Transform Methods in Imaging Processing - Lecture 2

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Abstract

In this second lecture we discuss the Discrete Fourier Transform (DFT), and its relation to filtering and convolution. Both signals and images will be considered. Applications to deblurring and digital image correlation will be discussed. Most of the discussion can be found in greater detail in the reference [1]

1 The Discrete Fourier Transform (DFT)

A good starting point for this discussion is the well-known exponential Fourier series expansion of a function, $x(t)$, on $[0,1]$.

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e_k(t) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k t}. \tag{1}$$

The Fourier coefficients can be obtained by scalar products:

$$c_k = \langle x(t), e_k(t) \rangle = \int_0^1 x(t)\overline{e^{2\pi i k t}} dt. \tag{2}$$

Now, in a typical sampling situation, we sample $x(t)$ at the N points

$$0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}.$$

to create a vector

$$X = \begin{bmatrix} x(0) \\ x(\frac{1}{N}) \\ \vdots \\ x(\frac{N-1}{N}) \end{bmatrix} = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} \tag{3}$$

and sample the wave forms $e_k(t)$ to create

$$E_k = \begin{bmatrix} e_k(0) \\ e_k(\frac{1}{N}) \\ \vdots \\ e_k(\frac{N-1}{N}) \end{bmatrix} = \begin{bmatrix} \exp(2\pi i \frac{k\cdot 0}{N}) \\ \exp(2\pi i \frac{k\cdot 1}{N}) \\ \vdots \\ \exp(2\pi i \frac{k\cdot (N-1)}{N}) \end{bmatrix}.$$

Note that we have the aliasing effects (confounding of frequencies):

$$\begin{aligned} E_{k+N} &= E_{k+N}, \\ E_{-k} &= \overline{E_{k,}} \end{aligned} \tag{4}$$

so that only $E_0, E_1, \ldots, E_{N-1}$ are to be considered.

Show the pictures for e_k and E_k on handout, and how k is a frequency?

What do we do for the c_k and what about the reconstruction formula

1. Let us start with the Fourier coefficients. We define

$$\widehat{X}(k) = X \bullet E_k = E_k^* X = \sum_{r=0}^{N-1} X(r) \exp\left(-2\pi i \frac{kr}{N}\right). \tag{5}$$

Show how $\widehat{X}(k) = X \bullet E_k$ captures frequency dft1demo3.m

Define the Discrete Fourier Transform $\widehat{X}$ as:

$$\widehat{X} = \begin{bmatrix} \widehat{X}(0) \\ \widehat{X}(1) \\ \vdots \\ \widehat{X}(N-1) \end{bmatrix} = \begin{bmatrix} E_0^* \\ E_1^* \\ \vdots \\ E_{N-1}^* \end{bmatrix} X. \tag{6}$$

Setting

$$F_N = \begin{bmatrix} E_0^* \\ E_1^* \\ \vdots \\ E_{N-1}^* \end{bmatrix}$$

we get a matrix equation.

$$\widehat{X} = F_N X. \tag{7}$$

For $N = 2, 3, 4, \ldots, N$, F_N is given by:

$$F_2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$F_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & z & z^2 \\ 1 & z^2 & z \end{bmatrix}, \; z = \exp\left(\frac{-2\pi i}{3}\right)$$

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & i \end{bmatrix},$$

$$F_N = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & z & z^2 & z^3 & \cdots & z^{N-1} \\ 1 & z^2 & z^{2\cdot 2} & z^{2\cdot 3} & \cdots & z^{2\cdot(N-1)} \\ 1 & z^3 & z^{3\cdot 2} & z^{3\cdot 3} & \cdots & z^{3\cdot(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z^{N-1} & z^{(N-1)\cdot 2} & z^{(N-1)\cdot 3} & \cdots & z^{(N-1)\cdot(N-1)} \end{bmatrix}, \; z = \exp\left(\frac{-2\pi i}{N}\right)$$

The equations 6, 7 are called the analysis equations since X is "analyzed" into its constituent frequencies. The E_k are called the analysis waveforms.

Now what is the analogue to the synthesis or reconstruction formula 1? Now, just as exponential functions are orthogonal,

$$\langle e_k(t), e_l(t) \rangle = \delta_{k,l}$$

so are the discrete waveforms (or are close)

$$\begin{aligned} E_k \bullet E_l &= \sum_{r=0}^{N-1} e^{2\pi i k \frac{r}{N}} \overline{e^{2\pi i l \frac{r}{N}}} \\ &= \sum_{r=0}^{N-1} \left(e^{2\pi i \frac{k-l}{N}}\right)^r \\ &= \frac{1 - \left(e^{2\pi i \frac{k-l}{N}}\right)^N}{1 - e^{2\pi i \frac{k-l}{N}}} \\ &= 0, \end{aligned}$$

because we have a sum of N'th roots of unity. Also

$$\begin{aligned} E_k \bullet E_k &= \sum_{r=0}^{N-1} e^{2\pi i k \frac{r}{N}} \overline{e^{2\pi i k \frac{r}{N}}} \\ &= \sum_{r=0}^{N-1} 1 \\ &= N \end{aligned}$$

Now write out a hypothesized reconstruction formula

$$X = \sum_{r=0}^{N-1} c_r E_r$$

and take dot products with the analysis waveforms

$$\begin{aligned} \widehat{X}(k) &= X \bullet E_k = \sum_{r=0}^{N-1} c_r E_r \bullet E_k = c_k E_k \bullet E_k = N c_k \\ c_k &= \frac{\widehat{X}(k)}{N}. \end{aligned}$$

Thus we obtain the reconstruction formula

$$X = \sum_{r=0}^{N-1} c_r E_r = \sum_{r=0}^{N-1} \frac{\widehat{X}(r)}{N} E_r = \sum_{r=0}^{N-1} \widehat{X}(r) \frac{E_r}{N}.$$

It is worth recording this as a theorem.

Theorem 1 *Let $X \in \mathbb{C}^N$ be any signal, then*

$$X = \sum_{r=0}^{N-1} \widehat{X}(k) \frac{E_k}{N}. \tag{8}$$

We call 8 a reconstruction or synthesis formula and $\frac{E_r}{N}$ the reconstruction or synthesis waveforms. Now the right hand side of 8 is a sum of column vectors and hence a matrix product

$$\begin{aligned}
X &= \left[\tfrac{E_0}{N}\ \tfrac{E_1}{N} \cdots \tfrac{E_{N-1}}{N}\right] \begin{bmatrix} \widehat{X}(0) \\ \widehat{X}(1) \\ \vdots \\ \widehat{X}(N-1) \end{bmatrix} \\
&= \frac{1}{N}\left[E_0\, E_1 \cdots E_{N-1}\right] \widehat{X} \\
&= \frac{1}{N} \begin{bmatrix} E_0^* \\ E_1^* \\ \vdots \\ E_{N-1}^* \end{bmatrix}^* \widehat{X} \\
&= \left(\frac{1}{N} F_N^*\right) \widehat{X}
\end{aligned}$$

i.e.,

$$X = \left(\frac{1}{N} F_N^*\right) \widehat{X}. \tag{9}$$

It is also reasonable to call 9 the reconstruction or synthesis equations. Note that

$$F_N^{-1} = \frac{1}{N} F_N^*$$

as may be proved from the orthogonality relations.

