MA 323 Geometric Modelling
Course Notes: Day 03
The Design Problem

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December 2nd, 2004

Yesterday, we introduced the model construction problem, and discussed two methods for creating curve models. Today, we introduce the design problem and apply the methods introduced yesterday to solve design problems. In principal, the model construction problem and the design problem look similar with slight differences. For instance, a solution to the model construction problem is algorithmic as it is a method for creating a model, while a solution to a design problem involves creating an actual model.

3.1 The Design Problem

In a design problem, a modeller is given information about the desired object and asked to create a model of the object. The difference between the design problem and the model construction problem is that the information that is provided in the design problem may not be in the form of geometric primitives (points). The information is typically in the form of design constraints. For instance, a typical design problem could be something like:

> Construct a closed non-self-intersecting curve that passes through the following points $\mathbf{q}_0$, $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, $\mathbf{q}_0$ in order, lies entirely within the rectangle $ABCD$, and contains the region R, (see diagram below).

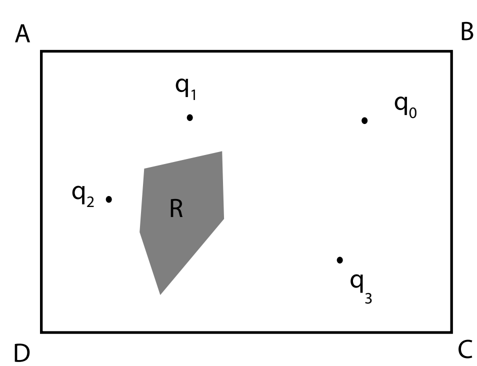

Figure 1: A Design Problem

In solving a design problem like the one above, a modeller is free to use any construction method or to combine various construction techniques to obtain a model. In fact, one can view the solution to the model construction problem as the specification of an algorithm or a method for constructing a model and the solution to a design problem as the application of a model construction method or methods to create a model.

A key difference between the model construction problem and the design problem is that in solving the design problem we get to choose the best construction method and the best geometric primitives. Some of the choice depends on the use of the model, and the method for implementing the model. These factors will be discussed in more detail later in the course. For the moment, we only have two construction methods available to use, one that creates a *smooth curve* and one that creates a *crooked (nonsmooth) curve*. Therefore, if we want a smooth curve we would use piecewise circular curves and if we do not want a smooth curve we would use a piecewise linear curves as they are easy to define. One could use the nonsmooth piecewise circular curves if one desired and if circumstances dictate that would yield a better result.

In the subsections below, we solve the design problem described above using piecewise linear curves and piecewise circular curves. When solving the problem, one should remember that we are free to choose the geometric primitives used in applying the construction algorithm. Moreover, it may be worthwhile to modify the algorithm by constructing specific geometric primitives to be used in the algorithm.

3.2 Solutions by Piecewise Linear Curves

Let's solve the design problem given above with piecewise linear curves. First notice, that connecting the points $\mathbf{q}_0$, $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ and $\mathbf{q}_0$ in order to obtain a closed non-self-intersecting curve, while remaining insider the quadrilateral $ABCD$, fails to contain the region R. We therefore need to use more points. For instance, adding additional points $\mathbf{q}_4$, $\mathbf{q}_5$ as indicated in the diagram below, we obtain a valid solution connecting the points in the order $\mathbf{q}_0$, $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_4$, $\mathbf{q}_5$, $\mathbf{q}_3$ and $\mathbf{q}_0$.

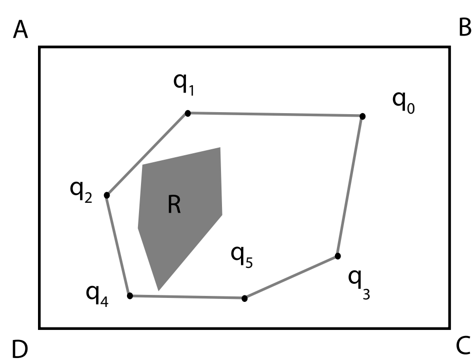

Figure 2: Solution to a Design Problem with Piecewise Linear Curves

This is only one possibly solution. There are an infinite number of possibly solutions corresponding to different choices of additional points, see diagrams below for additional solutions.

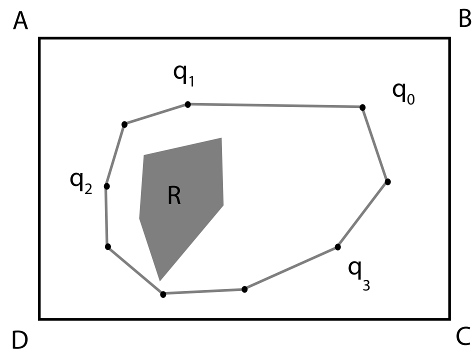

Figure 3: Another Solution with Piecewise Linear Curves

An obvious question should arise. Which of the many possible solutions is best? This is a model analysis problem. We need to quantify the measure of best. The measure could be aesthetic as in which solution yields the best looking curve. The measure could be minimization in terms of length or amount of information. The idea of constructing the best model is the hard problem in geometric modelling. We will discuss this problem in last subsection of this chapter when we discuss model analysis. Periodically, we revisit this notion of best solution throughout these notes.

