which ultimately restricts the dimension to $n \geq 3$).

Now we may state the bound for directed products involving L_z in dimensions $n \geq 2$.

Lemma 5.6. *Given any $r > 0$, there exists a distance $d = d(r) > 0$ such that each directed product in* (5.16) *satisfies the estimate*

$$\left\| \prod_{k=1}^{M} \big(L_z R_{i_k}(z^2)\big) f \right\|_B \leq C_{n,V,r}\, r^M \|f\|_B \tag{5.17}$$

uniformly over all $z > d^{-1}$ and all choices of M and δ satisfying $\delta \leq \frac{1}{20M}$.

Consequently, given any $c > 0$, there exists a number $M = m(c, V)$ and a partition of unity governed by $\delta = \frac{1}{20M}$ so that the sum over all directed products achieves the bound

$$\sum_{\substack{i_1 \ldots i_M \\ \text{directed}}} \left\| \prod_{k=1}^{M} \big(L_z R_{i_k}(z^2)\big) \right\|_{B \to B} \leq \tfrac{c}{2}$$

uniformly in $z > d^{-1}$.

Proof. In this proof, we will keep track of the superscripts $\pm$ on the resolvents. Also, we will write $\|V\|_{B^* \to B} = C_V$. There is no loss of generality if we assume that $r < C_n C_V$.

After a rotation, we may assume that every function Φ_{i_k} which appears in the product has support within a half-radian neighborhood of the north pole, where $x_n > \frac{2}{3}$. If $f \in B$ is supported on the half space $\{x_n > A\}$, then the support of $R^+_{i_k}(\lambda^2) f$ must be translated upward to $\{x_n > A + \frac{2}{3}d\}$. The short-range resolvent $R^+_d(\lambda^2)$ does not have a preferred direction; however if $f \in B$ is supported on $\{x_n > A\}$ then supp $R^+_d(\lambda^2) f \subset \{x_n > A - 2d\}$.

The purpose of keeping track of supports is that if $f \in B^*$ is supported away from the origin, in the set $\{|x| > A\}$, then the estimate in (3.13) can be improved to

$$\|Vf\|_B \lesssim \|\langle x\rangle^{1+} V \chi_{[|x|>A]}\|_{L^\infty} \|f\|_{B^*} \lesssim \langle A\rangle^{0-} \|f\|_{B^*}.$$

Note that we can choose $A = A(n, V, r)$ so that

$$\|Vf\|_B \le \frac{r^2}{C_n^2 C_V} \|f\|_{B^*}, \tag{5.18}$$

provided that f is supported in the set $\{|x| > A\}$.

Let χ be a smooth function supported on the interval $[-1, \infty)$ such that $\chi(x_n) + \chi(-x_n) = 1$. We will initially estimate the operator norm of $\big(\prod_k (L_z R^+_{i_k}(z^2))\big)\chi(x_n)$. Multiplication by $\chi(x_n)$ is bounded operator of approximately unit norm on B^*.

The support of $\chi(x_n) f$ lies in the half-space $\{x_n > -1\}$. Suppose every one of the indices i_k is numerical. Then each application of an operator $L_z R^+_{i_k}(z^2)$ translates the support upward by $\frac{2}{3}d$. For the first $\frac{3A}{2d}$ steps the operator norm of $L_z R^+_{i_k}(z^2)$ is bounded by $C_n C_V$. Thereafter it is possible to use the stronger bound of (5.18) because the support will have moved into the half-space $\{x_n > A\}$. The combined estimate is

$$\begin{aligned} \Big\| \prod_{k=1}^m (L_z R^+_{i_k}(z^2))\, \chi(x_n) f \Big\|_B &\le (C_n C_V)^m \Big(\frac{r^2}{(C_n C_V)^2}\Big)^{m - \frac{3a}{2d}} \|f\|_B \\ &= (C_n C_V)^{-m} (r^{-1} C_n C_V)^{\frac{3a}{d}} r^{2m} \|f\|_B. \end{aligned} \tag{5.19}$$

This is also valid for small m by our assumption that $r < C_n C_V$.