Remark 2 *Typically for a signal we think of the time domain of the signal X as being the integer point $r = 0, 1, 2, \ldots, N-1$ taken* $\bmod N$. *and the sample corresponding to r is $X(r)$ as in equation 3. We can extend sampling to any interval of integers of length N by periodicity*

$$X(r) = X(r \bmod N). \tag{10}$$

A typical alternative interval would be $-\frac{N}{2} < r \leq \frac{N}{2}$. The frequency domain of $\widehat{X}$ may be interpreted likewise. By construction the domain is $0 \leq k \leq N-1$. However if we define $\widehat{X}(k) = \widehat{X}(k) = X \bullet E_k$ for all integers then the aliasing relation automatically force

$$\widehat{X}(k) = \widehat{X}(k \bmod N). \tag{11}$$

Often the interval $-\frac{N}{2} < k \leq \frac{N}{2}$ is chosen for the frequency domain and then k has a reasonable interpretation as a frequency. In Matlab the relevant command is `fftshift`. *See the notes late in this section for plotting DFT's.*

Signal transform pairs The signal below is a composition of even and odd real signals. The even signal has a symmetric real part and the odd signal an anti symmetric purely imaginary part (see Exercise 21). Both the real and imaginary parts of the DFT were plotted in the lower pane.

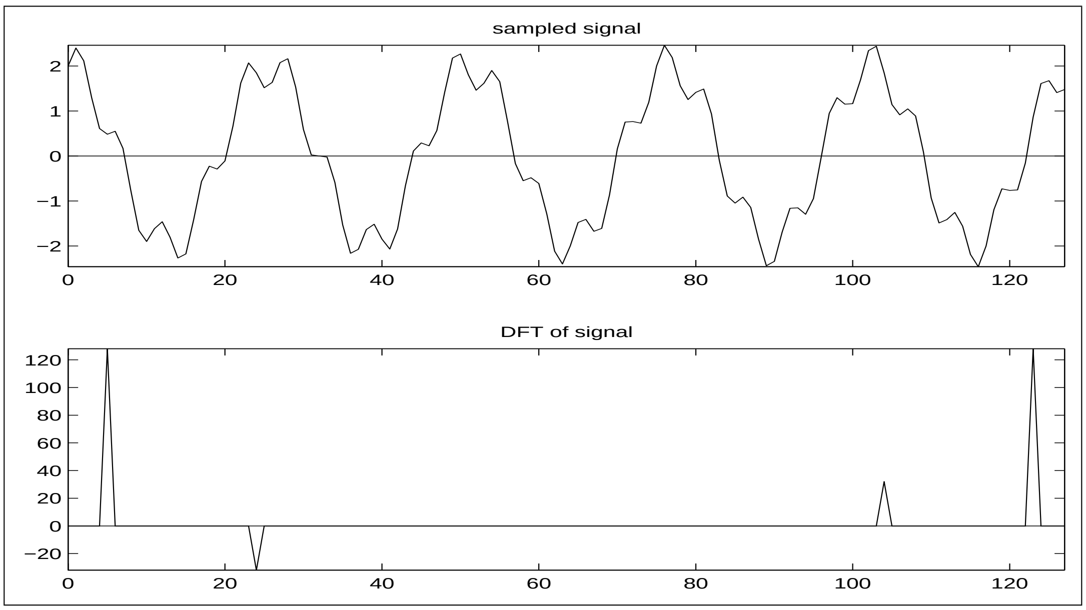

Fig. 1. $2\cos(5 \cdot 2\pi t) + 0.5\sin(24 \cdot 2\pi t)$ $(0 \le t \le 1)$ and its DFT on 128 points

Now lets look at some narrow and wide pulses. The amplitude of the transform is given by

$$\widehat{X}(k) = a\left|\frac{\sin(\pi\frac{kR}{N})}{\sin(\pi\frac{k}{N})}\right|, \ \widehat{X}(0) = aR \text{ (approximately)}.$$

The transform $\widehat{X}(k)$ has a peak at 0 and behaves like a sinusoid elsewhere The wider the pulse the narrower the peak and the faster the oscillation. The height of the pulses have been chosen so that area under the pulse curve is approximately 1. Thus the value of the DC coefficient,

$\widehat{X}(0)$, is approximately 1.

