1. Consider a situation in which a consumer consumes only two goods, \( x_1 \) and \( x_2 \), and take prices as exogenously given. Suppose that the consumer has an initial endowment of these two goods equal to \((\omega_1, \omega_2) = (2, 1)\). Assume the price of good 1 is \( p_1 \) and the price of good 2, \( p_2 = 1 \). Finally, suppose that the consumer’s preferences are given by \( u^A(x_1, x_2) = x_1 x_2 \). (The superscript A is only to distinguish this consumer from a second consumer that will be introduced in problem #2. Hence, it is irrelevant for the time being.)

a. Write out the budget constraint of this consumer and solve the budget constraint to give \( x_2 \) as a function of \( x_1 \).

b. Substitute this function into the utility function to give utility as a function of \( x_1 \).

c. Find the value of \( x_1 \) that maximizes the consumer’s utility. (Solve for the consumer’s demand function for \( x_1 \), that is \( x_1 \) as a function of \( p_1 \).)

d. Substitute the demand function for \( x_1 \) into the function you found in part (a) to find the optimal value of that is \( x_2 \) as a function of \( p_1 \).

e. Find the value of \( p_1 \) that will make this consumer willing to consume her initial endowment.

f. Suppose that \( p_2 = 2 \), rather than 1. Repeat the analysis above for this consumer and comment on the result.

2. (Continuation of #1.) Suppose that in addition to the consumer described in #1 we also have another consumer whose preferences are given by the utility function

\[ u^B(x_1, x_2) = x_1^2 x_2. \]

Suppose further that this consumer has an initial endowment of these two goods equal to \((\omega_1, \omega_2) = (1, 3)\). As in the first parts of #1, assume that \( p_2 = 1 \).

a. Repeat the analysis of parts (a) through (d) in #1 to derive this new consumer’s demand functions for \( x_1 \) and \( x_2 \) as a function of \( p_1 \).

b. Find the market demand by adding together the individual demand functions. That is

\[
X_1(p_1) = x_1^A(p_1) + x_1^B(p_1) \\
X_2(p_1) = x_2^A(p_1) + x_2^B(p_1)
\]

c. Find the value of \( p_1 \) for which the market demand for good 1 is exactly equal to the total endowment of good 1.

d. Confirm that at this price the market demand for good 2 is also equal to the total endowment of good 2.

e. Comment on your results.

3. Two old friends, Archie and Barney, have found themselves as the only two consumers in a little exchange economy which has only two commodities – cheese and wine. Archie’s initial endowment of these two commodities is \((C_A, W_A) = (3,2)\), and Barney’s initial endowment is \((C_B, W_B) = (1,6)\). They have identical utility functions of the form \( U_i(C_i, W_i) = C_i^p W_i \), where the subscripts denote the individual, using A for Archie and B for Barney.

a. Modify the Edgeworth box below to illustrate this initial allocation, and to illustrate one indifference curve for each of them, passing through this initial allocation. Use blue ink for Archie’s indifference curve and red ink for Barney’s.

b. At any Pareto optimal allocation where both consume some of each good, their marginal rates of substitution must be equal. Write an expression that states this condition in terms of the consumption of each good by each person.

c. On the diagram, show the locus of points that are Pareto efficient (the contract curve).
d. In this example, at any Pareto efficient allocation, where both persons consume both goods, the slope of Archie's indifference curve will be ________. Therefore, since we know that competitive equilibrium must be Pareto efficient, we know that at a competitive equilibrium,

\[
\frac{p_C}{p_W} = \text{________________________}. 
\]

e. What are Archie's and Barney's consumption bundles in a competitive equilibrium?

____________________________________________________________________________
4. Consider a pure exchange economy with two agents, A and B, and two goods, x and y, where the utility functions of the agents are

\[ U_A(x,y) = x^{0.4}y^{0.6} \]
\[ U_B(x,y) = x^{0.6}y^{0.4} \]

And the initial endowments of the two goods are

\[ \omega_A(x,y) = (30, 20) \]
\[ \omega_B(x,y) = (20, 30) \]

a. Derive and describe the Pareto efficient equilibrium of this economy under these conditions.

b. Construct an Edgeworth box that illustrates your answer in part (a).

