The Continuity Equation in Two Dimensions
Mathematical Modeling Week 8
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Flux Summary

We’ve seen that the total flux of a vector field \( \mathbf{q} \) over a curve \( S \) in the plane is given by the integral
\[
\int_S \mathbf{q} \cdot \mathbf{n} \, ds
\]
(1)

where \( ds \) is arc length and \( \mathbf{n} \) a unit normal vector on \( S \). The integral (1) measures net flux as positive in the direction \( \mathbf{n} \). If \( S \) is a closed curve (e.g., the boundary of some region \( D \)) then equation (1) measures the total outward flux of the vector field. We actually compute the flux by parameterizing \( S \) as
\[
x_1 = x_1(s), \quad x_2 = x_2(s)
\]
for \( a \leq s \leq b \) then
\[
ds = \sqrt{(x_1'(u))^2 + (x_2'(u))^2} \, du
\]
\[
\mathbf{n} = (x_2'(u)\mathbf{i} - x_1'(u)\mathbf{j}) / \frac{ds}{du}
\]
and the flux integral (1) becomes
\[
\int_a^b (q_1(x_1(u), x_2(u))x_2'(u) - q_2(x_1(u), x_2(u))x_1'(u)) \, du
\]
(2)

Exercise: Take \( S \) to be the unit circle and \( \mathbf{q}(x_1, x_2) = x_1\mathbf{i} + x_2\mathbf{j} \). Plot \( S \) and \( \mathbf{q} \) and use equation (2) to compute the flux across \( S \). Check the answer by using the obvious symmetries in the problem.

The Continuity Equation

Just as in one dimension, if \( \rho(x_1, x_2, t) \) is the planar density of the stuff then conservation implies a relation between \( \rho \) and \( \mathbf{q} \). As yesterday, I’ll suppress the dependence of \( \mathbf{q} \) on \( t \) and fill it in later. Suppose that \( D \) is some two-dimensional region (the control volume) and \( S \) is the boundary of \( D \). The total amount of stuff inside \( D \) is given by integrating the density \( \rho \) over \( D \),
\[
\int \int_D \rho(x_1, x_2, t) \, dx_1 \, dx_2.
\]
The rate at which the amount of stuff inside $D$ is changing is given by taking the time derivative of this integral,

$$\frac{\partial}{\partial t} \left( \int \int_D \rho(x_1, x_2, t) \, dx_1 \, dx_2 \right).$$

Just as in one dimension, if stuff is conserved inside $D$ then the rate at which this amount is changing should equal the rate at which the stuff is flowing across the boundary $S$, which is given by equation (1). Conservation of stuff then implies

$$\int \int_D \frac{\partial \rho}{\partial t}(x_1, x_2, t) \, dx_1 \, dx_2 = -\int_S \mathbf{q} \cdot \mathbf{n} \, ds. \quad (3)$$

Where we slipped the time derivative inside the integral on the left, permissible if $\rho$ is smooth enough. You should also contemplate why there’s a minus sign on the right.

Now according to the Divergence Theorem we have

$$\int_S \mathbf{q} \cdot \mathbf{n} \, ds = \int \int_D \left( \frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} \right) \, dx_1 \, dx_2. \quad (4)$$

Use this to replace the right side of equation (3) and collect all terms together on the left to obtain

$$\int \int_D \left( \frac{\partial \rho}{\partial t} + \frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} \right) \, dx_1 \, dx_2 = 0$$

over any region $D$ in the plane (at least, any region with a smooth enough boundary). What does this mean?

**Exercise:** If a continuous function $f(x_1, x_2)$ satisfies

$$\int \int_D f(x_1, x_2) \, dx_1 \, dx_2 = 0$$

for all regions $D$, argue that $f$ must be identically zero. Hint: if $f$ is continuous and NOT identically zero, you could find a small region $D$ over which $f$ doesn’t integrate to zero.
If you believe the statement in the exercise then you are forced to conclude
that
\[
\frac{\partial \rho}{\partial t} + \frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} = 0
\]
(5)
at each point \((x_1, x_2)\) (and each time \(t\)). This is the continuity equation in
two dimensions. Notice that if \(q\) were a function of only one variable \(x_1\) and
pointed only in the \(x_1\) direction (so \(q_2 \equiv 0\)) then equation (5) is exactly the
continuity equation from one dimension.

The divergence of a vector field and the continuity equation are often
written in a different form. The operator
\[
\nabla = \frac{\partial}{\partial x_1} \hat{i} + \frac{\partial}{\partial x_2} \hat{j}
\]
is called the “del” or “nabla” or “gradient” operator. It’s a vector-valued
differential operator. With this notation the divergence of a vector field \(q\)
can be written as \(\nabla \cdot q\), a dot product of two vectors. The continuity equation
becomes
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot q = 0.
\]
(6)

**Exercise:** Suppose that stuff isn’t conserved, so that the time rate of change
of the stuff inside a region \(D\) doesn’t necessarily equal the rate at which the
stuff flows across the boundary of \(D\). As in one dimension, this means that
stuff is created or destroyed in the region \(D\). Suppose that the rate at which
stuff is created or destroyed in \(D\) is given by an integral like
\[
\int \int_D k(x_1, x_2, t) \, dx_1 \, dx_2
\]
where \(k(x_1, x_2, t)\) is the “creation rate density”, i.e., measures how much stuff
is created per unit time per unit area. Go back over the derivation of the
continuity equation and figure out how \(k\) factors into the equation if stuff
isn’t conserved.

What is the physical significance of \(\nabla \cdot q\), the divergence? Consider a
steady-state (no time dependence) flow of some fluid in the plane. All par-
tials with respect to time are then zero. If you did the above exercise then
you can see that we must have \(\nabla \cdot q = k\); the divergence of \(q\) computed at a
point \(p\) measures the extent to which the flow involves creation or destruction
of material near the point \( p \).

**Exercises:**

1. Use Maple to plot the vector field \( \mathbf{q} = x_1 \mathbf{i} + x_2 \mathbf{j} \). It’s obvious that stuff is flowing away from \((0, 0)\) in every direction; “stuff” is being created in the vicinity of that point, so that the divergence of \( \mathbf{q} \) should be positive. Compute the divergence.

2. Use Maple to plot the vector field \( \mathbf{q} = 2\mathbf{i} - 3\mathbf{j} \). Imagine this as governing the steady-state mass-flow of some fluid across the plane—would the fluid be conserved? What does this say about the divergence of \( \mathbf{q} \)? Compute the divergence.