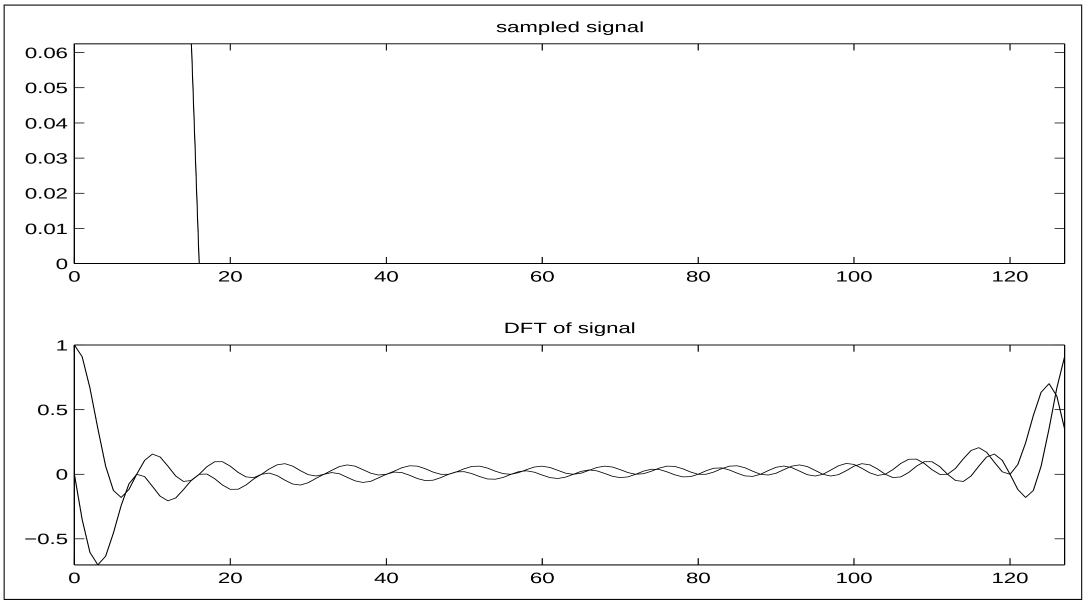

Fig 3.13 Narrow pulse and DFT on 128 points

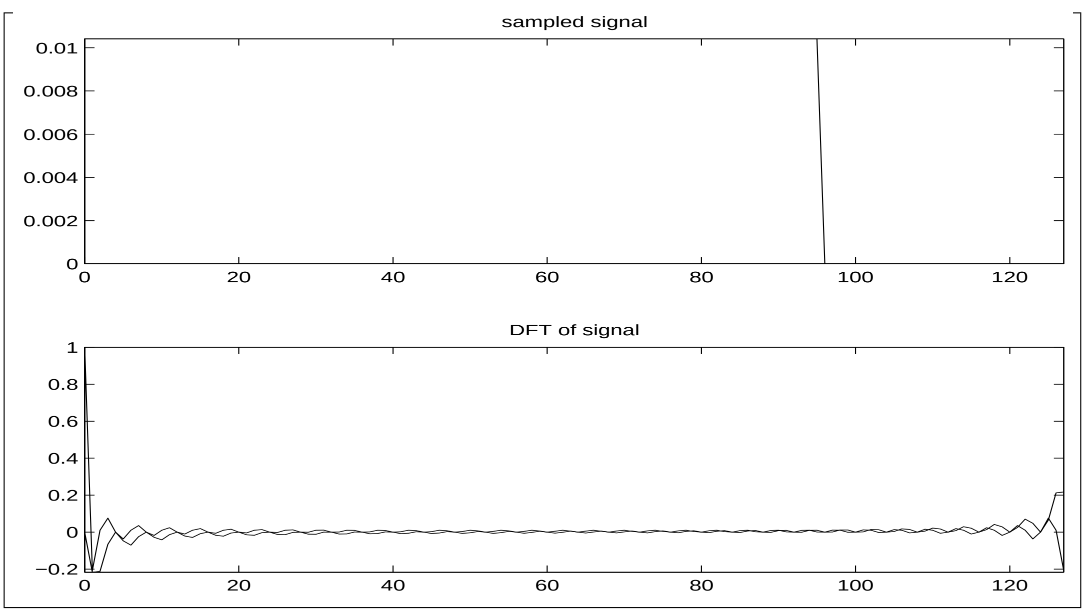

Fig 2. Wide pulse and DFT on 128 points

Since we will be dealing with noise it will be helpful to know what noise and the DFT of noise looks like. We will make use of the idea when we show how to use a DFT to denoise a picture. The noise signal in Figure 3 was created by plotting random values uniformly distributed between -1 and 1. This means that the mean of X will be approximately 0. Observe that almost every frequency occurs in the spectrum, with no apparent pattern. Simply put, the DFT of random data is again random.

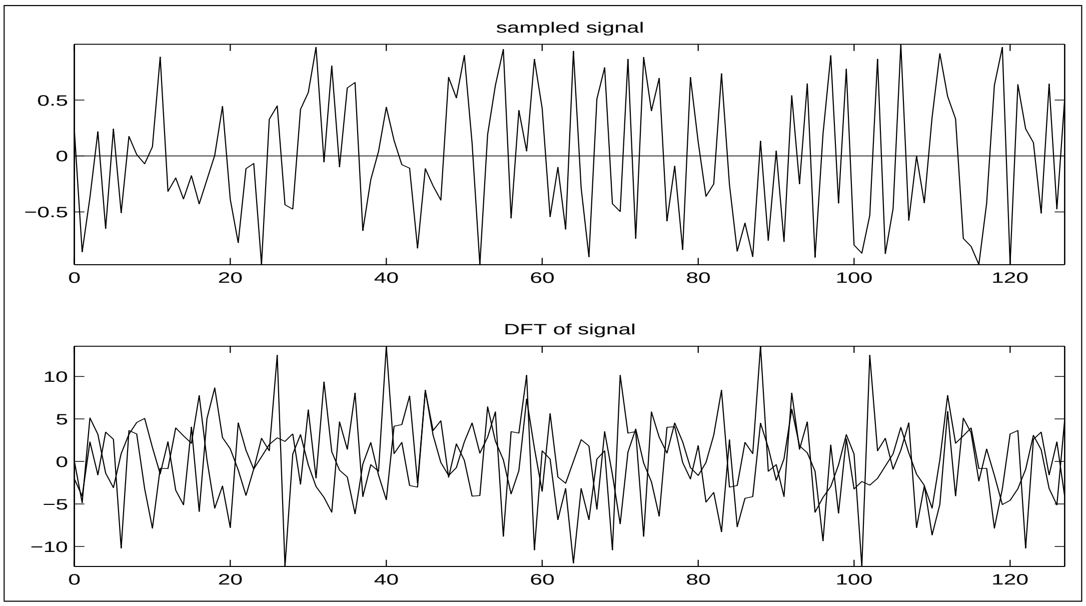

Fig 3. Noise and DFT on 128 points

Suggestions on plotting $DFT's$ In graphing DFT's we usually do the following:

- Just plot the magnitude and not look at the phase.
- To enhance small frequencies plot $\ln\left(1 + \left|\widehat{X}\right|\right)$. This is more important for images
- Do an "fftshift" For $N = 2m$ use the model wave forms $E_{-m+1}, \ldots, E_m$ in that order to create the DFT and use $-m+1, \ldots, m$ to label the horizontal axis. For $N = 2m+1$ we use $E_{-m}, \ldots, E_m$ and label the axis with $-m, \ldots, m$. In each case we get a complete set of model wave forms though they are reordered. Low frequencies are at the centre and high frequencies at the outer edges.

In Figure 4 the magnitude of the of the DFT of a narrow pulse is plotted with an fftshift. It allows us to see symmetries more easily.

Geometrically we cut the DFT graph in half and shift the right half to the left of the left half. the symmetry of the DFT and the energy of the DFT is concentrated in low frequencies.

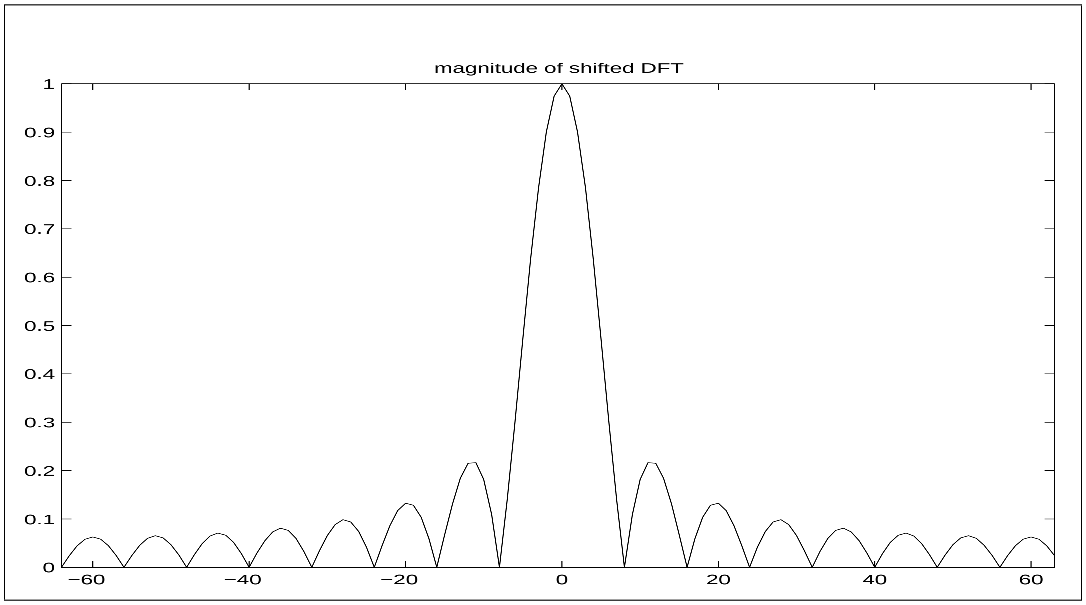

Fig. 4 Magnitude of DFT of narrow pulse in fftshift format

Finally let us finish with the signal and DFT pair of a real audio signal, showing both the magnitude and the logarithm of the magnitude. The following train whistle signal comes from Matlab. The main thing to observe is that most of the energy of this signal is carried by a small portion of the frequencies.