If each directed resolvent $R^+_{\Phi_i}(\lambda^2)$ is seen as taking one step forward, then the short-range resolvent $R^+_d(\lambda^2)$ may take as many as three steps back. Suppose a directed product includes exactly one index $i_k = d$. This will have the most pronounced effect if it occurs near the beginning of the product, delaying the upward progression of supports by a total of 4 steps. In this case one combines (5.18), and Lemma 5.4 to obtain

$$\Big\| \prod_{k=1}^m (L_z R^+_{i_k}(z^2))\, \chi(x_n) f \Big\|_B \le (C_n C_V)^m d \Big(\frac{r^2}{(C_n C_V)^2}\Big)^{m - (\frac{3A}{2d} + 4)} \|f\|_B.$$

Notice that this estimate agrees with the one in (5.19) up to a factor of $d(r^{-1}C_nC_V)^8$. By setting $d = d(r) = \big(\frac{r}{C_nC_V}\big)^8$, the bound in (5.19) is strictly larger. Similar arguments yield the same result for any directed product with one or more instances of the short-range resolvent $R_d^+(\lambda^2)$.

To remove the spatial cutoff, write

$$\prod_{k=1}^{m} L_z R_{i_k}^+(z^2) \;=\; \Big(\prod_{k=1}^{m/2} L_z R_{i_k}^+(z^2)\Big)(\chi(x_n)+\chi(-x_n))\Big(\prod_{k=\frac{m}{2}+1}^{m} L_z R_{i_k}^+(z^2)\Big).$$

Consider the $\chi(x_n)$ term. By (5.19), the first half of the product carries an operator norm bound of $(C_n\,C_V)^{-\frac{m}{2}}(r^{-1}C_nC_V)^{\frac{3A}{d(r)}}r^m$. The second half contributes at most $(C_nC_V)^{m/2}$. Put together, this product has an operator norm less than $C_{n,V,r}\,r^m$, where $C_{n,V,r} = (r^{-1}C_nC_V)^{\frac{3A}{d(r)}}$.

The $\chi(-x_n)$ term has nearly identical estimates, by duality. The adjoint of any directed resolvent $R_\Phi^+(z^2)$ is precisely $R_{\widetilde{\Phi}}^-(z^2)$, with $\widetilde{\Phi}$ being the antipodal image of Φ. Because the order of multiplication is reversed, one applies the geometric argument above to the adjoint operators $R_{i_k}^-(z^2)L_z^*$ (modulo the antipodal map).

According to Lemma 5.5 there are at most $\delta^{1-n}(C_n)^m$ directed products of length m. To prove (5.6), it therefore suffices to let $r = \frac{1}{2C_n}$, and $\delta = \frac{1}{20m}$ so that the sum of the operator norms of all directed products is bounded by $20^{n-1}C_{n,V}m^{n-1}2^{-m}$. This can be made smaller than $\frac{\epsilon}{2}$ by choosing m sufficiently large. □

As for the undirected products, recall that their defining feature is the presence of adjacent resolvents $R_i^+(\lambda^2)$ oriented in distinct directions. The resulting oscillatory integral has no region of stationary phase, and therefore exhibits improved bounds at high energy provided the potential $V(x)$ is smooth. The following lemma follows from Lemma 4.9 in [22] by minor changes in the proof.

Lemma 5.7. *Let Φ_1 and Φ_2 be chosen from a partition of unity of S^{n-1} so that their supports are separated by a distance greater than 10δ. Suppose $V \in C^\infty(\mathbb{R}^n)$ with compact support. Then for each $j \geq 0$, and any $N \geq 1$,*

$$\big\| L_z R_{d,\Phi_2}^+(z^2)(L_zR_d^+(z^2))^j L_z R_{d,\Phi_1}^+(z^2)\big\|_{B\to B} = \mathcal{O}(z^{-N}) \tag{5.20}$$

as $z \to \infty$ and similarly for $R^-(z^2)$.

