Review

Remember from last week, a whole number is said to be prime if it cannot be divided by any other whole number, except itself and 1, no remainders or fractions allowed! If a number is not prime, it is said to be composite.

Remember also that last week we used the Sieve of Eratosthenes to generate a list of all the primes up to 100. They are


The number 1 is not considered to be a prime!

Interesting Facts about Primes

1. Euclid proved (2300 years ago) that there is no largest prime number; they go on and on.

2. The largest prime yet discovered has 9,808,358 digits; just writing it out would take over 100 pages! It was found on September 4, 2006, just a couple months ago. There is a prize of $100,000 for the first prime that has over ten million digits, so this last one was just short of the prize. But a new largest prime is found every few months, so someone will soon claim the prize.

3. Between 1 and 100 there are 25 primes (count them in the list above). But as you go farther out in the integers, the primes become more and more scarce, though they never run out. Between 1,000,001 and 1,000,100 there are only 6 primes, namely

\[1,000,003, \ 1,000,033, \ 1,000,037, \ 1,000,039, \ 1,000,081, \ 1,000,099.\]

Between 1,000,000,001 and 1,000,000,100 there are only 4 primes, namely

\[1,000,000,000,003, \ 1,000,000,000,037, \ 1,000,000,000,061, \ 1,000,000,000,063, \ 1,000,000,000,091.\]

4. A pair of primes that differ by 2 are called twin primes, like 29 and 31, or 1,000,037 and 1,000,039. No one knows if there is a largest pair of twin primes.

5. It seems that every even number can be written as a sum of two primes, usually in more than one way. For example, \(80 = 7 + 73\) (also \(80 = 13 + 67 = 19 + 61 = 37 + 43\)). This is called “Goldbach’s Conjecture”. Using computers, mathematicians have checked that every even number up to 300,000,000,000,000,000 can be written as a sum of two primes! But no one has proved that it is always possible.

Activities

1. **Factoring**: Start with the number I give you. Divide it by 2, if possible. Take what’s left (the “quotient”) and try dividing it by 2 again, and repeat until you can no longer divide by 2. Keep a record of how many times you were able to divide by 2.
Now take what’s left and try dividing it by 3, over and over, until you can’t divide by 3 anymore; keep track of how many times you can divide by 3. Repeat this procedure with 5, 7, 11, and all the others primes (use our list of primes). Keep going until you have “divided it down” to 1.

You’ve now factored the original number, that is, figured out how to write it as a product of prime numbers. This is called the “prime factorization” of the number. Write your prime factorization below:

The Fundamental Theorem of Arithmetic states that every whole number can be factored into prime numbers, and in only one way (aside from the order in which you write the primes). So for example, we can write

$$868 = 2 \times 2 \times 7 \times 31$$

and in any order ($868 = 2 \times 7 \times 31 \times 2$, for example), but we will never by able to factor 868 into a different set of primes; there will always be two 2’s, a 7, and a 31.

Notice the numbers which are “close” (even differ by 1) may have completely different prime factorizations.

2. Factoring is Hard: Factoring a number into its prime pieces is a LOT harder than going the other way (multiplying primes together to construct a composite number).

(a) To see this, find a partner. Choose two primes from our list (pick two primes between, say, 23 and 97) and multiply them together. Don’t let your partner know which primes you used. Write the product of your primes on a piece of paper and hand it to your partner. You partner should do the same and hand a number to you.

(b) Take the number your partner gives you and try to figure out which primes were multiplied together to produce the number.

Not so easy, is it?

3. Euclid’s Proof: Pick a number that’s a multiple of 2 (that is, an even number). Add one to your number. Is the result divisible by 2?

Pick a number that’s a multiple of 3. Add one to your number. Is the result divisible by 3?

Pick a number that’s a multiple of 4. Add one to your number. Is the result divisible by 4?

Do you see that this will always work? If you understand this, you can understand Euclid’s proof that there is no largest prime. Here’s how it goes. Suppose you thought there is a largest prime—say, for example 103. You are mistaken. To see this, multiply together all the primes up to and including 103, then add 1. The number is

$$2 \times 3 \times 5 \times 7 \times \cdots \times 101 \times 103 + 1.$$ 

There are two possibilities: this number is prime or it is composite. If it’s prime then 103 was NOT the largest prime (since the number above is larger than 103). On the other hand, if the number is composite then it can be divided by some prime. But the number isn’t divisible by any prime up to 103. So there must be some prime bigger than 103!

Of course, if you replace 103 by whatever you think is the largest prime, the argument above shows that you were wrong—there is no largest prime!