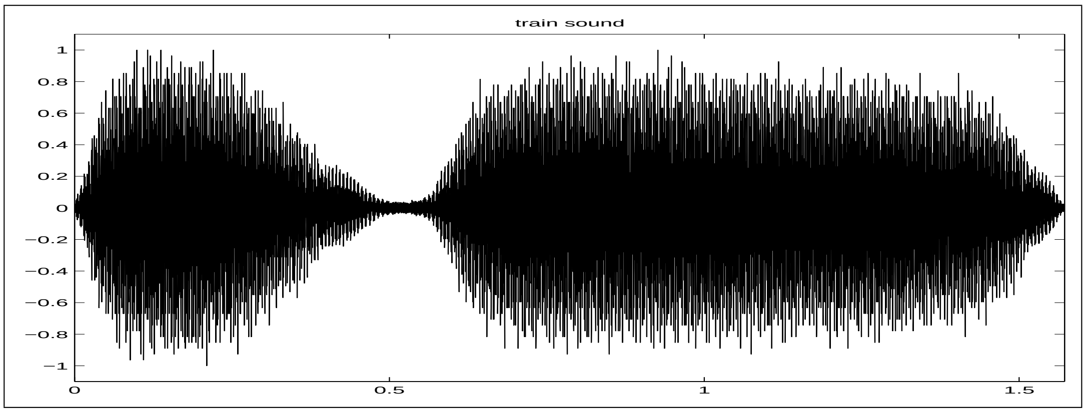

Fig 5. Train whistle

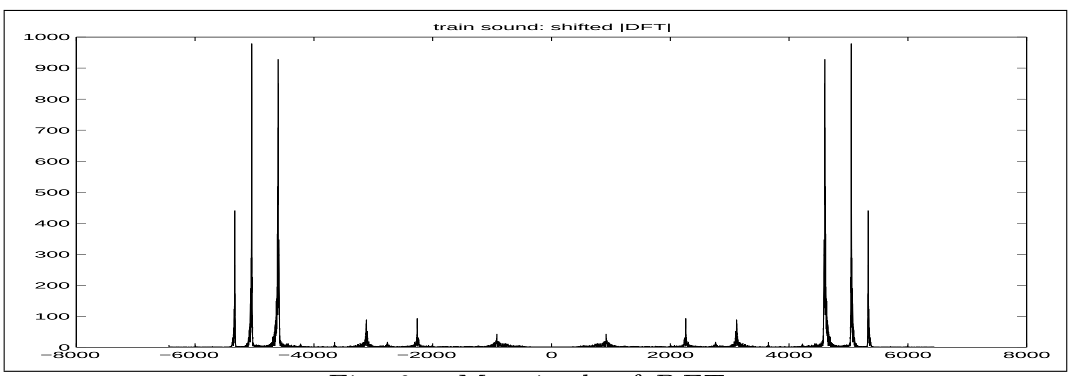

Fig. 6.a. Magnitude of DFT

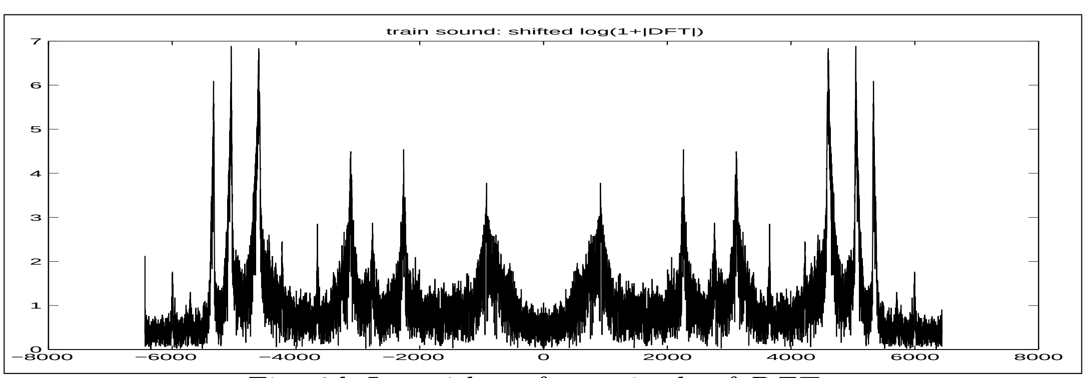

Fig. 6.b Logarithm of magnitude of DFT

2 The 2D discrete Fourier transform

The 2D model for images is an $m \times n$ matrix X which we identify with a function on $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$. This simply means that we have wraparound labeling of the rows and columns. On the computer screen the top and bottom edges are identified as well as the left and right sides. The waveforms are "tensor product" functions that we would create on any Cartesian product (e.g., the Cartesian plane). On the plane, two exponentials of period 1 in x and y give a doubly periodic product

$$\begin{aligned} e_k(x)e_l(y) &= \exp(2\pi ikx)\exp(2\pi ily) \\ &= \exp(2\pi i(kx+ly)) \end{aligned}$$

For a discrete wave we construct $\mathcal{E}_{k.l}$, a discrete $2D$ waveform by

$$\begin{aligned} \mathcal{E}_{k.l}(r,s) &= E_{m,k}(r)E_{n,l}(s) \\ &= \exp\left(2\pi i\frac{kr}{m}\right)\exp\left(2\pi i\frac{ls}{m}\right) \\ &= \exp\left(2\pi i\left(\frac{kr}{m}+\frac{ls}{m}\right)\right) \end{aligned}$$

As matrices we have

$$\mathcal{E}_{k.l} = E_{m,k}E^t_{n,l}$$

an $m \times n = (m \times 1) \times (1 \times n)$ matrix product.