Note that under the conditions of Lemma 5.7 each undirected product in (5.16) satisfies the bound

$$\Big\| \prod_{k=1}^{M} \big(L_z R_{i_k}(z^2)\big) \Big\|_{B\to B} = \mathcal{O}(z^{-N}) \tag{5.21}$$

for any $N \geq 1$. We now show by approximation that vanishing still holds for merely continuous V, but without any control over the rate.

Lemma 5.8. *Let Φ_1 and Φ_2 be chosen as in Lemma 5.7. Suppose V is a continuous function with $V \in Y$. Then each undirected product in* (5.16) *satisfies the limiting bound*

$$\lim_{z\to\infty} \Big\| \prod_{k=1}^{M} \big(L_z R_{i_k}(z^2)\big) \Big\|_{B\to B} = 0. \tag{5.22}$$

Proof. For any small $\gamma > 0$, there exists a smooth approximation $V_\gamma \in C^\infty(\mathbb{R}^n)$ of compact support so that $\|V - V_\gamma\|_{B^*\to B} < \gamma$ and $\|V_\gamma\|_{B^*\to B} < 2\|V\|_{B^*\to B}$. Define the operator $L_{z,\gamma}$ accordingly. We have

$$\Big\| \prod_{k=1}^{M} \big(L_z R_{i_k}(z^2)\big) - \prod_{k=1}^{M} \big(L_{z,\gamma} R_{i_k}(z^2)\big) \Big\|_{B\to B} \lesssim \gamma(2\|V\|_{B^*\to B})^{M-1}$$

uniformly in $z \geq 1$. Thus, by (5.21),

$$\limsup_{z\to\infty} \Big\| \prod_{k=1}^{M} \big(L_z R_{i_k}(z^2)\big) \Big\|_{B\to B} \lesssim \gamma(2\|V\|_{B^*\to B})^{M-1}.$$

Sending $\gamma \to 0$ finishes the proof. □

Proof of Lemma 5.1. Lemma 5.6 provides a recipe for selecting a value of M, together with a partition of unity $\{\Phi_i\}$ and a short-range threshold d, so that the sum over all directed products in (5.16) will be an operator of norm less than $\frac{1}{4}$. This fixes the number of undirected products as approximately $\delta^{M(1-n)} = (20M)^{M(n-1)}$. For each of these, Lemma 5.8 asserts that its operator norm tends to zero as $\lambda \to \infty$. The same is true for the finite sum over all undirected products of length M. In particular it is less than the directed product estimate provided $z > z_1(M)$ is sufficiently large. □

6. Extension to the complex plane

This section provides a short proof of Corollary 1.3 via a perturbation argument. We first record a strong statement of continuity in the limiting absorption principle.

Proposition 6.1. *Let $\lambda_n \to 1$ and $\varepsilon_n \to 0^+$. Then for each $\sigma > \frac{1}{2}$,*

$$\lim_{n\to\infty} \|R_0(\lambda_n + i\varepsilon_n) - R_0^+(1)\|_{L^2_\sigma \to L^2_{-\sigma}} = 0.$$

Proof. By the triangle inequality and scaling relations,

$$\begin{aligned}\|R_0(\lambda_n + i\varepsilon_n) &- R_0^+(1)\|_{L^2_\sigma \to L^2_{-\sigma}} \\ &\leq \|R_0(\lambda_n + i\varepsilon_n) - R_0^+(\lambda_n)\|_{L^2_\sigma \to L^2_{-\sigma}} + \|R_0^+(\lambda_n) - R_0^+(1)\|_{L^2_\sigma \to L^2_{-\sigma}} \\ &\lesssim \|R_0(1 + i\tfrac{\varepsilon_n}{\lambda_n}) - R_0^+(1)\|_{L^2_\sigma \to L^2_{-\sigma}} + \|R_0^+(\lambda_n) - R_0^+(1)\|_{L^2_\sigma \to L^2_{-\sigma}}.\end{aligned}$$

