Unbounded Operators
MA 466
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Not all interesting linear operators on a Hilbert space are bounded—
differentiation is the “prime” example (ha ha–get it?) Let $T$ be a linear
operator with domain $D(T)$ where $D(T)$ is a subspace of a Hilbert space $H$.
We’ll also assume that $D(T)$ is dense in $H$, so that the closure of $D(T)$ is
all of $H$. In this case $T$ is said to be densely defined. Of course $T$ could be
bounded, but then it can be extended to a bounded operator on all of $H$,
(by continuity: if $x \in H$ but $x$ isn’t in $D(T)$, choose a sequence $x_n \in D(T)$
which converges to $x$, set $T(x) = \lim_n T(x_n)$; it’s well-defined). So we’ll stick
to the case in which $T$ is unbounded.

**Example 1:** Let $H = L^2(a, b)$ (with functions taking real values). Let $T$
denote differentiation with domain $D(T) = \{ f \in H : f \in C^1[0, 1] \}$. The
set $D(T)$ is dense in $H$. Also, $T$ is unbounded, for the sequence $f_n = \sin(nx)$
is bounded in $L^2(a, b)$, but $T(f_n) = n \cos(nx)$ is unbounded.

One can define the adjoint operator for an unbounded densely-defined
operator $T$. What we’d like that $T^*$ satisfy $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all
$x \in D(T)$ and all $y \in D(T^*)$. Before we define $T^*$ we should figure out what
to take for $D(T^*)$. In fact we’ll choose $D(T^*)$ to be all $y \in H$ such that the
linear functional $x \rightarrow \langle Tx, y \rangle$ is continuous for $x \in D(T)$ (this doesn’t say
what $D(T^*)$ is very explicitly though). Since $D(T)$ is dense this allows us
to extend the functional to all $x \in H$ as outline in the first paragraph. The
Riesz representation then dictates the existence of a unique $g \in H$ such that
$\langle Tx, y \rangle = \langle x, g \rangle$ for all $x \in H$. We define the adjoint as $T^*y = g$. This
uniquely determines $T^*$ on its domain.

**Example 2:** This continues Example 1. Let $g$ be a $C^1[0, 1]$ function (so
$g \in L^2(0, 1)$) with $g(0) = g(1) = 0$; let us use $V$ to denote this subspace of
$L^2(0, 1)$ (it’s also dense). Then

$$\langle Tf, g \rangle = \int_0^1 f'(x)g(x) \, dx.$$
Integrate by parts to find that
\[ \langle Tf, g \rangle = - \int_0^1 f(x)g'(x) \, dx. \]
If we take \( T^*(f) = -f' \), then we have \( \langle Tf, g \rangle = \langle f, T^*g \rangle \) for all \( f \in C^1[0,1] \) and \( g \in V \). The domain \( V \) may not be as large as possible (that is, it may be possible to enlarge the class of \( g \) for which we obtain \( \langle Tf, g \rangle = \langle f, T^*g \rangle \)), but in any case we’ve defined an adjoint for \( T \) on a dense subspace of \( H \).

An operator is said to be “symmetric” if \( \langle Tx, y \rangle = \langle x, Ty \rangle \) for all \( x, y \in D(T) \) (note that \( x \) AND \( y \) must both be in \( D(T) \)). If it’s the case that \( D(T^*) = D(T) \) then the operator is said to be self-adjoint.

Example 3: This is a small variation on Example 2. Let \( H = L^2(0,1) \) where the functions may take values in \( \mathbb{C} \). Let \( T \) be the operator \( T(f) = if' \) with domain \( D(T) = \{ f \in H : f \in C^1([0,1]), f(0) = f(1) \} \). The adjoint of \( T \) is defined much as in Example 2: Let \( g \) be a \( C^1([0,1]) \) function (so \( g \in L^2(0,1) \)) with \( g(0) = g(1) \) (same conditions as \( f \)). Then
\[ \langle Tf, g \rangle = \int_0^1 i f'(x) \overline{g(x)} \, dx. \]
Integrate by parts to find that
\[ \langle Tf, g \rangle = -i \int_0^1 f(x) \overline{g'(x)} \, dx = \langle f, Tg \rangle. \]
In other words, \( T \) is self-adjoint.

It turns out that one can define the spectrum of an unbounded operator (similar to bounded: if \( (T - \lambda I) \) is injective on \( D(T) \) and onto \( H \) with bounded inverse, then \( \lambda \) is said to be in the resolvent set of \( T \); the spectrum \( \sigma(T) \) consists of all \( \lambda \) NOT in the resolvent set). The spectrum of an unbounded operator is always closed, but may not be bounded, and MAY be empty.

In Example 3, with \( T(f) = if' \) with the requirement \( f(0) = f(1) \), it’s easy to see that the functions \( \phi_n(x) = \cos((\pi/2 + 2\pi n)x) \) are eigenfunctions with eigenvalues \( \lambda_n = \pi/2 + 2\pi n \) (which tend to infinity).