Show movies of waveforms parametrized in polar form. `wavefft2cband.m`

The 2D transform of an image X has the same style of definition

$$\begin{aligned} \widehat{X}(k,l) &= X \cdot \mathcal{E}_{k.l} \\ &= \sum_{r=0}^{m-1}\sum_{s=0}^{n-1} X(r,s)\overline{\mathcal{E}_{k.l}(r,s)} \\ &= \sum_{r=0}^{m-1}\sum_{s=0}^{n-1} X(r,s)\overline{E_{m,k}(r)}\;\overline{E_{n,l}(s)} \\ &= \sum_{r=0}^{m-1}\sum_{s=0}^{n-1} X(r,s)\overline{\exp\left(2\pi i\frac{kr}{m}\right)}\;\overline{\exp\left(2\pi i\frac{ls}{m}\right)}. \end{aligned}$$

Recalling that

$$F_N = \begin{bmatrix} E^*_{N,0} \\ E^*_{N,1} \\ \vdots \\ E^*_{N,N-1} \end{bmatrix}.$$

then it is easy to prove that the $2D$ transform has a nice factorization

$$\widehat{X} = F_m X F_n^t = F_m X F_n \tag{12}$$

and hence the inverse transform is given by

$$X = \frac{1}{mn} F_m^* \widehat{X} F_n^*.$$

Though 12 can be easily proven from the definition of matrix multiplication *it* is simpler to think of the $2D$ DFT as follows;

- calculate the DFT of each column of X, put the resulting transforms into a matrix Y
- calculate the DFT of the rows of Y. Put the rows into a final result Z.

In symbols

$$\begin{aligned}
&X \to F_m X = Y \\
&Y \to Y^t \to F_n Y^t \to (F_n Y^t)^t = Y F_n^t = (F_m X) F_n^t = Z
\end{aligned}$$

This also tells us that we can calculate a 2D DFT as n m-dimensional DFT's plus m n-dimensional DFT's

There is a reconstruction formula analogous to Proposition 6

Theorem 3 *Let X be any $m \times n$ matrix, then*

$$X = \sum_{k=0}^{m-1} \sum_{l=0}^{n-1} \widehat{X}(k,l) \frac{\mathcal{E}_{k.l}}{mn}$$

Let us finish with show two image DFT pairs one natural the other unnatural.

Fig 7. Original Image X

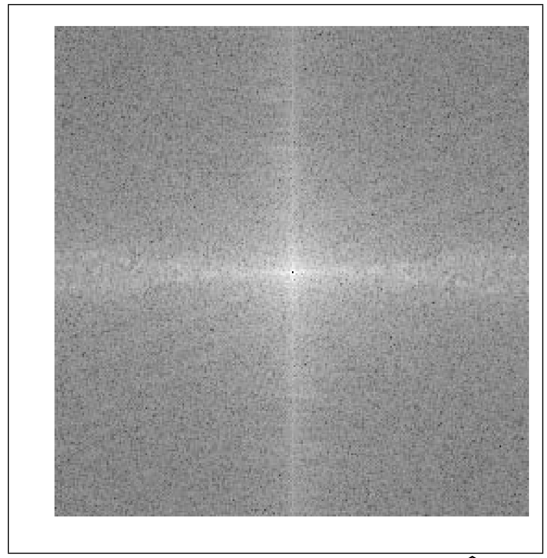

Fig 8. DFT of image, $\log(1+|\widehat{X}|)$

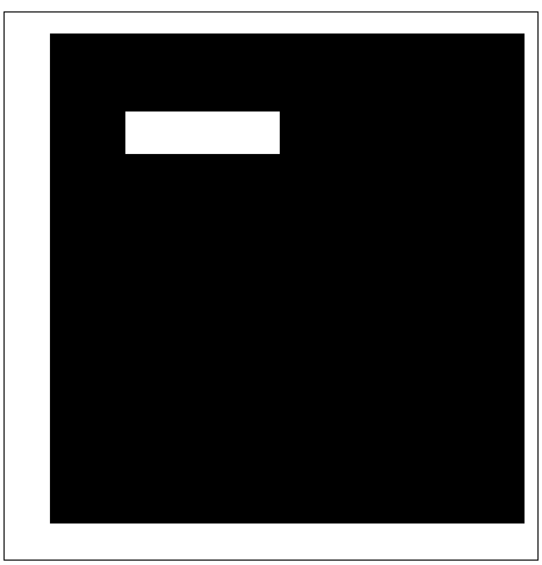

Fig 9. Original Image X

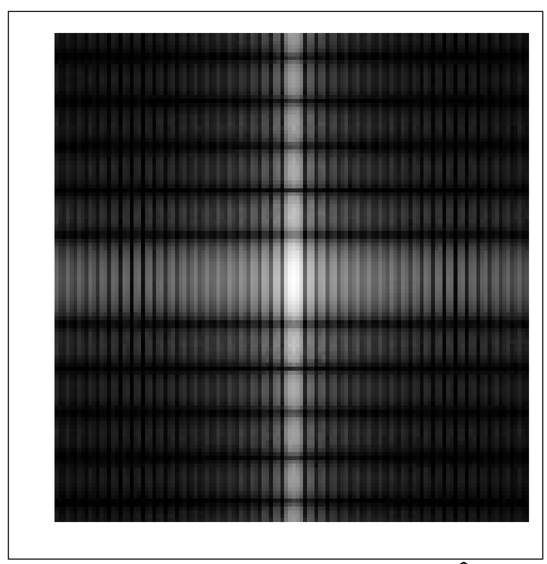

Fig 10. shifted DFT $\log(1+|\widehat{X}|)$

The last two image illustrate something about factorizable images. See exercise 24.

3 Filtering and convolution

One approach to understanding filtering is smoothing a signal using a moving average. So, given a sampled signal $X = [\ldots, X(-1),\ X(0),\ X(1), \ldots]$ defined at integer time points, we can define a two point moving average Y by

$$Y(r) = \frac{X(r) + X(r-1)}{2} = \frac{1}{2}X(r) + \frac{1}{2}X(r-1) \tag{13}$$

We are typically dealing with a finitely sampled signal $X = [X(0),\ X(1),\ \ldots, X(N-1)]^t$. To take care of the boundary values we use a wraparound or circular definition coming from the group structure of $\mathbb{Z}/N\mathbb{Z}$.

$$Y(r) = \frac{X(r) + X(r-1)}{2} = \frac{1}{2}X(r) + \frac{1}{2}X(r-1) \bmod N \tag{14}$$

where the mod N refers to the indices. Thus for an eight point signal we get:

$$\begin{bmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \\ Y(4) \\ Y(5) \\ Y(6) \\ Y(7) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} X(0) + X(7) \\ X(1) + X(0) \\ X(2) + X(1) \\ X(3) + X(2) \\ X(4) + X(3) \\ X(5) + X(4) \\ X(6) + X(5) \\ X(7) + X(6) \end{bmatrix}.$$

The filtering or smoothing can be represented by a matrix multiplication:

$$\begin{bmatrix} Y(0) \\ Y(1) \\ Y(2) \\ Y(3) \\ Y(4) \\ Y(5) \\ Y(6) \\ Y(7) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix}.$$

As a second example we can detect local variation by differencing

$$Z(r) = \frac{X(r) - X(r-1)}{2} = \frac{1}{2}X(r) - \frac{1}{2}X(r-1) \bmod N \tag{15}$$

and

$$\begin{bmatrix} Z(0) \\ Z(1) \\ Z(2) \\ Z(3) \\ Z(4) \\ Z(5) \\ Z(6) \\ Z(7) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} X(0)-X(7) \\ X(1)-X(0) \\ X(2)-X(1) \\ X(3)-X(2) \\ X(4)-X(3) \\ X(5)-X(4) \\ X(6)-X(5) \\ X(7)-X(6) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \\ X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix}.$$

Show "movie" of moving average and differencing hilopassdemo.m

Here is the final scene of the movie, using the filter above for a signal on 128 points with a low frequency, a moderate frequency, and noise.