c. Derive and describe the Pareto efficient equilibrium of this economy if the only change to the above conditions is that A’s preferences are the same as B’s:

\[ U_A(x,y) = x^{0.6}y^{0.4} \]
\[ U_B(x,y) = x^{0.6}y^{0.4} \]

5. Charlotte and Wilber are two agents in a two-agent, two commodity pure exchange economy where apples and bananas are the two commodities. Charlotte loves apples and hates bananas. Her utility function is \( U_C(a, b) = a - \frac{1}{2}b^2 \), where \( a \) is the number of apples she consumes and \( b \) in the number of bananas she consumes. Wilber likes both apples and bananas. His utility function is \( U_W(a, b) = a + 2\sqrt{b} \). Charlotte has an initial endowment of no apples and 8 bananas. Wilber has an initial endowment of 16 apples and 8 bananas.

a. On the graph below, mark the initial endowment and label it E. Use red ink to draw the indifference curve for Charlotte that passes through this point. Use blue ink to draw the indifference curve for Wilbur that passes through this point.

b. If Charlotte hates bananas and Wilber likes them, how many bananas can Charlotte be consuming at a Pareto optimal allocation? On the diagram, use black ink to mark the locus of Pareto optimal allocations of apples and bananas between Charlotte and Wilbur.

c. We know that a competitive equilibrium allocation must be Pareto optimal and the total consumption of each good must equal the total supply, so we know that at a competitive equilibrium, Wilbur must be consuming how many bananas? \__________ \ If Wilbur is consuming this number of bananas, his marginal utility for bananas will be \__________ \ and his marginal utility of apples will be \__________ \. If apples are the numeraire, then the only price of bananas at which he will want to consume exactly 16 bananas is \__________ \. In competitive equilibrium for this Charlotte-Wilbur economy, Wilbur will consume \__________ \ bananas and \__________ \ apples, while Charlotte will consume \__________ \ bananas and \__________ \ apples.

d. This problem illustrates something about one of the two welfare theorems discussed on pages 602 – 606 of Varian’s *Intermediate Microeconomics*. What does it illustrate?
6. Suppose a production process is described by a Cobb-Douglas production function \( f(v_1, v_2) = v_1^{1/2}v_2^{3/2} \).

a. Write an expression for the marginal product of \( v_1 \). Does the marginal product of \( v_1 \) increase, decrease, remain constant for small increases in \( v_1 \), holding \( v_2 \) fixed?

b. Write an expression for the marginal product of \( v_2 \). Does the marginal product of \( v_2 \) increase, decrease, remain constant for small increases in \( v_2 \), holding \( v_1 \) fixed?

c. Does an increase in the amount of \( v_2 \) used in production increase, leave unchanged, or decrease the marginal product of \( v_1 \)?

d. Write an expression for the marginal rate of technical substitution (MRTS). Does this technology exhibit a diminishing MRTS?

e. What kind of returns to scale does this production function exhibit?

7. You manage a crew of 160 workers who could be assigned to make either of two products. Product A requires 2 workers per unit of output. Product B requires 4 workers per unit of output.

a. Write an equation to express the combinations of products A and B that could be produced using exactly 160 workers. On the diagram below, use blue ink to shade in the area depicting the combinations of A and B that could be produced with 160 workers. (Assume that it is also possible for some workers to do nothing at all.)

b. Suppose now that every unit of product A that is produced requires the use of 4 shovels as well as 2 workers and that every unit of products B requires 2 shovels and 4 workers. On the graph you have just modified, use red ink to shade in the area depicting combinations of A and B that could be produced with 180 shovels if there were no worries about the labor supply. Write down an equation for the set of combinations of A and B that require exactly 180 shovels.
c. If you have 160 workers and 180 shovels, what is the largest amount of product A that you could produce? If you produce this amount, you will not use your entire supply of one of the inputs. Which one? How many will be left unused?

8. Hydraulics Ltd. has designed a pipeline that provides a throughput of 70,000 gallons of water per 24-hour period. If the diameter of the pipeline were increased by 1 inch, throughput would increase by 4,000 gallons per day. Alternatively, throughput could be increased by 6,000 gallons per day using the original pipe diameter with pumps that had 100 more horsepower.

a. Estimate the marginal rate of technical substitution between pump horsepower and pipe diameter.

b. Provide a verbal interpretation of the MRTS: If pipe diameter were reduced by 1 inch, how much would horsepower have to be increased in order to keep output (or “throughput”) constant?

c. Assuming the cost of additional pump size is $600 per horsepower and the cost of larger diameter pipe is $200,000 per inch, does the original design exhibit the property required for optimal input combinations? If so, why? If not, how would you change it and why?

