Introduction to Vectors[1]

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There's going to be a race! Specifically, a triathlon. The first leg consists of a 2000 meter swim in the river pictured in Figure 1, beginning at the point labeled "A". To alleviate a crowding problem, half of the athletes will swim downstream 1000 meters to point B and then swim back upstream, returning to A. The other athletes will swim 1000 meters cross-current to point C and then return cross-current to A. The current in the river is a constant 0.5 meter per second to the left, as shown in Figure 1. All of the athletes are capable of swimming at a speed greater than this (relative to the surrounding water).

A question is raised concerning the fairness of this procedure. Does either group of athletes have an advantage?

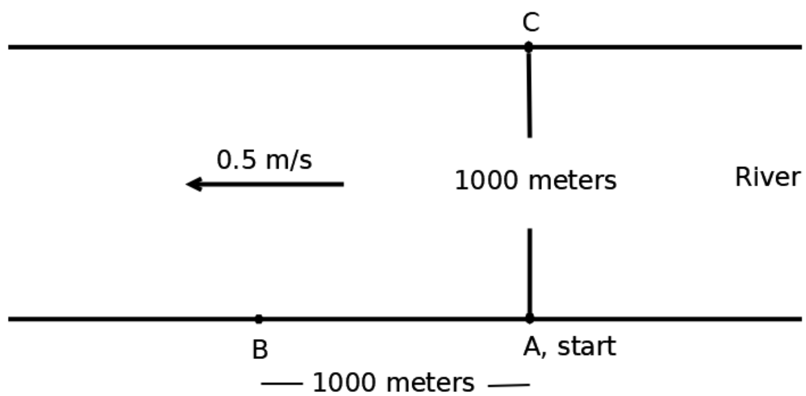

Figure 1: Triathlon setup.

Scalars and Vectors

Physical quantities like mass, speed, electrical charge, and length are called *scalar* quantities—they can be represented by a single number (with units). Other types of physical quantities cannot be represented by just one number. They have both a *magnitude* and a *direction* and are called *vector quantities*. For example, the velocity of an object cannot be expressed by a single number—you have to say not only how fast it's going, but what direction it's going.

One way to express velocity (in two dimensions) is to say how fast the object is going in both the x and y directions. Consider an object which is moving 2 meters per second in the positive x direction and 3 meters per second in the negative y direction. Let's call the velocity of the object $\vec{v}$; people often put arrows over quantities which a vectors. In this case we could write

$$\mathbf{v} =< 2, -3 >$$

to indicate that the object is moving at 2 m/s in the x direction, -3 m/s in the y direction. The boldface quantity $\mathbf{v}$ denotes a *vector*. The brackets $<>$ are common notation for vectors. You might

[1] based on a problem in "Problem Solving for Engineers and Scientists", by Raymond Friedman

also see them written as $\mathbf{v} = 2\vec{i} - 3\vec{j}$, especially in physics, or even as $\mathbf{v} = (2, -3)$ (like a point in the plane).

Geometrically, a velocity (or any vector) can be represented by an arrow. Suppose the object is at a given position some time. One second later the object will have moved 2 meters in the x direction, -3 in the y. We can represent the velocity geometrically by connecting the initial to the later position with an arrow which points in the direction of motion.

Exercises 1

- Given that speed is total distance traveled per unit time, what is the speed of an object with velocity $< 3, 8 >$?
- What is the speed of an object with velocity $< a, b >$?

Adding Velocities and Vectors

Consider a person swimming at velocity $\mathbf{v} =< 1, -2 >$ meters per second *relative to the surrounding water*. In other words, every second the swimmer moves 1 meter to the right and 2 meters down, relative to the water (which may itself be in motion relative to the land).

Exercises 2

- Suppose that the velocity of the water with *respect to the shore* is $\mathbf{w} =< 1, 1 >$ meters per second. What is the velocity of the swimmer with respect to the shore?
- More generally, suppose the swimmer has a velocity of $\mathbf{v} =< v_1, v_2 >$ meters per second *with respect to the water*, and the water is moving at $\mathbf{w} =< w_1, w_2 >$ meters per second *relative to the shore*. What is the velocity of the swimmer relative to the shore?

Swimming Time for the Down and Back Group

In analyzing the fairness of the race let's suppose, for simplicity, that a swimmer is capable of sustaining a speed of 1 meter per second relative to the water. NOTE that 1 m/s is a scalar, not a vector. The swimmer's direction has not been specified.

Exercises 3

- In the triathlon problem stated at the start of the handout, what is the velocity vector $\mathbf{w}$ of the water *with respect to the shore*?
- Suppose the swimmer moves 1 m/s relative to the water, in the negative x direction, heading from point A to B. What vector $\mathbf{v}$ describes the swimmer's velocity *relative to the water*? What vector describes the swimmer velocity *relative to the shore*?

  What is the swimmer's speed *relative to the shore*, and how long will it take to get from A to B?
- Repeat the previous part, but now assume that the swimmer is making the return trip, moving 1 m/s in the positive x direction, relative to the water.
- What is the total time for this swimmer to go from A to B and back?

Time for the Across and Back Group

Suppose now that the swimmer is heading from point A to point C. Again, the swimmer is capable of sustaining 1 m/s relative to the water and the water flows 0.5 m/s in the negative x direction *relative to the land.*

Exercises 4

- Suppose the swimmer's velocity $\mathbf{v}$ *relative the water* is 1 m/s straight up. what is his/her velocity relative to the shore? Will the swimmer reach point C?
- OK, so we'd better adjust. Suppose the swimmer adopts velocity $\mathbf{v} =< v_1, v_2 >$ relative to the water, and hence has velocity $\mathbf{v} + \mathbf{w}$ relative to the shore. We need to adjust v_1 and v_2 so the swimmer makes it to point C, while maintaining 1 m/s through the water.
  - Write out the vector $\mathbf{v} + \mathbf{w}$ explicitly (it involves unknowns v_1 and v_2). What condition(s) are required on v_1 and v_2 if the swimmer is to make it to point C, while maintain exactly 1 m/s relative to the water all the way?
- Having determined $\mathbf{v}$ (the swimmer's velocity relative to the water) now find the swimmer's velocity relative to the shore (and hence points A and C.) Find the speed of the swimmer and determine the time to cross from A to C.
- Determine the time to cross back to A from C.

Now that you've determined the "down and back" and "across and back" times for a swimmer capable of sustaining 1 m/s, you can answer the fairness question:

Exercises 5

- Does either group have an advantage?
- Repeat the analysis but assuming that the swimmer can swim at a speed of "a" m/s where a is some unspecified positive constant, and that the current in the river moves from right to left at "b" m/s (with $b > 0$). You should also assume that $a > b$ (why?) Are there any values of a and b for which the race is fair? Does one group ALWAYS have the advantage? Can you PROVE your conclusion for all values of a, b?