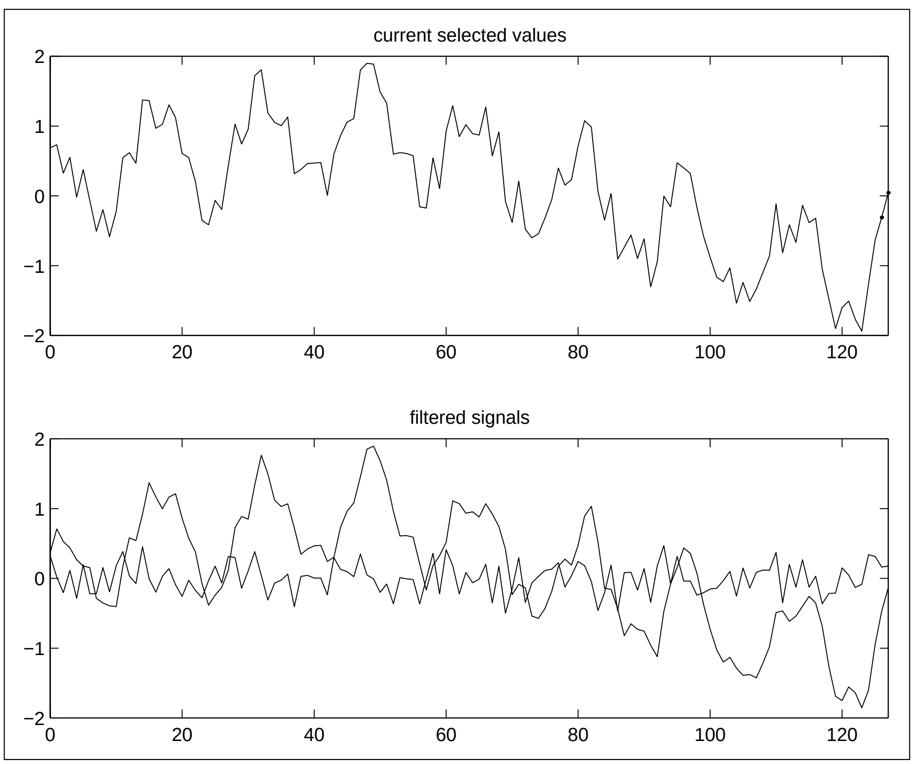

Fig 11. Smoothing an differenceing - high and low pass filtering

The general definition of convolution is the following:

Definition 4 *Let $X, Y \in L^2(\mathbb{Z}/N\mathbb{Z}) = \mathbb{C}^N$ be two vectors defined on $\mathbb{Z}/N\mathbb{Z} = \{0, 1, \ldots, N-1\}$ then the (circular) convolution $Z = Y * X$ of X and Y is defined by*

$$Z(r) = \sum_{s=0}^{N-1} Y(s)X(r-s) \bmod N.$$

Remark 5 *Typically we assume that Y is a mask of small support i.e, Y has only a few non-zero entries. The $\mathbb{Z}$ version of convolution is implemented in Matlab by* `conv`*. Scripts implementing circular convolution are available from the webpage [2].*

Proposition 6 *The circular convolution makes $L^2(\mathbb{Z}/N\mathbb{Z}) = \mathbb{C}^N$ into an associative, commutative algebra which is essentially the convolution algebra of $L^2(\mathbb{Z}/N\mathbb{Z})$. The identity element is the function f, satisfying $f(0) = 1$ but zero otherwise. The map $X \to Y * X$ is a linear transformation*

$$X \to Y * X = M_Y X$$

whose representing matrix is given by the circulant matrix

$$M_Y = \begin{bmatrix} Y(0) & Y(N-1) & Y(N-2) & \cdots & Y(1) \\ Y(1) & Y(0) & Y(N-1) & \cdots & Y(2) \\ Y(2) & Y(1) & Y(0) & \cdots & Y(3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Y(N-1) & Y(N-2) & Y(N-3) & \cdots & Y(0) \end{bmatrix}$$

4 The 2D convolution

There is analogues to convolution for images. Again it is best introduced through the concept of local averaging. To this end let f be the following mask function defined on $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$

$$f(r,s) = 1/9 \text{ if } r \equiv -1, 0, 1 \bmod m, \text{ and } s \equiv -1, 0, 1 \bmod n$$
$$f(r,s) = 0, \text{ otherwise}$$

We will use f to define a smoothing operator for images, similar to the running average, the operator will replace a pixel with the average of it and it eight neighbours. So if X is an $n \times m$ image we define $f * X$ by

$$\begin{aligned} f * X(r,s) &= \frac{1}{9} \sum_{u=-1}^{1} \sum_{v=-1}^{1} X(r-u, s-v) \\ &= \sum_{u=0}^{m-1} \sum_{v=0}^{n-1} f(u,v) X(r-u, s-v) \end{aligned}$$

where the sum is taken mod m in r and mod n is s. From the definition of f the second sum always has the 9 non-zero term appropriately chosen.

$\boxed{\text{Show movie of filtering with the 3 by 3 mask. smoothimage.m}}$

All the work we did previously with the $1D$ convolution works here as well. The main result we get is the following.

Proposition 7 *The circular convolution makes $L^2(\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}) = M_{m,n}(\mathbb{C})$ ($\mathbb{C}$-vector space of $m \times n$ matrices) into a associative, commutative algebra which is essentially the convolution algebra of $L^2(\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z})$. The identity element is matrix with a 1 in the upper left corner and all other entries zero.*

There is a link to certain masks that are variously called separable, factorizable or pure tensor product masks.

Definition 8 *A function $f \in L^2(\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z})$ is called separable, factorizable or a pure tensor product, if there are $g \in L^2(\mathbb{Z}/m\mathbb{Z})$ and $h \in L^2(\mathbb{Z}/n\mathbb{Z})$ such that*

$$f(r, s) = g(r)h(s).$$

Remark 9 *A matrix $X \in M_{\mathbb{C}}(m, n)$ corresponds to a separable function if and only if it has a factorization*

$$X = YZ^t$$

where, Y and Z are column matrices of the appropriate size.

Proposition 10 *Suppose the mask (matrix) $f \in L^2(\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z})$ is separable*

$$f(r, s) = g(r)h(s).$$

Then the linear transformation on $M_{m,n}(\mathbb{C})$ defined by $X \to f * X$ factors as

$$f * X = H_g Z H_h^t$$

for appropriate convolution matrices M_g and M_h

5 Frequency response and the convolution theorem

In the last section we saw that smoothing keeps low frequencies and eliminate high ones and that differencing does the opposite. How can we predict the behavior of an arbitrary filter, or even more to the point design a filter to achieve certain specifications. The idea is to study the frequency response of the filter. This useful concept leads to an intuitive proof of the convolution theorem.