9. A firm uses a single input to produce a commodity according to the production function \( f(v) = v^{1/2} \). The commodity sells for $100 per unit. The production input costs $50 per unit.

a. Write out the firm’s profit function. What is the profit maximizing level of output? How many units of the input factor will the firm use at this level of output? How much profit does it make when it maximizes profits?

b. Suppose that the firm is taxed $20 per unit of its output and the price of its input is subsidized by $10. Revise the firm’s profit function to reflect these new realities. What is the profit maximizing level of output? How many units of the input factor will the firm use at this level of output? How much profit does it make when it maximizes profits?

c. Suppose that instead of the unit taxes and subsidies, the firm is taxed at 50% of its profits. Write out the firm’s after-tax profit function. What is the profit maximizing level of output? How many units of the input factor will the firm use at this level of output? How much profit does it make when it maximizes profits?

10. A production process is described by a Cobb-Douglas production function \( f(v_1, v_2) = v_1^{1/2}v_2^{1/2} \). The price of its output is 4. The price of the first input factor is \( w_1 \) and the price of the second input factor is \( w_2 \).

a. Write the two equations that say that the value of the marginal product of each factor is equal to its wage.

b. The two equations in part (a) imply that if \( w_1 = 2w_2 \), then the optimal input ratio, \( v_1/v_2 \), is _______.
11. Suppose that the production function for a competitive firm is \( Q = K^{0.5}L^{0.5} \). The firm sells its output at a price of $10, and can hire labor at a wage of $5. Capital is fixed at 25 units. What is the profit-maximizing quantity of labor?

12. Carefully modify the diagram below so that it provides a clear representation of a general equilibrium for a two-agent \((A, B)\), two-good \((x, y)\) economy with the following characteristics:

- The economy’s PPF is concave and passes through the points (0, 16), (10, 12), (15, 10), (18, 5) and (20, 0).
- \( p_y/p_x = -1 \)
- Agent A’s utility function is \( U_A(x, y) = x^{0.3}y^{0.7} \).
- Agent B’s utility function is \( U_B(x, y) = x^{0.7}y^{0.3} \).

Your modified diagram should include the following:

a. The PPF
b. A marginal rate of transformation line, the slope of which reflects the price ratio, and which is tangent to the PPF.
c. A contract curve tracing out the Pareto set and reflecting the Cobb-Douglas preferences given above.
d. A consumption equilibrium with \((x^A_1, x^A_2) = (5, 7)\) and \((x^B_1, x^B_2) = (10, 3)\).
e. Consumer indifference curves at equilibrium.
f. Consumer terms of trade (or budget) line.
g. Is the consumption equilibrium “fair” in Varian’s sense of the term (pp. 639 – 640)?
h. Show mathematically that a different consumption equilibrium, with \((x^A_1, x^A_2) = (3, 5)\) and \((x^B_1, x^B_2) = (12, 5)\) is not “fair” in Varian’s sense of the term.
13. Suppose you have a two consumer, two-commodity economy (goods \( x \) and \( y \)), that the economy’s PPF is given by \( y = 20 - 2x^2/2700 \), and that the consumers have identical utility functions \( U(x, y) = x^{0.5} y^{0.5} \).
Calculate the general equilibrium. Recall that the general equilibrium must satisfy the condition that 
\[
\text{MRS} = \frac{p_1}{p_2} = \text{MRT},
\]
and would specify production amounts for \( x \) and \( y \), as well as a price ratio.
(Hint: If consumers are identical, perhaps you can treat the problem as if there is just one consumer.)

