1 History and Background

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   - Attenuation of x-rays
   - Geometry
   - The Radon Transform and Sinogram

3 Inverting the Radon Transform
   - Unfiltered Backprojection
   - Filtered Backprojection
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First CT Scanners

First practical scanners built in the late 1960’s.
First CT Scanners

Images took hours to process/render, and were crude:

![CT Scan Image](image-url)
Modern CT Scanners

Modern scanners are fast and high-resolution:
The mathematics underlying the model for a CT scanner is much older.

- Based on the Radon (and Fourier) transforms, dating back the early 20th century (and farther).
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- Based on the Radon (and Fourier) transforms, dating back the early 20th century (and farther).
- Most of it is easy enough to do in a Calc 2 class!
Mathematical Model

We fire x-rays through a body at many angles and offsets, measure beam attenuation (output/input intensity):
Attenuation of x-rays

- Suppose $L(s)$, $a \leq s \leq b$ parameterizes a line with respect to arc length.
Attenuation of x-rays

- Suppose \( L(s), \ a \leq s \leq b \) parameterizes a line with respect to arc length.
- Let \( I(s) \) be the intensity of the x-ray along \( L \), with \( I(a) = I_a \) (known input intensity).
Suppose $L(s)$, $a \leq s \leq b$ parameterizes a line with respect to arc length.

Let $I(s)$ be the intensity of the x-ray along $L$, with $I(a) = I_a$ (known input intensity).

We suppose the x-ray beam is attenuated according to

$$I'(s) = -\lambda(L(s))I(s)$$

as it passes through the body. The function $\lambda$ is called the attenuation coefficient. We want to find $\lambda$. 
Attenuation of x-rays

\[ I'(s) = -\lambda(L(s)) \cdot I(s) \text{ with } I(a) = I_a \text{ known.} \]

We measure the output \( I(b) \).
Solving the Attenuation DE

The DE \( l'(s) = -\lambda(L(s))l(s) \) with \( l(a) = l_a \) is easy to solve via separation of variables. We find

\[
l(s) = l_a \exp\left(-\int_a^s \lambda(L(t)) \, dt\right).
\]
The DE $I'(s) = -\lambda(L(s))I(s)$ with $I(a) = I_a$ is easy to solve via separation of variables. We find

$$I(s) = I_a \exp\left(-\int_a^s \lambda(L(t)) \, dt\right).$$

If we know (measure) $I(b)$ then we can compute

$$\int_a^b \lambda(L(t)) \, dt = -\ln(I(b)/I(a)).$$

We can find the integral on the left, for any line through the body.
Attenuation Example

Some line integrals:

![Diagram showing line integrals with values 0.23, 0.92, 0.71, and 2.92]
Geometry and Notation

Suppose $L(s) = p + sn^\perp$, where

- $n = \langle \cos(\theta), \sin(\theta) \rangle$ dictates line normal vector, $\theta \in [0, \pi)$.
- $p = rn$, $r \in (-1, 1)$ is “offset” from the origin.

Note $-\sqrt{1 - r^2} < s < \sqrt{1 - r^2}$. 
The Radon Transform

In summary, by firing x-rays through the body, we can compute the integral

\[ d(r, \theta) = \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} \lambda(L(s)) \, ds \]

for \( 0 \leq \theta < \pi, -1 < r < 1 \).

The quantity \( d(r, \theta) \) is called the “Radon Transform” of \( \lambda \).
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Is this enough to determine \( \lambda? \) How?
The Sinogram

CT target and its sinogram:
The Sinogram

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CT target and its sinogram:
Observation: Every x-ray through a high attenuation region will yield a large line integral.
For any fixed point \((x_0, y_0)\) in the body, the line \(L(s)\) with normal at angle \(\theta\) is given non-parametrically by

\[ x \cos(\theta) + y \sin(\theta) = r \]

with \(r = x_0 \cos(\theta) + y_0 \sin(\theta)\):
Each point on the curve \( r = x_0 \cos(\theta) + y_0 \sin(\theta) \) in the sinogram corresponds to a line through \((x_0, y_0)\) in the target.
The average value of the Radon transform $d(\theta, r)$ over all lines through $(x_0, y_0)$ is then

$$\tilde{\lambda}(x_0, y_0) = \int_{0}^{\pi} d(\theta, x_0 \cos(\theta) + y_0 \sin(\theta)) \, d\theta.$$ 

This is called the backprojection of $d(\theta, r)$. 

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Kurt Bryan
Inverse Problems 4: The Mathematics of CT Scanners
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Unfiltered Backprojection Example 1
Unfiltered Backprojection Example 2

![Unfiltered Backprojection Example 2](image_url)
Unfiltered Backprojection Example 3
Unfiltered Backprojection is Blurry

- Straight backprojection ("unfiltered" backprojection) gives slightly blurry reconstructions.
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- Straight backprojection ("unfiltered" backprojection) gives slightly blurry reconstructions.
- Unfiltered backprojection is only an approximate inverse for the Radon transform.
- There’s another step needed to compute the true inverse (and get sharper images).
Filtered Backprojection