Proposition 11 *Let$f \in L^2(\mathbb{Z}/N\mathbb{Z})$ and let $E_k = E_{N.k}$ be a pure wave form. Then*

$$f * E_k = \widehat{f}(k)E_k$$

or E_k is an $\widehat{f}(k)$ - eigenvector for the circulant of convolution matrix H_f :

$$H_f E_k = f * E_k = \widehat{f}(k)E_k.$$

Proof. Let $Z = f * E_k$ then by definition

$$\begin{aligned}
Z(r) &= \sum_{s=0}^{N-1} f(s)E_k(r-s) \\
&= \sum_{s=0}^{N-1} f(s)\exp\left(2\pi i\frac{k(r-s)}{N}\right) \\
&= \exp\left(2\pi i\frac{kr}{N}\right)\sum_{s=0}^{N-1} f(s)\overline{\exp\left(2\pi i\frac{ks}{N}\right)} \\
&= \widehat{f}(k)E_k(r).
\end{aligned}$$

Since this is true for each r then $Z = \widehat{f}(k)E_k$. ■

Remark 12 *Write $\widehat{f}(k) = re^{i\theta}$ then $f * E_k = re^{i\theta}E_k$ has an amplitude gain (or attenuation) by a factor of r and a phase shift caused by the factor $e^{i\theta}$. To see what we mean by gain or attenuation observe that*

$$\|X * E_k\| = \left\|\widehat{X}(k)E_k\right\| = \left|\widehat{X}(k)\right| \|E_k\| = r\|E_k\|,$$

and that the peaks of real and imaginary pars of $X * E_k$ will be roughly r times the peaks of the real and imaginary parts of the signal.

The following picture illustrates both the attenuation and phase shift

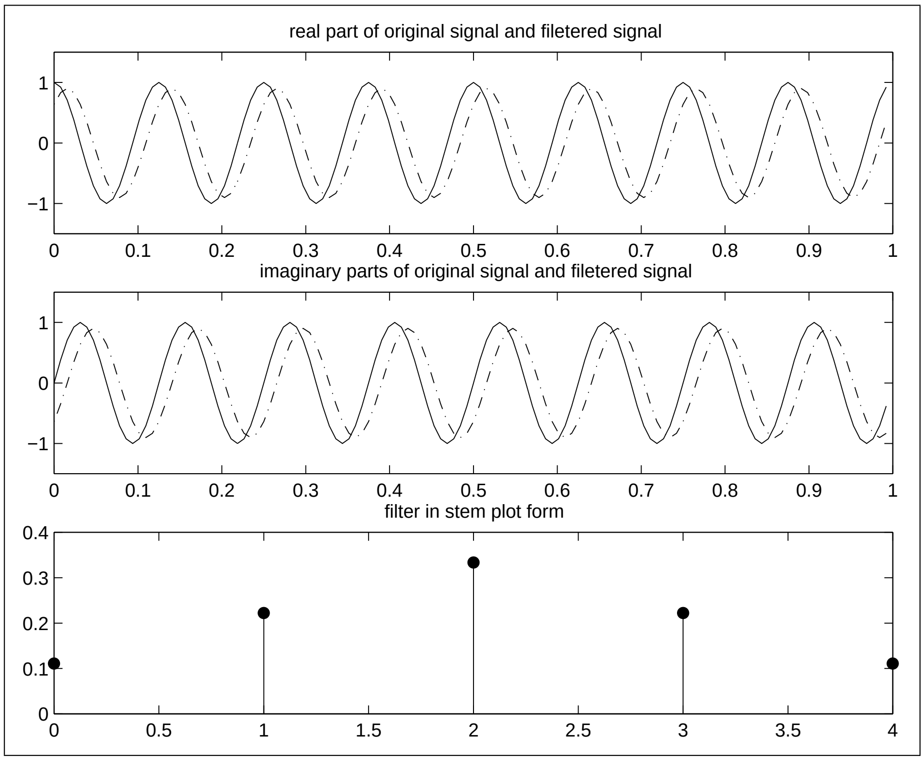

Fig 12. Exponential wave before and after filtering.

The effect of attenuation is understood by considering the plot of $\widehat{f}(k)$ as a function of k usually plotted for k in a symmetric range $-\frac{N}{2} \leq k \leq \frac{N}{2}$. Here for example are the plots of the two point averager and the two point differencer. We have included the phase shift also.