Both of the differences in the last line converge to zero by the limiting absorption principle and continuity of the resolvent respectively. □

Proof of Corollary 1.3. By the standard resolvent identities (5.2) and (3.2)

$$\begin{aligned}\mathcal{R}_V(\lambda + i\gamma) &= \mathcal{R}_0(\lambda + i\gamma)\big[I + V\mathcal{R}_0(\lambda + i\gamma)\big]^{-1} \\ &= (D_m + \lambda + i\gamma)R_0((\lambda + i\gamma)^2 - m^2)\big[I + V\mathcal{R}_0(\lambda + i\gamma)\big]^{-1}\end{aligned}$$

The free Dirac resolvent is controlled by rescaling by λ. For the gradient term,

$$\|\nabla R_0((\lambda + i\gamma)^2 - m^2)\|_{L^2_\sigma \to L^2_{-\sigma}} \leq \|\nabla R_0((1 + i\tfrac{\gamma}{\lambda})^2 - (\tfrac{m}{\lambda})^2)\|_{L^2_\sigma \to L^2_{-\sigma}} \lesssim 1,$$

with the last inequality following from continuity of the Schrödinger resolvent at 1. Similarly,

$$\begin{aligned}|m + \lambda + i\gamma|\, &\|R_0((\lambda + i\gamma)^2 - m^2)\|_{L^2_\sigma \to L^2_{-\sigma}} \\ &\leq \Big(1 + \frac{m}{\lambda} + \frac{\gamma}{\lambda}\Big)\|R_0((1 + i\tfrac{\gamma}{\lambda})^2 - (\tfrac{m}{\lambda})^2)\|_{L^2_\sigma \to L^2_{-\sigma}} \lesssim 1\end{aligned}$$

provided $\frac{\gamma}{\lambda}$ and $\frac{m}{\lambda}$ are sufficiently small. Boundedness of $\mathcal{R}_V(\lambda + i\gamma)$ therefore rests on the behavior of $[I + V\mathcal{R}_0(\lambda + i\gamma)]^{-1}$.

The crucial bound (5.4) shows that

$$\|[I + V\mathcal{R}_0^+(\lambda)]^{-1}\|_{L^2_\sigma \to L^2_\sigma} \lesssim 1 + \|V\mathcal{R}_0^+(\lambda)\|^{M-1}_{L^2_\sigma \to L^2_\sigma} \leq C(V)$$

for all $\lambda > \lambda_0$. We would like to expand

$$\big[I + V\mathcal{R}_0(\lambda + i\gamma)\big]^{-1} = \big[(I + V\mathcal{R}_0^+(\lambda)) + V\big(\mathcal{R}_0(\lambda + i\gamma) - \mathcal{R}_0^+(\lambda)\big)\big]^{-1} \tag{6.1}$$

via a Neumann series, and this can be done provided the operator norm of $V(\mathcal{R}_0(\lambda + i\gamma) - \mathcal{R}_0^+(\lambda))$ is less than $\frac{1}{C(V)}$.

The difference of Dirac resolvents can be expanded out as

$$\mathcal{R}_0(\lambda + i\gamma) - \mathcal{R}_0^+(\lambda) = (D_m + \lambda + i\gamma)R_0((\lambda + i\gamma)^2 - m^2) - (D_m + \lambda)R_0^+(\lambda^2 - m^2)$$

$$= i\gamma R_0^+(\lambda^2 - m^2) + (D_m + \lambda + i\gamma)\big(R_0((\lambda+i\gamma)^2 - m^2) - R_0^+(\lambda^2 - m^2)\big).$$

