SOLVING SYSTEMS OF DIFFERENTIAL EQUATIONS IN THE CASE OF A DEFECTIVE COEFFICIENT MATRIX

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Abstract. We provide a natural derivation and motivation of the form of solution to a system of differential equations with defective coefficient matrix. We avoid much of the technical linear algebra machinery and instead use a scalar integrating factor and convert to a higher order but easily solved system. From here, we derive the necessary form of solutions using calculus and basic matrix algebra.

When solving systems of differential equations in the case of a defective coefficient matrix, that is, when the $n \times n$ matrix does not have a set of n linearly independent eigenvectors, many texts use a largely unmotivated algorithmic approach involving finding generalized eigenvectors and multiplying by appropriate powers of t. Alternatively, some texts delve deep into theoretical linear algebra using the Jordan block form to find the form of solution. We propose a derivation that motivates the form of the solution without requiring theoretical linear algebra. Our goal is to use basic calculus and matrix algebra to provide insight into how and why the generalized eigenvectors and powers of t appear. We do this by reducing to the zero eigenvalue case and then solving a much simpler but higher order system.

We consider a system of n first order, constant coefficient, homogeneous differential equations in matrix form:

$$\vec{x}\,'(t) = A\vec{x}(t). \tag{1}$$

Here $\vec{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T$. The standard approach in an introductory systems of differential equations course, [1, 2, 3] is to solve the system of differential equations by finding the eigenvalues and eigenvectors of the coefficient matrix A. That is, if $A\vec{v} = \lambda\vec{v}$, then $\vec{x}(t) = e^{\lambda t}\vec{v}$ solves the system (1). This suffices to find all possible solutions, provided the $n \times n$ matrix A has n linearly independent eigenvectors. When the matrix does not have n linearly independent eigenvectors, the solution method is more complicated.

1. The basic set-up

To best illustrate the structure of solutions to a system with a defective matrix, we consider the case of a system of equations (1) where A is defective with a repeated eigenvalue λ and only a single eigenvector. We first consider the case when the repeated eigenvalue is zero. We later show that we can reduce to this case if $\lambda \neq 0$.

Iterating (1) once yields

$$\frac{d^2}{dt^2}\vec{x}(t) = \frac{d}{dt}\left(\frac{d}{dt}\vec{x}(t)\right) = \frac{d}{dt}\left(A\vec{x}(t)\right) = A^2\vec{x}(t).$$

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When A has only zero as an eigenvalue,[1] we have $A^n = 0$. We repeatedly differentiate until we arrive at

$$\frac{d^n}{dt^n}\vec{x}(t) = A^n\vec{x}(t) = \vec{0}. \tag{2}$$

This gives an n^{th} order system (2) such that solutions of (1) are also solutions of this new system. Further, system (2) is easy to solve by directly integrating n times. Hence, solutions to the n^{th} order equation (2) are polynomials of order at most $n-1$:

$$\vec{x}(t) = \vec{c}_0 + t\vec{c}_1 + t^2\vec{c}_2 + \cdots + t^{n-2}\vec{c}_{n-2} + t^{n-1}\vec{c}_{n-1}$$

where $\vec{c}_j$ may be any constant vectors. For example, $\vec{x} = \vec{c}_0$, $\vec{x} = \vec{c}_0 + t\vec{c}_1$, $\vec{x} = \vec{c}_0 + t\vec{c}_1 + t^2\vec{c}_2$ are solutions provided $n \geq 3$. This gives us the form of solutions to the original system (1) and motivates the presence of the powers of t. However, there are solutions to (2) that are not solutions to (1). Note that the solution set to (1) has dimension n, while the solution set of (2) is higher dimensional. To give a solution to (1), the vectors $\vec{c}_j$ cannot be entirely arbitrary. Moreover, we seek a full set of n linearly independent solutions to the first order system (1).

We now derive relationships the coefficient vectors in the polynomial solution to (2) must satisfy in order to solve the original system (1). For convenience, we divide by a factorial and re-write a representative solution as follows:

$$\vec{x}_k(t) = \frac{t^{k-1}}{(k-1)!}\vec{v}_1 + \cdots + \frac{t^j}{j!}\vec{v}_{k-j} + \cdots + t\vec{v}_{k-1} + \vec{v}_k = \sum_{j=0}^{k-1}\frac{t^j}{j!}\vec{v}_{k-j}, \tag{3}$$

for $1 \leq k \leq n$.

For $k = 1$: Taking $k = 1$ in (3), we define $\vec{x}_1(t) = \vec{v}_1$. By substituting this into (1), we see that if $\vec{x}_1(t)$ is a solution to (1), then $\vec{0} = \vec{x}_1{}'(t) = A\vec{v}_1$. That is, $\vec{v}_1$ is an eigenvector for A with eigenvalue $\lambda = 0$ as expected.

For $k = 2$: Taking $k = 2$ in (3), we define $\vec{x}_2(t) = t\vec{v}_1 + \vec{v}_2$. Again, substituting into (1) we have

$$\vec{v}_1 = \vec{x}_2{}'(t) = A(t\vec{v}_1 + \vec{v}_2) = t(A\vec{v}_1) + A\vec{v}_2.$$

By equating powers of t on each side, we see that if $\vec{x}_2(t)$ is a solution to (1), then $A\vec{v}_1 = \vec{0}$ and $A\vec{v}_2 = \vec{v}_1$.

