Homework 1

MA430

As always, give full justification for your answers!

1. Show that $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$ (assuming $\mathbf{A}$ is invertible). In particular, $\mathbf{A}^T$ is invertible if $\mathbf{A}$ is invertible. Hint: This is almost a one-liner, if you can see through the smoke. Here are the steps—you provide the justification for each step:

$$\begin{aligned} (\mathbf{A}^{-1})^T \mathbf{A}^T &= (\mathbf{A}\mathbf{A}^{-1})^T \\ &= \mathbf{I}^T \\ &= \mathbf{I}. \end{aligned}$$

   Therefore $(\mathbf{A}^T)^{-1}$ is $(\mathbf{A}^{-1})^T$.

2. Which of these are subspaces of $\mathbb{R}^3$? If the set is a subspace, prove it. If not, provide a counterexample.

   (a) The set of all vectors (x, y, z) with $x + y + z = 0$.

   (b) The set of all vectors (x, y, z) with $x + y + z^2 = 0$.

   (c) The set of all vectors (x, y, z) with $xyz = 0$.

   (d) The set of all vectors (x, y, z) with $y = z = 0$.

3. Find a basis for the subspace of $\mathbb{R}^4$ that consists of all vectors that satisfy

$$5x_1 - 2x_2 + x_3 + x_4 = 0.$$

   What is it's dimension? Prove it!

4. Let $\mathbf{A}$ be the matrix defined by

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 5 \\ -1 & 0 & 3 \\ -1 & 4 & 11 \end{bmatrix}.$$

   and let $\mathbf{b} =< b_1, b_2, b_3 >$. Perform Gaussian elimination (feel free to use Maple) to solve $\mathbf{A}\mathbf{x} = \mathbf{b}$ and find conditions on b_1, b_2, b_3 that guarantee consistency of the system, and so find which $\mathbf{b}$ are in the "range" of $\mathbf{A}$. Hint: this condition involves a linear combination of b_1, b_2, b_3. What does this subset of $\mathbb{R}^3$ look like, geometrically?

5. (a) Suppose $\mathbf{A}$ is a 3 by 7 matrix. Why must $\mathbf{A}$ have a non-trivial nullspace?

   (b) Suppose $\mathbf{A}$ is a 3 by 7 matrix and $\mathbf{A}\mathbf{x} = \mathbf{b}$ is not solvable for a certain vector $\mathbf{b}$ in $\mathbb{R}^3$. What can you say about the nullity of $\mathbf{A}$?

6. Find a basis for the column space, row space, and nullspace of the matrix defined by

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 1 & 1 & 2 & 0 & 3 \\ 3 & -1 & 2 & 4 & 5 \\ 5 & 0 & 5 & 5 & 10 \\ 4 & 2 & 6 & 2 & 10 \end{bmatrix}$$

   by emulating the techniques in the text. Use Maple or Matlab for Gaussian elimination! Verify that the Rank-Nullity Theorem holds here.

7. Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ are mutually orthogonal nonzero vectors in some vector space V (with an inner product, of course). Show that this set of vectors must be linearly independent. Hint: Start with

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_r\mathbf{v}_r = 0$$

   and then dot each side above with $\mathbf{v}_k$ for each $k = 1, 2, \ldots, r$. What can you conclude about each c_k?

8. Show that it is impossible to find a set of 100 mutually orthogonal (nonzero) vectors in $\mathbb{R}^{99}$. Hint: See last exercise.