3.3 Solution by Piecewise Circular Curves

In the previous subsection, we solved a design problem with piecewise linear curves, let's now solve it with piecewise circular curves. If we do not desire a globally smooth curve, the solution can be done in a similar manner to the solutions above using piecewise linear curves, see figure below. All that must be done is introduce additional points and then check that the resulting curve stays within the desired region.

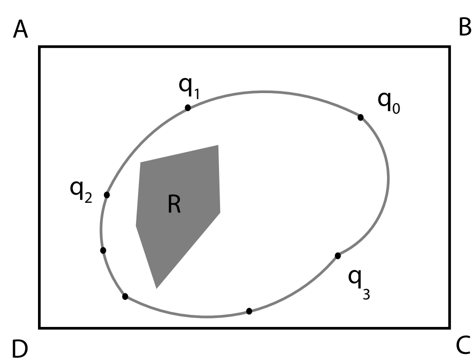

Figure 4: A Solution with a Nonsmooth Circular Curve

However, if we desire to use a smooth piecewise circular curve, we have a much harder problem. First notice, if we use $\mathbf{q}_0$, $\mathbf{q}_1$, $\mathbf{q}_2$ to define the first arc, and then continue the process described for constructing a smooth piecewise circular curve using the point $\mathbf{q}_3$. The second arc will contain the region R but will pass outside the quadrilateral $ABCD$, see diagram below. This problem in the solution can be solved easily by introducing additional points.

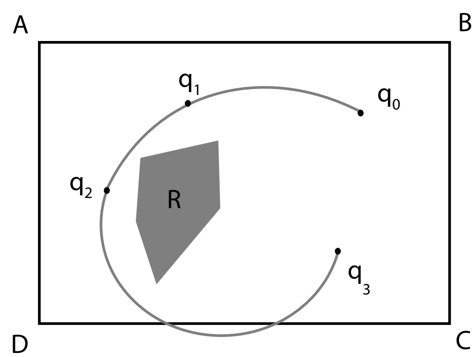

Figure 5: Problem with Solution with Smooth Piecewise Circular Curves

However, the more challenging problem with our current methods is to construct a smooth closed curve. One has to choose exactly the right additional points to arrange a closed circular curve to be smooth at the starting/ending point of the construction. In particular, using $\mathbf{p}_0, \mathbf{p}_1, \cdots, \mathbf{p}_m$ as the points and $\mathbf{l}$ as the initial tangent line, one needs to choose the last point $\mathbf{p}_m$ so that the tangent line of the arc constructed from $\mathbf{p}_{m-1}$ and $\mathbf{p}_m$ at $\mathbf{p}_m$ has tangent line $\mathbf{l}$. This is a severe limitation. We are restricting the point p_{m-1} from anywhere in a plane to lying on a specific curve. In fact, as we shall see, the point $\mathbf{p}_m$ must lie on a circle determined by $\mathbf{p}_0$, $\mathbf{p}_{m-1}$ and the tangent lines to the curve at $\mathbf{p}_0$ and $\mathbf{p}_{m-1}$. This is not optimal as the tangent line at $\mathbf{p}_{m-1}$ depends on all the previous points.

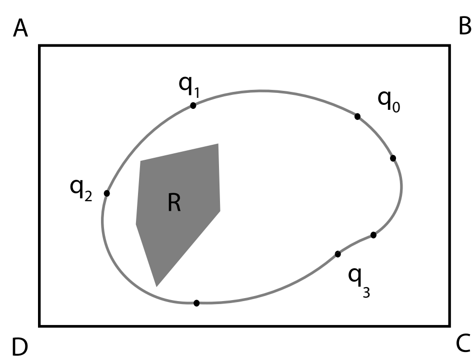

Figure 6: A Solution with Smooth Piecewise Circular Curves

An alternate method of solving such a design problem is modify the construction algorithm, by considering a different problem. Instead of constructing a piecewise circular curve by specifying points and using essentially a tangent line at the first point, consider the more symmetric problem of constructing a smooth circular curve that passes through points $\mathbf{p}_0$ and $\mathbf{p}_1$ and has prescribed tangent lines l_0 and l_1 at $\mathbf{p}_0$ and $\mathbf{p}_1$ respectively. This can be used to break our original problem into a bunch of smaller problems, as we can choose a tangent line at each of our original points (and each additional point if necessary). Once we have a construction method for this type of problem, we can solve the original problem by applying the same technique on the data sets $\{\mathbf{q}_0, l_0, \mathbf{q}_1, l_1\}$, $\{\mathbf{q}_1, l_1, \mathbf{q}_2, l_2\}$, $\{\mathbf{q}_2, l_2, \mathbf{q}_3, l_3\}$, and $\{\mathbf{q}_3, l_3, \mathbf{q}_0, l_0\}$, inserting additional points and tangent lines where necessary. The composite piecewise circular curve produced in this manner is smooth because we have common tangent line at point $\mathbf{q}_i$, and we will choose the orientations compatibly.