Each of these terms can be bounded using the same scaling arguments as above, with the end result

$$\|\mathcal{R}_0(\lambda+i\gamma) - \mathcal{R}_0^+(\lambda)\|_{L^2_\sigma \to L^2_{-\sigma}} \tag{6.2}$$
$$\lesssim \frac{\gamma}{\lambda}\|R_0^+(1-(\tfrac{m}{\lambda})^2)\|_{L^2_\sigma\to L^2_\sigma} + \Big(1+\frac{m}{\lambda}+\frac{\gamma}{\lambda}\Big)\|R_0((1+i\tfrac{\gamma}{\lambda})^2 - (\tfrac{m}{\lambda})^2) - R_0^+(1-(\tfrac{m}{\lambda})^2)\|_{L^2_\sigma\to L^2_\sigma}.$$

Proposition 6.1 asserts the existence of constants $\delta > 0$ and $K < \infty$ such that if $0 \le \frac{\gamma}{\lambda}, \frac{m}{\lambda} < \delta$, then $\|R_0^+(1-(\frac{m}{\lambda})^2)\| \le K$ and $\|R_0((1+i\frac{\gamma}{\lambda})^2-(\frac{m}{\lambda})^2)-R_0^+(1-(\frac{m}{\lambda})^2)\| \le \frac{1}{10\|V\|C(V)}$. The latter inequality holds because both arguments in the free resolvent reside in a small neighborhood of 1. With the additional restrictions that $\delta < \min(1, \frac{1}{5K\|V\|C(V)})$, it follows that

$$\|\mathcal{R}_0(\lambda+i\gamma) - \mathcal{R}_0^+(\lambda)\|_{L^2_\sigma\to L^2_{-\sigma}} < \frac{1}{2\|V\|C(V)}.$$

Composing with pointwise multiplication by V completes the estimate for (6.1).

When $|\gamma| > \|V\|$, the perturbed resolvent can be estimated by much more elementary means. Here we can use self-adjointness of D_m to bound $\|\mathcal{R}_0(\lambda+i\gamma)\|_{L^2\to L^2} \le \frac{1}{|\gamma|}$, after which it follows that the Neumann series for $[I + V\mathcal{R}_0(\lambda+i\gamma)]^{-1}$ converges even in the space of bounded operators on L^2. One concludes that $\|\mathcal{R}(\lambda+i\gamma)\|_{L^2\to L^2} \le \frac{1}{|\gamma|(1-|\gamma|\,\|V\|)}$.

Within the cones $0 < |\gamma| < \delta|\lambda|$ with $|\lambda| > \lambda_1$, the perturbed resolvent can instead be bounded using (1.8) and the identity

$$\mathcal{R}(\lambda+i\gamma) = \mathcal{R}_0(\lambda+i\gamma) - \mathcal{R}_0(\lambda+i\gamma)V\mathcal{R}_0(\lambda+i\gamma) + \mathcal{R}_0(\lambda+i\gamma)V\mathcal{R}(\lambda+i\gamma)V\mathcal{R}_0(\lambda+i\gamma).$$

The first two terms are bounded operators on L^2 with norms less than $\frac{1}{|\gamma|}$ and $\frac{\|V\|}{|\gamma|^2}$ respectively. In the third term we use the decay of V to map L^2 to L^2_σ, or from $L^2_{-\sigma}$ to L^2, then the composition is again a bounded operator on L^2 with norm no greater than $\frac{C\|V\|^2}{|\gamma|^2}$.

Put together, the perturbed Dirac resolvent $\mathcal{R}(\lambda+i\gamma)$ exists as a bounded operator on L^2 whenever $|\gamma| > \|V\|$ or when $|\lambda| > \max(\lambda_1, \frac{\|V\|}{\delta})$ and $\gamma \ne 0$. □

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Department of Mathematics, University of Illinois, Urbana, IL 61801, U.S.A.
E-mail address: berdogan@math.uiuc.edu

Department of Mathematics, University of Cincinnati, Cincinnati, OH 45221 U.S.A.
E-mail address: goldbeml@ucmail.uc.edu

Department of Mathematics, Rose-Hulman Institute of Technology, Terre Haute, IN 47803, U.S.A.
E-mail address: green@rose-hulman.edu