14. Suppose the utility possibility frontier for two individuals is given by \( U_A + 2U_B = 200 \). On the diagram below, plot the frontier.

a. In order to maximize a “Nietzschean social welfare function”, 
\[ W(U_A, U_B) = \max \{ U_A, U_B \} \]
on the utility possibility frontier shown in the diagram, one would set \( U_A = \) ___________ and \( U_B = \) ___________.

b. If instead we use a Rawlsian criterion, 
\[ W(U_A, U_B) = \min \{ U_A, U_B \} \]
then the social welfare function is maximized on the diagram where \( U_A = \) ___________ and \( U_B = \) ___________.

c. Suppose that social welfare is given by 
\[ W(U_A, U_B) = U_A^{0.5} U_B^{0.5} \]. In this case, social welfare would be maximized when \( U_A = \) ___________ and \( U_B = \) ___________.
15. Roger and Gordon have identical utility functions, \( U(x, y) = x^2 + y^2 \). There are 10 units of \( x \) and 10 units of \( y \) to be divided between them.

   a. Draw an Edgeworth box showing some of their indifference curves (Roger = blue, Gordon = red) and mark the Pareto Optimal allocations with black ink. What is it about their preferences that generates a “peculiar” outcome?

   b. What are the fair allocations in this case?

16. Friedrich consumes economics books and other things. His utility function for economics books, \( x \), and money to spend on other things \( (m) \) is

   \[
   U(x, m) = 100x - x^2/2 + m
   \]

   a. What kind of utility function does Friedrich have? (i.e., what kind of preferences does he have?)

   b. What is his inverse demand function for economics books?

   c. If the price of economics books is 50, how many economics books will he consume?

   d. If the price of economics books is 80, how many economics books will he consume?

   e. Suppose that Friedrich has $4,000 in annual income. What is his total utility if the price of economics books is 50?

   f. Suppose that Friedrich still has $4,000 in annual income. What is his total utility if the price of economics books rises to 80?

   g. So … when the price of economics books rises from 50 to 80, Friedrich’s utility changes by what amount?

   h. What is the change in (net) consumer’s surplus when the price of economics books rises from 50 to 80?

17. Evaluate the welfare impact on a consumer with income of $40,000 of a price change in gasoline from $4 to $6.

   Consider two different assumptions about the consumer’s utility function:

   a. The utility function \( U(x_1, x_2) = x_1^{0.05} x_2^{0.95} \), where \( x_1 = \) gasoline \( x_2 = \) everything else, and \( p_2 = 1 \).

   b. The utility function \( U(x_1, m) = 14x_1 - 0.01x_1^2 + m \), where \( x_1 = \) gasoline \( x_2 = \) everything else, \( p_2 = 1 \), and \( m = \) income available for all purchases other than gasoline.

   In each case, calculate the optimal consumption bundle prior to the price change, the optimal consumption bundle after the price change, the compensating differential, the equivalent differential, and the change in consumer surplus.

18. The demand curve for ski lessons \( D(p) = 100 - 2p \), and the supply curve is given by \( S(p) = 3p \).

   a. What is the equilibrium price and quantity?

   b. A tax of $10 per ski lessons is imposed on consumers. Solve for the new price to skiers and the new equilibrium quantity of ski lessons.

   c. A senator from a mountainous state suggests that although consumers of ski lessons are rich and deserve to be taxed, ski instructors are poor and deserve a subsidy. He proposes a $6 subsidy on production of ski lessons while maintaining the $10 tax on purchases of ski lessons. Show that this policy would be no different than a tax of $4 without any subsidy.
19. Suppose that the price elasticity for oatmeal is constant and equal to $-1$. When the price of oatmeal is $10 per unit, the total amount demanded is 6,000 units.

   a. Write out an equation for the demand function. Graph this function below with blue ink.
   b. If the supply is perfectly inelastic at 5,000 units, what is the equilibrium price? Add the supply curve on your graph and label the equilibrium with an E.
   c. Suppose that the demand curve shifts outward by 10%. Write down the new demand function. Suppose further that supply increases by 5%. Solve for the new equilibrium price and quantity. Use red ink to draw the new demand curve and the new supply curve.
   d. By what percentage (approximately) did the equilibrium price rise?
   e. Write a general rule: With this kind of demand and supply, when demand shifts outward by $x\%$ and supply shifts outward by $y\%$, equilibrium price rises by approximately _______$\%$.

20. Suppose that market demand is $D(p) = 3,200 - 600p$ and $S(p) = 200p$. For each of the following levels of a unit tax, determine how much revenue the government will raise from the tax and the size of the dead weight loss of the tax.

   a. Tax = 1
   b. Tax = 3
   c. Tax = 5

   d. Compare the outcomes and summarize what you can say about the behavior of tax-generated revenue and deadweight losses as a function of the level of a tax.