Kaleidoscopic Tilings on Surfaces, Group Algebras and Separability

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Credits

• Some of this work has been done and will be done with numerous undergraduates
Talk 2 Outline

- Groups again and group actions
- Group algebras
- Tilings and tiling groups again
- Chains, boundaries, and homology
- Separability
Groups and group action
Definition for these talks

• A group $G$ is set of transformations of a set or space $X$, such that $G$ is closed under composition and taking of inverses.

• If $g, h \in G$ then $gh = g \circ h \in G$ and $g^{-1} \in G$

• For $x \in X$ and $g \in G$ define the action of $g$ on $x$ by $gx = g(x)$.

• Note that $(gh)x = g \circ h(x) = g(hx)$.

• Typically a group will preserve some property of the space – e.g., distance preserving or rigid motion.

• Subgroups are subsets of $G$ closed under compositions and taking inverses.
Groups and group actions
Examples

- $X = \{1,2,3\ldots,n\}$ and $G$ is a group of permutations of $X$.
- $X$ is a regular $n$-gon and $G = D_n$ is the dihedral group of symmetries of $X$.
- $X$ is tiled surface and $G$ and $G^*$ are the tiling groups acting on $X$ as geometric transformations.
- $G$ acts on subsets of $X$, e.g. $k$-subsets, vertices, edges, tiles, faces, lines, or circles.
- The tiling groups act as permutation groups on vertices edges and tiles.
Group algebras

- Let $k$ be a field and $G$ a finite group.
- Group algebra $k[G]$ is the set of formal $k$-sums of element of $G$.
  \[ k[G] = \{ \sum_{g \in G} a_g g \} \]
- Let $\alpha = \sum_{g \in G} a_g g$, $\beta = \sum_{g \in G} b_g g$
- Sums: $\alpha + \beta = \sum_{g \in G} (a_g + b_g) g$
- Products:
  \[ \alpha \beta = (\sum_{g \in G} a_g g) (\sum_{h \in G} b_h h) = \sum_{g \in G} \sum_{h \in G} (a_g b_h g h) \]
Group algebra - sample calculations

- \( G = \{ 1, g, g^2 \} = \text{cyclic group of order 3} \).
- \( \alpha = 1 + g + g^2, \beta = 1 - g \).
- \( \alpha \beta = 1 + g + g^2 - 1g - gg - g^2g \)
  \[ = 1 + g + g^2 - g - g^2 - 1 = 0 \]
- Set \( \varepsilon = (1 + g + g^2)/3, \alpha = a\,1 + b\,g + c\,g^2 \)
- Then \( \alpha \, \varepsilon = \varepsilon \, \alpha = (a + b + c) \, \varepsilon \)
Group algebra – action modules

- $X$ be a set on which $G$ acts
- $k[X] = \{ \Sigma_{x \in X} a_x x \}$ is a $k$-vector space
- $k[X]$ is a $k[G]$ module
- Let $\alpha = \Sigma_{g \in G} a_g g$, $\nu = \Sigma_{x \in X} b_x x$
- The $\alpha \nu = (\Sigma_{g \in G} a_g g) (\Sigma_{x \in X} b_x x)$
  $= \Sigma_{g \in G} \Sigma_{x \in X} (a_g b_x g x)$
- $(gh)x = g(hx)$ implies $(\alpha \beta)\nu = \alpha (\beta \nu)$
(2,3,5) – tiling – soccer ball
(2,4,4) - tiling of the torus
Separability

- A reflection in a tiling is separating if the mirror of the reflection separates the tiling into two pieces.
- Problem: determine from the tiling group whether a reflection is separating
The tiling group - 1

Full Tiling Group for triangle (a finite group)

\[ G^* = \langle p, q, r \rangle \]

Group Relations

\[ p^2 = q^2 = r^2 = 1. \]

\[ (pq)^l = (qr)^m = (rp)^n = 1. \]
The tiling group - 2

Observe/define:
\[ a = pq, b = qr, c = rp \]

Tiling Group:
\[ G^* = < p, q, r > \]

Orientation Preserving Tiling Group:
\[ G = < a, b, c > \]
Chain modules for a tiling - 1

• Let $X_n$ be the $n$ dimensional components of the tiling of $S$.
  • $n=0$, vertices,
  • $n=1$, edges,
  • $n=2$, faces.
• $C_n = k[X_n]$
• $\partial : C_n \rightarrow C_{n-1}$ is the boundary map
Chain modules for a tiling - 2

- $\partial \Delta_0 = e_p + e_q + e_r$
- $\partial (q \Delta_0) = -a^{-1}e_p - e_q - b e_r$
- See picture on next slide
- Extend by $G$-linearity
- $\partial \zeta \Delta_0 + \partial \eta q \Delta_0 = \zeta \partial \Delta_0 + \eta \partial (q \Delta_0)$
- Don’t need to worry about $n=1$ or $n=0$. 
The master tile

\[ \Delta_0 \]

\[ \varphi \Delta_0 \]

\[ e_p \]

\[ e_q \]

\[ e_r \]
Separability criterion

- Let $k = \{0,1\}$, $\mathcal{A} = k[G]$.
- Let $\mathcal{M}_\phi = \alpha e_p + \beta e_q + \gamma e_r$ sum of all the edges in the mirror of the reflection $\phi$.
- Then $q$ is separating if and only if there are $\zeta$ and $\eta$ satisfying
  \[ \partial \zeta \Delta_0 + \partial \eta q \Delta_0 = \alpha e_p + \beta e_q + \gamma e_r \]
- or $\eta(1-a^{-1}) = \beta - \alpha$, $\eta(1-b) = \beta - \gamma$
- Note: $\mathcal{M}_\phi$ can be easily computed.
Solvability of equations in the group algebra

- Let $\mathcal{A} = k[G]$.
- Then $\eta \alpha_1 = \beta_1$, $\eta \alpha_2 = \beta_2$ if and only if
- $\alpha_1 \gamma_1 + \alpha_2 \gamma_2 = 0$ implies $\beta_1 \gamma_1 + \beta_2 \gamma_2 = 0$ for all $\gamma_1, \gamma_2$
- The point is that we might only need to check a small number of equations.