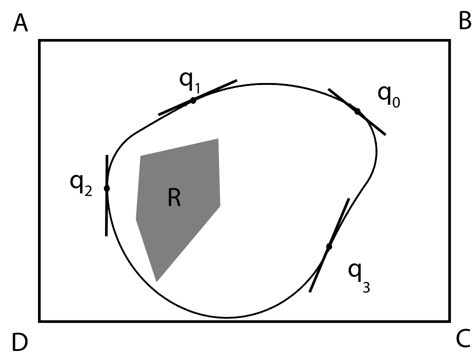

Figure 7: A Solution with a Smooth Circular Curve

The solution to our new problem, find a piecewise circular curve that passes through two given points with prescribed tangent lines, requires the construction of two circular arcs, and is called a *biarc*. The construction of a biarc relies on a modification of the construction of a smooth piecewise circular curve. One instance of a biarc can be constructed easily using simple geometric constructions, see below. However, there are other constructions of biarcs, see exercises. In particular, there is a one parameter family of biarcs based on the construction of a particular circle.

GEOMETRIC CONSTRUCTION OF A BIARC: To explain the construction of a biarc, consider the diagram below. The given information for the construction of a biarc is the points p_0 and p_1 and the tangent lines l_0 and l_1. We let Q be the midpoint of the line segment p_0p_1, and q be the desired joint point of the piecewise circular curve with q on the perpendicular bisector of p_0p_1. The points r_0 and r_1 are chosen to be on the tangent lines l_0 and l_1 such that the tangent line of the biarc at q is r_0r_1. We specify the biarc by providing a construction of the desired joint point q. This follows by some of the geometric properties of the construction of the requisite circles.

We first note that $|p_0r_0| = |r_0q|$ by the construction of a circle passing through p_0 and q with tangent l_0 at p_0. This means the triangle $\triangle p_0qr_0$ is an isosceles triangle so that $\angle r_0p_0q = \angle r_0qp_0$. For the same reason the triangle $\triangle p_1qr_1$ is an isosceles triangle and $\angle r_1p1q = \angle r_1qp_1$.

To derive the condition on the point q, we need to specify the angle $\phi = \angle qp_0Q = \angle qp_1Q$. First, we let θ_0 be the angle $\angle r_0p_0Q$ and $\theta 1$ be the angle $\angle r_1p_1Q$. Next, we note the angle $\angle p_1qQ = \angle p0qQ = \frac{\pi}{2} - \phi$ by the construction of the isosceles

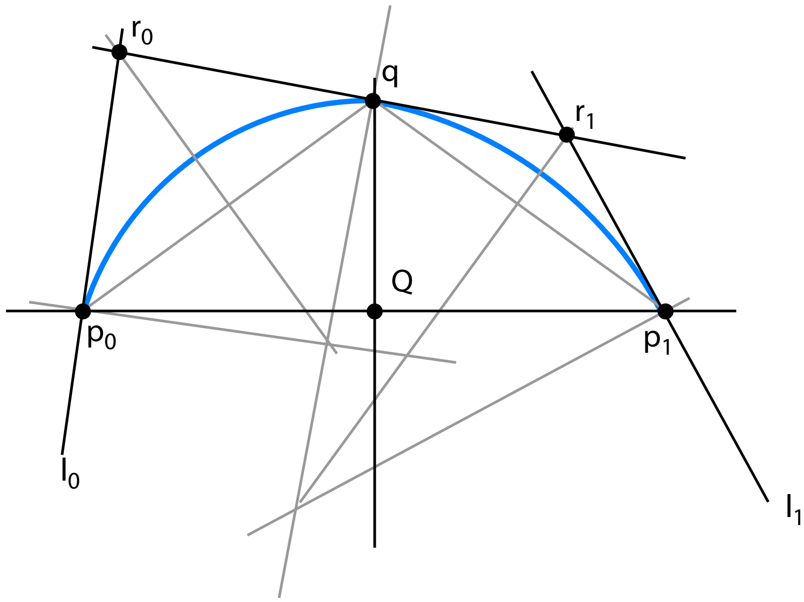

Figure 8: Construction of a Biarc

triangle $\triangle p_0 q p_1$. From the definitions of ϕ, θ_0, θ_1, and the arguments in the previous paragraph, we have $\angle r_0 q p_0 = \theta_0 - \phi$ and $\angle r_1 q p_1 = \theta_1 - \phi$. The angle sum

