Homework 2

MA430

As always, give full justification for your answers!

1. The Cauchy-Schwarz inequality states that $| < \mathbf{v}, \mathbf{w} > | \leq \|\mathbf{v}\| \|\mathbf{w}\|$ for any inner product and associated norm.

   (a) Suppose that in fact $\mathbf{v} = k\mathbf{w}$ for some constant k (so $\mathbf{v}$ and $\mathbf{w}$ are parallel). Show that equality is attained, i.e., $| < \mathbf{v}, \mathbf{w} > | = \|\mathbf{v}\| \|\mathbf{w}\|$.

   (b) Now for the converse: suppose that equality is attained in Cauchy-Schwarz. Show that $\mathbf{v} = k\mathbf{w}$ for some constant k. Follow these steps.

      i. Nothing to prove yet, just recall from Calc 3 that given any vectors $\mathbf{v}$ and $\mathbf{w}$ we can write

      $$\mathbf{v} = \mathbf{v}_{||} + \mathbf{r} \tag{1}$$

      where $\mathbf{v}_{||}$ is parallel to $\mathbf{w}$ and $\mathbf{r}$ is orthogonal to $\mathbf{w}$. In fact, this can be done by taking $\mathbf{v}_{||} = k\mathbf{w}$ where $k = < \mathbf{v}, \mathbf{w} > / \|\mathbf{w}\|^2$; then $\mathbf{r} = \mathbf{v} - k\mathbf{w}$ is orthogonal to $\mathbf{w}$ (easy to check).

      ii. Use (1) to show that

      $$\|\mathbf{v}\|^2 = k^2 \|\mathbf{w}\|^2 + \|\mathbf{r}\|^2. \tag{2}$$

      Hint: $\|\mathbf{v}\|^2 = < \mathbf{v}, \mathbf{v} >$.

      iii. Use (1) to show that

      $$| < \mathbf{v}, \mathbf{w} > | = |k| \|\mathbf{w}\|^2. \tag{3}$$

      iv. Use the assumption that $| < \mathbf{v}, \mathbf{w} > | = \|\mathbf{v}\| \|\mathbf{w}\|$ use (3) to show that $\|\mathbf{v}\| = |k| \|\mathbf{w}\|$ and then combine this with (2) to conclude that $\mathbf{r} = \mathbf{0}$. How does this show what we want?

2. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ where $\mathbf{v}_1 = (1, 1, 0), \mathbf{v}_2 = (-1, 1, 1)$, and $\mathbf{v}_3 = (1, -1, 2)$ are vectors in $\mathbb{R}^3$.

   a. Verify that S is orthogonal with respect to the usual inner product. This shows S must be a basis for $\mathbb{R}^3$.

b. Write the vector $\mathbf{w} = (3, 4, 5)$ as a linear combination of the basis vectors in S. Verify that the linear combination you obtain actually reproduces $\mathbf{w}$!

c. Rescale the vectors in S to unit length to produce an equivalent set S' of orthonormal vectors.

d. Write the vector $\mathbf{w} = (3, 4, 5)$ as a linear combination of the basis vectors in S'.

3. There are infinitely many other inner products on $\mathbb{R}^n$ besides the standard dot product, and they can be quite useful too.

   Let $\mathbf{d} = (d_1, d_2, \ldots, d_n) \in \mathbb{R}^n$. Suppose that $d_k > 0$ for $1 \leq k \leq n$.

   a. Let $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ be vectors in $\mathbb{R}^n$. Show that the function

   $$(\mathbf{v}, \mathbf{w})_d = \sum_{k=1}^{n} d_k v_k w_k$$

   defines an inner product on $\mathbb{R}^n$. Write out the corresponding norm.

   b. Let $\mathbf{d} = (1, 5)$ in $\mathbb{R}^2$, and let $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ with $\mathbf{v}_1 = (2, 1), \mathbf{v}_2 = (5, -2)$. Show that S is orthogonal with respect to the $(,)_d$ inner product.

   c. Find the length of each vector in S with respect to the norm induced by this inner product.

   d. Write the vector $\mathbf{w} = (-2, 5)$ as a linear combination of the basis vectors in S. Verify that the linear combination you obtain actually reproduces $\mathbf{w}$!

4. Suppose V is a vector space over $\mathbb{R}$ with an inner product and with a norm $\|\mathbf{v}\| = \sqrt{(\mathbf{v}, \mathbf{v})}$ that comes from the inner product.

   a. Show that this norm must satisfy the *parallelogram* identity

   $$2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2 = \|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2.$$

   b. Let $\mathbf{v} = (2, 5)$ and $\mathbf{w} = (3, 1)$. Compute each of $\|\mathbf{v}\|_\infty$, $\|\mathbf{w}\|_\infty$, $\|\mathbf{v} + \mathbf{w}\|_\infty$, and $\|\mathbf{v} - \mathbf{w}\|_\infty$, and verify that the parallelogram identity from part (a) does not hold. Hence the supremum norm $\|\cdot\|_\infty$ cannot come from an inner product. (Actually, almost any choice for $\mathbf{v}$ and $\mathbf{w}$ will work.)

c. Repeat part (b) with the ℓ^1 norm $\|\cdot\|_1$.

5. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 3 \\ -2 & -1 & 4 & 1 \end{bmatrix}$$

(a) Find a basis for $N(\mathbf{A})$. Use the Matlab `null` command.

(b) Find a basis for $R(\mathbf{A}^T)$. Use the Matlab `orth` command (it computes not just a basis for the column space, but an orthonormal basis!) Alternatively, note that the columns of $\mathbf{A}^T$ are themselves a basis for the column space!

(c) Verify that every basis vector for $N(\mathbf{A})$ is orthogonal to every basis vector for $R(\mathbf{A}^T)$. Thus any vector in $N(\mathbf{A})$ is orthogonal to every vector in $R(\mathbf{A}^T)$ and vice-versa, i.e., $N(\mathbf{A}) = R(\mathbf{A}^T)^\perp$.

6. Let V be the subspace of $\mathbb{R}^3$ with basis $\mathbf{a}_1 = (1,0,0), \mathbf{a}_2 = (0,1,0)$. Compute the projection of an arbitrary point (a,b,c) in $\mathbb{R}^3$ onto P. Then slap your head and say "doh"! Then project (a,b,c) onto $V^\perp$, the orthogonal complement of V. Slap your head again.

7. Repeat the last problem with subspace V spanned by $\mathbf{a}_1 = (1,1,3)$ and $\mathbf{a}_2 = (-1,0,4)$. Probably don't need to slap your head this time.

8. Find the least squares solutions $\mathbf{x}$ to the overdetermined system

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 2 & 3 & -4 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}.$$

9. Find the minimal norm solution to the underdetermined system

$$\begin{bmatrix} 1 & 2 & -1 & 0 & 4 \\ 0 & 2 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Verify that $\mathbf{x} = [0\ 1\ 0\ 0\ 0]^T$ is the sparsest possible solution to this system. Is it the minimal norm solution?