Vanishing Cycles and Kaleidoscopic Quadrilateral Tilings

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Credits

• All of this work has been done jointly with undergraduates
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Outline - 1

- Tilings of surfaces – examples, definition
- Hyperbolic geometry
- Kaleidoscopic tilings
- Tiling group $G^*$
- Riemann-Hurwitz equation
Outline - 2

- Tiling theorems
- Variation of quadrilaterals
- Vanishing cycles
- Number of vanishing cycles
Icosahedral-Dodecahedral tiling
(2,3,5) – tiling – soccer ball
(2,4,4) - tiling of the torus
(2,2,2,2) - tiling of the torus
(3,3,4) - tiling of the hyperbolic plane
(2,2,3,3) - tiling of the hyperbolic plane
Tiling: definition

- Let $S$ be a surface of genus $\sigma$.
- **Tiling:** Covering by polygons “without gaps and overlaps”
- **Kaleidoscopic:** Symmetric via reflections in edges.
- **Geodesic:** Edges in tiles extend to geodesics in both directions
- terminology: $(l,m,n)$-triangle, $(k,l,m,n)$-quadrilateral
Hyperbolic geometry -1

- Refer to tiling pictures
- Points, lines and angles
- Reflections
Hyperbolic geometry -2

- formula for area of a triangle

$$\pi - \left( \frac{\pi}{l} + \frac{\pi}{m} + \frac{\pi}{n} \right)$$

$$= \pi \left( 1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n} \right)$$
Hyperbolic geometry -3

- formula for area of a quadrilateral

\[
2\pi - \left( \frac{\pi}{k} + \frac{\pi}{l} + \frac{\pi}{m} + \frac{\pi}{n} \right)
\]

\[
= \pi \left( 2 - \frac{1}{k} - \frac{1}{l} - \frac{1}{m} - \frac{1}{n} \right)
\]
The tiling group - triangle - 1
The tiling group - triangle - 2

Full Tiling Group for triangle (a finite group)

\[ G^* = \langle p, q, r \rangle \]

Group Relations

\[ p^2 = q^2 = r^2 = 1. \]

\[ (pq)^l = (qr)^m = (rp)^n = 1. \]
The tiling group - quadrilateral - 2
The tiling group - quadrilateral - 2

Full Tiling Group for quadrilateral (a finite group)

\[ G^* = \langle p, q, r, s \rangle \]

Group Relations

\[ p^2 = q^2 = r^2 = s^2 = 1. \]

\[ (pq)^k = (qr)^l = (rp)^m = (ps)^m = 1. \]
Riemann Hurwitz equation - triangles

Let $S$ be a surface of genus $\sigma$ and $|G^*|$ the number of triangles:

$$\frac{4\sigma - 4}{|G^*|} = 1 - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}$$
Riemann Hurwitz equation - quadrilaterals

Let $S$ be a surface of genus $\sigma$ and $|G^*|$ the number of quadrilaterals:

$$\frac{4\sigma - 4}{|G^*|} = 1 - \frac{1}{k} - \frac{1}{l} - \frac{1}{m} - \frac{1}{n}$$
Tiling theorem - triangles

A surface $S$ of genus $\sigma$ has a tiling with tiling group:

$$G^* = < p, q, r >$$

if and only if

- the group relations hold, and
- the Riemann Hurwitz equation holds.

Therefore Tiling Problems can be solved via group computation.
A surface $S$ of genus $\sigma$ has a tiling with tiling group:

$$G^* = \langle p, q, r, s \rangle$$

if and only if

- the group relations hold, and
- the Riemann Hurwitz equation holds.
Variation of quadrilaterals

- Matlab show
- Parameter space for the quadrilateral looks like the real line
- As you go to infinity in the parameter space the lengths of pairs of opposite side goes to infinity
Construction of vanishing cycles

- The perpendicular bisectors of opposite sides of a quadrilateral, $B_{p,r}$ and $B_{q,s}$ generates a geodesic on the surface, called a vanishing cycle.
- Let $D_{p,r} = \langle p, r \rangle$ and $D_{q,s} = \langle q, s \rangle$ These groups map the vanishing cycles to themselves just like a dihedral group maps a circle to itself.
Vanishing cycles do vanish

- The hyperbolic length of $B_{p,r}$ goes to zero as the lengths of $p,r$ go to infinity
- The hyperbolic length of $B_{q,s}$ goes to zero as the lengths of $q,s$ go to infinity
- Maple “show and tell” of a vanishing cycle in a family of surfaces
Number of vanishing cycles

- The number of $p,r$ vanishing cycles is
  \[ |G^*|/| D_{p,r} | \]
- The number of $q,s$ vanishing cycles is
  \[ |G^*|/| D_{q,s} | \]