If $d(\theta, r)$ is the “raw” sinogram, first construct $\tilde{d}(\theta, r)$ by expanding into a Fourier series with respect to $r$:

$$d(\theta, r) = \sum_{k=-\infty}^{\infty} c_k e^{i\pi kr} \quad \text{with} \quad c_k = \int_{-1}^{1} d(\theta, r)e^{-i\pi kr} \, dr,$$

then

\[1\] in the continuous case, a Fourier integral transform
If $d(\theta, r)$ is the “raw” sinogram, first construct $\tilde{d}(\theta, r)$ by expanding into a Fourier series\(^1\) with respect to $r$:

$$d(\theta, r) = \sum_{k=-\infty}^{\infty} c_k e^{i\pi kr} \text{ with } c_k = \int_{-1}^{1} d(\theta, r)e^{-i\pi kr} \, dr,$$

then set

$$\tilde{d}(\theta, r) \sum_{k=-\infty}^{\infty} |k| c_k e^{i\pi kr}.$$

In signal processing terms, we apply a high-pass “ramp” filter to $d$, in the $r$ variable. Finally, backproject.

\(^1\)in the continuous case, a Fourier integral transform
Filtered Backprojection Example 1
Filtered Backprojection Example 2
Filtered Backprojection Example 3
Suppose $\lambda$ depends only distance from the origin, so $\lambda = \lambda(\sqrt{x^2 + y^2})$: 
The Radial Case

In this case it’s easy to see that the Radon transform depends only on $r$, not $\theta$. For a line at distance $r$ from the origin

$$d(r) = \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} \lambda(\sqrt{r^2 + s^2}) ds = 2 \int_{0}^{\sqrt{1-r^2}} \lambda(\sqrt{r^2 + s^2}) ds.$$
In summary, if we are given the function $d(r)$ for $0 \leq r \leq 1$

$$d(r) = 2 \int_0^{\sqrt{1-r^2}} \lambda(\sqrt{r^2 + s^2}) \, ds$$

can we find the function $\lambda$?
In summary, if we are given the function $d(r)$ for $0 \leq r \leq 1$

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can we find the function $\lambda$?

With a couple change of variables, this integral equation can be massaged into a “well-known” integral equation.
The Radial Case

Start with

\[ d(r) = 2 \int_{0}^{\sqrt{1-r^2}} \lambda(\sqrt{r^2 + s^2}) \, ds. \]
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\[ d(r) = 2 \int_0^{\sqrt{1-r^2}} \lambda(\sqrt{r^2 + s^2}) \, ds. \]

Make \( u \)-substitution \( u = \sqrt{r^2 + s^2}, \) (so \( s = \sqrt{u^2 - r^2}, \) and \( ds = u \, du/\sqrt{u^2 - r^2} \)), obtain

\[ d(r) = 2 \int_r^1 \frac{u \lambda(u)}{\sqrt{u^2 - r^2}} \, du. \]
The Radial Case

Rewrite as

\[ d(r) = 2 \int_{r}^{1} \frac{u \lambda(u)}{\sqrt{u^2 - r^2}} \, du \]

\[ = 2 \int_{r}^{1} \frac{u \lambda(u)}{\sqrt{(1 - r^2) - (1 - u^2)}} \, du. \]
The Radial Case

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\[ = 2 \int_r^1 \frac{u \lambda(u)}{\sqrt{(1 - r^2) - (1 - u^2)}} \, du. \]

Define \( z = 1 - r^2 \) (so \( r = \sqrt{1 - z} \)), substitute \( t = 1 - u^2 \) (so \( u = \sqrt{1 - t} \), \( du = -\frac{1}{2\sqrt{1-t}} \, dt \)) to find

\[ d(\sqrt{1 - z}) = \int_0^z \frac{\lambda(\sqrt{1 - t})}{\sqrt{z - t}} \, dt. \]
Inverting the Radon Transform

Unfiltered Backprojection

Filtered Backprojection

An Easy Special Case

The Radial Case

\[ d(\sqrt{1-z}) = \int_0^z \frac{\lambda(\sqrt{1-t})}{\sqrt{z-t}} \, dt \]

define \( g(z) = d(\sqrt{1-z}) \) and \( \phi(t) = \lambda(\sqrt{1-t}) \). We obtain

\[ \int_0^z \frac{\phi(t)}{\sqrt{z-t}} \, dt = g(z) \]

known as Abel’s equation. It has a closed-form solution!
The solution to
\[ \int_0^z \frac{\phi(t)}{\sqrt{z-t}} \, dt = g(z) \]
is
\[ \phi(t) = \frac{1}{\pi} \frac{d}{dt} \left( \int_0^t \frac{g(w) \, dw}{\sqrt{t-w}} \right). \]
(Recall \( g(w) = d(\sqrt{1-w}) \)). We solve for \( \phi(t) \) and recover \( \lambda(r) = \phi(1-r^2) \).
Suppose $d(r) = \frac{\sqrt{1-r^2}}{3} (14 + 4r^2)$. Then

$$g(z) = d(\sqrt{1-z}) = \frac{\sqrt{z}}{3} (18 - 4z)$$

and

$$\phi(t) = \frac{1}{\pi} \frac{d}{dt} \left( \int_0^t \frac{g(w) \, dw}{\sqrt{t-w}} \right) = 3 - t.$$ 

Finally

$$\lambda(r) = \phi(1 - r^2) = 2 + r^2.$$
www.rose-hulman.edu/~bryan/invprobs.html