$$\angle r_0 q p_0 + \angle p_0 q Q + \angle Q q p_1 + \angle p_1 q r_1 = \pi$$

gives the equation

$$\theta_0 - \phi + \pi/2 - \phi + \pi/2 - \phi + \theta_1 - \phi = \pi$$

since $r_0 r_1$ is a straight line. This implies $\theta_0 + \theta_1 = 4\phi$ or $\phi = (\theta_0 + \theta_1)/4$.

In the above construction (diagram), we fixed the point $\mathbf{q}$ to lie on the perpendicular bisector of the segment $\mathbf{p}_0\mathbf{p}_1$. This is not necessary. We could use any point on the circle through $\mathbf{p}_0$, $\mathbf{q}$, $\mathbf{p}_1$. Showing this is left as a challenge exercise. The important observation from this is that in constructing biarc one has a degree of freedom. We note that one can easily modify the construction of a biarc to construct a triarc or other number of intermediate arcs to allow a larger degree of freedom for the individual curve segments. However, there is normally no reason for considering other arcs.

It is easy to solve the design problem using biarcs. The extra degrees of freedom that comes from choosing tangent lines at each of the given points allows us to easily construct a solution, see figure below. Of course, though the solution below only adds tangent lines, it may be necessary to also choose additional points and tangent lines.

3.4 Comments on solution to the design problem

Notice that in the solutions of the design problem provided above, the amount of information used to solve the problem is up to the designer. This is unlike in the construction problem, where the designer needs to supply a certain amount of information to construct the model specified by the construction method. This freedom allows one to consider the idea of a *best* solution. One of the designers problems is to decide what is the appropriate definition of *best solution* for the problem. The objective definition of best will depend on the circumstance that shaped the problem.

Typically, the concept of a best solution is mathematically expressed as a minimization problem. The definition of best in this sense then relies on function to be minimized, the so-called objective function. In this course, we will focus on problems where the objective function is defined geometrically; that is aesthetic objective functions that are not rooted in an application domain terminology. Sometimes we will use aesthetic objective functions in an informal sense, as which looks better, and sometimes we will consider aesthetic objective functions via calculation, minimizing length, area, curvature, et cetera.

Furthermore, notice that in the solution to the design problem considered we actually considered a new model creation algorithm. This is a feature of a modelling application. The problems dictate the methods used to solve them. Sometimes one can fit a previous model creation method to solve a different problem and other times one needs to consider a new model creation algorithm, possibly modifying a previous model creation algorithm.

Exercises

REWRITE EXERCISES

1. (Computational) Consider the design problem: construct a curve that passes through the points $[0, 0]$, $[0, 1]$, $[2, 2]$, $[3, 0]$ that is tangent to the line $y = (x + 3)/4$
   (a) Solve the design problem using a piecewise circular curve.
   (b) Determine the length of the curve.
   (c) Can you solve the problem with a piecewise circular curve or a piecewise linear curve with a shorter length. How?
2. (Computational) Consider the design problem: find a smooth closed curve that is tangent to each of the line segments in the figure below

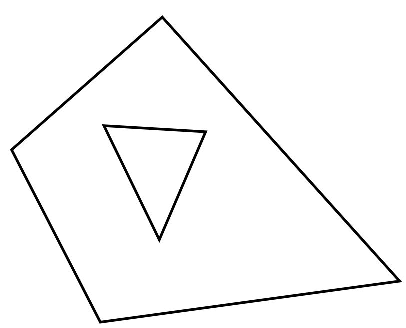

Figure 9: A Design Problem

   (a) Solve the problem using seven biarcs.
   (b) Solve the problem using any number of biarcs.

(c) Which of the two solutions is more aesthetically pleasing? Why?

3. (Algorithmic) Given points p_0, p_1, p_2 and tangent lines l_0 and l_2 at p_0 and p_2 respectively. Construct a piecewise circular curve that passes through the given points and has the given tangent lines. Your solution should consist of a deterministic method for constructing the curve. There should be no choosing of arbitrary points or lines. Any point or line needed must be constructed.

4. (Interactive) Complete the interactive exercises associated with the applet on the course web-page associated to BiArcs.

5. (Challenge) Given the points p_0, p_1 with tangent lines l_0 at p_0 and l_1 at p_1. Construct the point q as in the given construction of a biarc. Show that is possible to construct a biarc that passes through any point on the circle through p_0, q and p_1.