Divisible tilings in the Hyperbolic Plane

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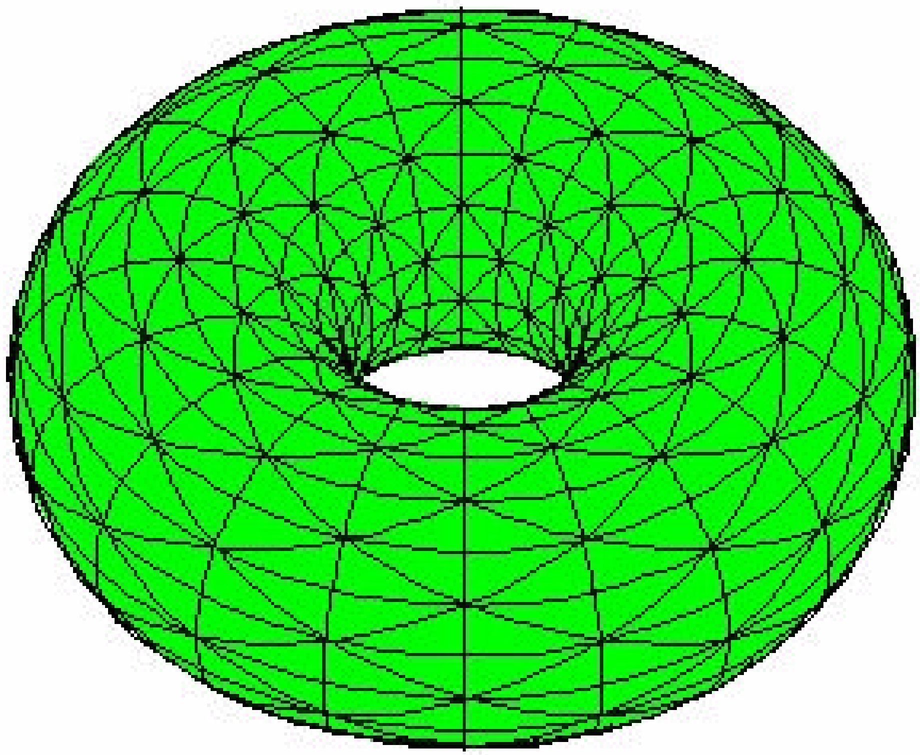
Divisible Tilings

- Divisible euclidean tilings
- hyperbolic geometry, area of a polygon, tilings
- divisible tilings, formula for K
- free and bound vertices, special K
- combinatorial search
- geometric search
- further questions

Divisible euclidean tilings

- show examples
- show tiling on a surface (further question)

(2,4,4) -tiling of the torus



Kaleidoscopic Tiling of the Plane: Definition by example

- **Tiling:** Covering by polygons "without gaps and overlaps"
- **Kaleidoscopic:** Symmetric via reflections in edges.
- **Geodesic** edges extend to lines in the tiling
- Kaleidoscopic polygons if and only angles

of the form $\frac{p}{n}$ where n is an integer

Kaleidoscopic Tiling of the Plane: Terminology

- terminology
  - (l,m,n) -triangle,
  - (s,t,u,v)-quadrilateral

Hyperbolic geometry

- Points, lines and angles
- reflections - show picture
- formula for area of a triangle

$$\pi - \left(\frac{\pi}{l} + \frac{\pi}{m} + \frac{\pi}{n}\right)$$

$$= \pi\left(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}\right)$$

Hyperbolic geometry

- formula for area of a quadrilateral

$$2p - \left(\frac{p}{s} + \frac{p}{t} + \frac{p}{u} + \frac{p}{v} \right)$$

$$= p \left(2 - \frac{1}{s} - \frac{1}{t} - \frac{1}{u} - \frac{1}{v} \right)$$

Tilings and divisible tilings - 1

- (2,3,7) and (3,3,4) example of tilings
- show divisible tilings created from (2,3,7)-example
- Divisible tiling if tiling can be kaleidoscopically subdivided by a finer tiling

Tilings and divisible tilings - 2

- Divisible tilings can be found by kaleidoscopically subdividing a kaleidoscopic tile by another kaleidoscopic tile

Formula for K

- K = number of tiles required to tile a larger tile

$$K = \frac{p\left(2 - \frac{1}{s} - \frac{1}{t} - \frac{1}{u} - \frac{1}{v}\right)}{p\left(1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}\right)}$$

Free and bound vertices, Special K

- Free vertices, infinite families of tilings, show example
- bound vertices, finitely many examples
- If K > special K there are only bound vertices
- Special K = 12

Combinatorial search K <= special K

- Show associated Catalan and hub polygons
- work out K=4, combinatorially
- C(12) = 208012 so there are that many Catalan polygons
- use dihedral symmetry to reduce to 7528

Geometric/Algebraic search K > special K

- show (2,3,7) example for (7,7,7)
- show failed (2,3,7) tiling of (7,7,7,7)
- show algebraically why (2,3,7) tiles (7,7,7)

Universal Word Relations

$$p^2 = q^2 = r^2 = 1.$$

$$(pq)^l = (qr)^m = (rp)^n = 1,$$