A Homework Problem

Let A be the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_3$. We'll write things in multiplicative notation, elements $(p, q) \in A$ with $p = 1, -1$ (second roots of unity) and $q = 1, \omega, \omega^2$ with $\omega = e^{2\pi i/3}$ (third roots). Group operations are written multiplicatively with component by component multiplication, e.g., $(1, \omega) \cdot (-1, \omega) = (-1, \omega^2)$. In fact, let's name the elements of A as

$$\begin{aligned} a_1 &= (1,1) \\ a_2 &= (1,\omega) \\ a_3 &= (1,\omega^2) \\ a_4 &= (-1,1) \\ a_5 &= (-1,\omega) \\ a_6 &= (-1,\omega^2). \end{aligned}$$

Note a_1 is the identity element in A.

Here are some characters on A:

$$\begin{aligned} \chi_1(p,q) &= 1 \\ \chi_2(p,q) &= p \\ \chi_3(p,q) &= q \\ \chi_4(p,q) &= pq \\ \chi_5(p,q) &= q^2 \\ \chi_6(p,q) &= pq^2 \end{aligned}$$

In fact this is all of them. To see this, note that for any $\chi \in \widehat{A}$ we clearly need $\chi(-1, 1)$ to be a 2nd root of unity since $\chi((1, 1)^2) = \chi(1, 1) = 1$, so either $\chi(-1, 1) = 1$ or $\chi(-1, 1) = -1$ (two choices). Similarly we need $\chi(\omega, 1)^3 = 1$, so $\chi(\omega, 1) = 1$ or $\chi(\omega, 1) = \omega$ or $\chi(\omega, 1) = \omega^2$ (3 choices). And you can indeed verify that every one of these six choices is legit, and gives exactly the table of characters above. Note that the choice of $\chi(-1, 1)$ and $\chi(1, \omega)$ dictates the value of χ on all other elements of A, via $\chi(p, q) = \chi(p, 1)\chi(1, q)$.

Problems

1. Exhibit an explicit isomorphism from A to $\widehat{A}$.
2. Verify that $< \chi_2, \chi_2 >= 6$ and $< \chi_2, \chi_5 >= 0$, straight from the definition.

3. Let $f \in \ell^2(A)$ be the function with $f(a_1) = 2, f(a_2) = i, f(a_4) = -3$, and $f(a_k) = 0$ for $k = 3, 5, 6$. Compute $\widehat{f}(\chi)$ for each of the characters above. You can leave powers of ω in the answer (but note $\omega^3 = 1$!)

4. Recall that any character δ in the bidual $\widehat{\widehat{A}}$ is in fact of the form $\delta = \delta_a$, where δ_a is defined by
$$\delta_a(\chi) = \chi(a). \tag{1}$$
Let δ be the mapping from $\widehat{A}$ to $\mathbb{C}$ defined by $\delta(\chi_1) = 1, \delta(\chi_2) = 1, \delta(\chi_3) = \omega^2, \delta(\chi_4) = \omega, \delta(\chi_5) = \omega, \delta(\chi_6) = \omega$. Test a few (or all!) of the relations $\delta(\chi_j \chi_k) = \delta(\chi_j)\delta(\chi_k)$ to check that δ is in fact an element of the bidual. What element of A does δ correspond to in equation (1)?

5. For the δ in question (4), with f as in the question (3), compute $\widehat{\widehat{f}}(\delta)$. For real fun, compute this quantity cleverly AND from the definition!