For general k:

$$\vec{x}_k(t) = \frac{t^{k-1}}{(k-1)!}\vec{v}_1 + \frac{t^{k-2}}{(k-2)!}\vec{v}_2 + \cdots + t^1\vec{v}_{k-1} + \vec{v}_k = \sum_{j=0}^{k-1}\frac{t^j}{j!}\vec{v}_{k-j}. \tag{4}$$

[1]One can show if A has only zero eigenvalues then $A^n = 0$ using the fact that for any $\vec{x}$, the set $\{\vec{x}, A\vec{x}, A^2\vec{x}, \ldots, A^n\vec{x}\}$ is linearly dependent. The proof is not the focus of this article, we omit it for the sake of brevity. Alternatively, one can invoke the Cayley-Hamilton Theorem, see [2, Section 7.5].

Differentiating and substituting $\vec{x}_k(t)$ into (1) we arrive at:

$$\sum_{j=1}^{k-1} \frac{t^{j-1}}{(j-1)!}\vec{v}_{k-j} = \vec{x}_k{}'(t) = A\vec{x}_k(t) = \sum_{j=0}^{k-1} \frac{t^j}{j!} A\vec{v}_{k-j}.$$

By equating the coefficients of t^{k-1} on both sides we see that if $\vec{x}_k(t)$ is a solution to (1), then $A\vec{v}_1 = \vec{0}$. For t^{k-2}, $A\vec{v}_2 = \vec{v}_1$. This pattern continues, for the coefficients on $t^{k-\ell}$ (taking $j = \ell + 1$ in $\vec{x}_k{}'(t)$ and $j = \ell$ in $A\vec{x}_k(t)$ respectively), we must have $A\vec{v}_\ell = \vec{v}_{\ell-1}$ for each $1 \leq \ell \leq k$. In fact, repeated multiplication by A naturally derives "generalized eigenvector" equations, where for each $1 \leq \ell \leq k$,

$$A^m \vec{v}_\ell = \begin{cases} \vec{v}_{\ell-m} & m < \ell \\ \vec{0} & m \geq \ell \end{cases} . \tag{5}$$

Note these generalized eigenvector equations are necessary conditions. It is straightforward to check that under these conditions $\vec{x}_1(t), \vec{x}_2(t), \ldots, \vec{x}_n(t)$ are, in fact, solutions to (1).

To show that the solutions $\vec{x}_1(t), \vec{x}_2(t), \ldots, \vec{x}_n(t)$ are linearly independent we consider the equation

$$c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_{n-1}\vec{x}_{n-1}(t) + c_n\vec{x}_n(t) = \vec{0}.$$

We begin by looking at powers of t, starting with the largest. Since $\vec{x}_n(t)$ is the only term with a t^{n-1} we must have $c_n = 0$. Then $\vec{x}_{n-1}(t)$ is the only term with a t^{n-2}, hence $c_{n-1} = 0$. Continuing this process, we see that $c_j = 0$ for each j. Thus the solutions are linearly independent.

2. The general case

We now show that we can reduce to the case when the repeated eigenvalue is zero. We consider the scalar integrating factor $e^{-\lambda t}$, and note that if $\vec{y}(t)$ solves the system of equations $y'(t) = A\vec{y}(t)$, then $\vec{x}(t) = e^{-\lambda t}\vec{y}(t)$ solves the system $\vec{x}\,'(t) = (A - \lambda I)\vec{x}(t)$. Substituting $\vec{x}(t)$ into the system and using that $y'(t) = A\vec{y}(t)$ by assumption, we have

$$\begin{aligned} \vec{x}\,'(t) &= e^{-\lambda t}\vec{y}\,'(t) - \lambda e^{-\lambda t}\vec{y}(t) = e^{-\lambda t}A\vec{y}(t) - \lambda e^{-\lambda t}\vec{y}(t) \\ &= (A - \lambda I)e^{-\lambda t}\vec{y}(t) = (A - \lambda I)\vec{x}(t). \end{aligned}$$

Hence $\vec{x}(t)$ solves the system as claimed. Finally, since A has eigenvalue λ, the matrix $(A-\lambda I)$ has eigenvalue zero. Therefore it suffices to consider the case when $\lambda = 0$.

If a system of differential equations (1) has a defective coefficient matrix A, with repeated eigenvalue λ we need only "undo" the effect of the integrating factor. We multiply the form derived in the zero eigenvalue case by the exponential $e^{\lambda t}$. That is, solutions of $\vec{x}\,'(t) = A\vec{x}(t)$ take the form

$$\vec{x}_k(t) = e^{\lambda t} \sum_{j=0}^{k-1} \frac{t^j}{j!} \vec{v}_{k-j},$$

where $\vec{v}_1$ an eigenvector of A, and the remaining "generalized eigenvectors" $\vec{v}_2, \ldots, \vec{v}_k$ satisfy the relationship(s) in (5) for each $1 \leq \ell \leq k$. This process may be employed if A has distinct eigenvalues, though it requires a Jordan-block decomposition and conjugation.

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