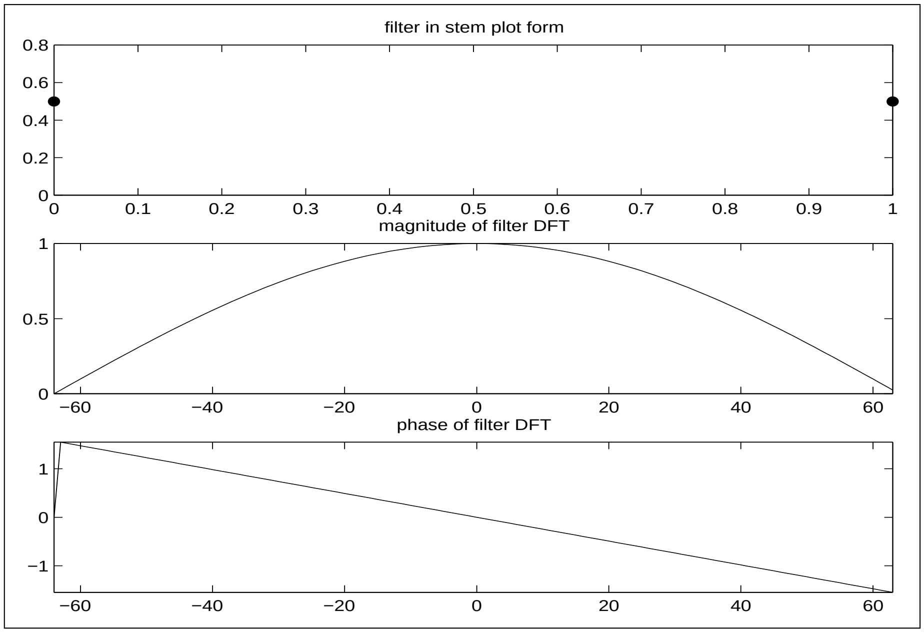

Fig. 13 Frequency reponse of 2 point averager

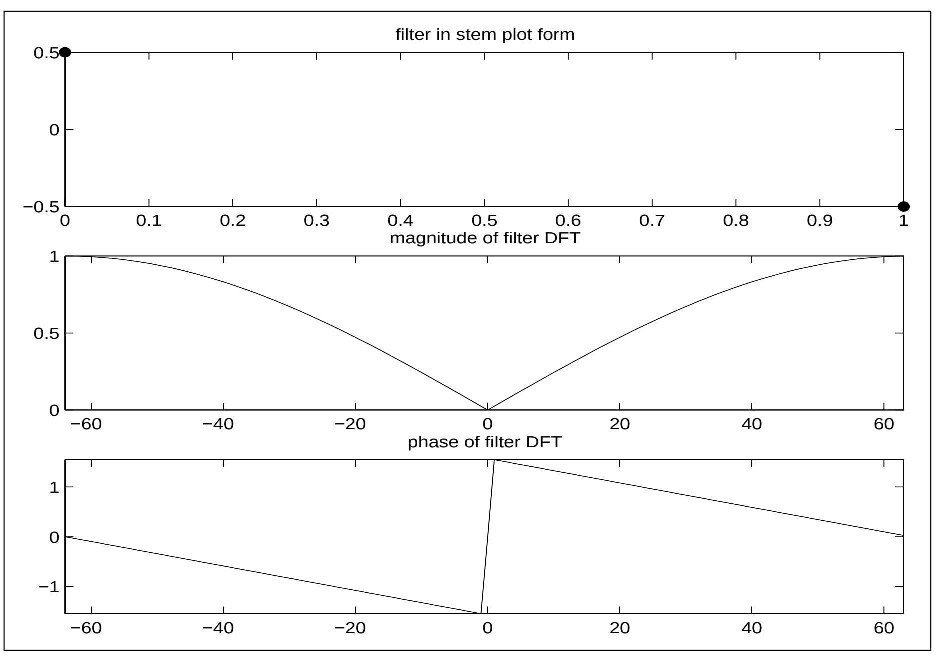

Fig. 14 Frequency reponse of 2 point differencer

The analogue of Proposition 6 also holds for matrices

Proposition 13 *Let X be a matrix and $\mathcal{E}_{k.l}$ a pure exponential wave form of the same size. Then*

$$X * \mathcal{E}_{k,l} = \widehat{X}(k,l)\mathcal{E}_{k,l}.$$

We are now ready for the convolution theorem

Theorem 14 *Let X and Y be either two vectors or matrices of the same size. Then*

$$\widehat{X * Y} = \widehat{X}\widehat{Y}$$

where the last multiplication is component wise multiplication.

Proof. We will just show it for vectors. By the reconstruction formula

$$Y = \sum_{r=0}^{N-1} \widehat{Y}(k)\frac{E_k}{N}.$$

Then by the frequency response result

$$\begin{aligned} X * Y &= \sum_{r=0}^{N-1} \widehat{Y}(k)\frac{X * E_k}{N} \\ &= \sum_{r=0}^{N-1} \widehat{Y}(k)\widehat{X}(k)\frac{E_k}{N} \end{aligned}$$

One more application of the reconstruction formula yields

$$X * Y = \sum_{r=0}^{N-1} \widehat{X * Y}(k)\frac{E_k}{N}.$$

Since the wave forms are a basis then

$$\widehat{X * Y}(k) = \widehat{X}(k)\widehat{Y}(k),$$

which is the statement of the theorem. ■

Remark 15 *The theorem says that the Fourier transform is an algebra isomorphism from the convolution algebra to the algebra of function of point wise multiplication on the index set. In particular, the convolution identity is mapped the function which is identically on the index set.*

6 Applications

Image restoration Suppose that we have a blurred photographic image Y and suppose that we want to unblur it. A simple model for blurring is

$$Y = h * X + E$$

where X is the true image h is some blurring mask and E is additive noise. What we would like to do is find an inverse or unblurring mask g such that

$$gY = g * (h * X) + g * E$$

satisfies:

- $g * (h * X)$ is a good approximation for X
- $g * E$ is fairly small.

Now suppose that $\widehat{g}\widehat{h} = 1$ (point wise for every pair in the frequency domain) or that $\widehat{g} = 1/\widehat{h}$ and that $\widehat{g}\widehat{E}$ is reasonably small then

$$\begin{aligned}
\widehat{g}\widehat{Y} &= \widehat{g}(\widehat{h}\widehat{X} + \widehat{E}) \\
&= \widehat{g}\widehat{h}\widehat{X} + \widehat{g}\widehat{E} \\
&\widehat{X} + \widehat{g}\widehat{E}
\end{aligned}$$

If $\mathcal{F}^{-1}$ denotes the inverse Fourier transform then

$$\begin{aligned}
\mathcal{F}^{-1}(\widehat{g}\widehat{Y}) &= \mathcal{F}^{-1}(\widehat{X} + \widehat{g}\widehat{E}) \\
&= \mathcal{F}^{-1}(\widehat{X}) + \mathcal{F}^{-1}(\widehat{g}\widehat{E}) \\
&= X + g * E.
\end{aligned}$$

This says we can unblur in three stages

- Compute $\widehat{Y}$
- Determine $\widehat{g}$ and then $\widehat{g}\widehat{Y}$
- Compute the inverse transform of $\widehat{g}\widehat{Y}$ and then take its real part.

There are several "gotchas" in this scheme.

- h and therefore $\widehat{h}$ is unknown
- $\widehat{h}$ probably has zeros or small values
- $\widehat{E}$ is unknown.

Here is a crude first attempt

- Estimate h as a mask supported on or near line segment in the spatial domain for linear motion blurring. You can guess this by looking at the blurred picture. In addition you may be able perform experiments on the camera to determine blurring due to lens aberration. A combination of various factor may be required to propose a good model for $\widehat{h}$.

- Construct an ϵ-pseudo inverse

$$\widehat{g}(k,l) = \frac{1}{\widehat{h}(k,l)} \text{ if } \left|\widehat{h}(k,l)\right| \geq \varepsilon$$
$$\widehat{g}(k,l) = 0 \text{ if } \left|\widehat{h}(k,l)\right| < \varepsilon$$

  for an appropriately chosen ϵ. There are more sophisticated inverses than this.

An example of deblurring is given on the webpage [2].

Digital Image Correlation To come.

7 Exercises

Exercise 16 *If $f(t)$ is a function on $[0,1]$, convince yourself that*

$$\int_0^1 |f(t)|^2 \, dt$$

is a reasonable measure of the energy of the signal. For instance think of $f(t)$ as the voltage across a unit resistor. Then the total energy dissipated over the unit of time is $\int_0^1 |f(t)|^2 \, dt$.

Exercise 17 *Thinking of*

$$X = \begin{bmatrix} f(0) \\ f(\frac{2}{N}) \\ f(\frac{2}{N}) \\ \\ f(\frac{N-1}{N}) \end{bmatrix}$$

show that $\frac{X \cdot X}{N}$ is a reasonable value for the energy of X.

Exercise 18 *Show that all the exponential waveforms, e_k,on $[0,1]$, and the discrete waveforms $E_{N,k}$ all have unit energy.*

Exercise 19 *Carry the previous two exercises functions on the unit square and for matrices.*

Exercise 20 *For a real signal X show that*

$$\widehat{X}(N-k) = \widehat{X}(-k) = \overline{\widehat{X}(k)}$$

Exercise 21 *For a Hermitian symmetric signal, $X(-k) = \overline{X(k)}$, show that $\widehat{X}$ is real and for a skew symmetric signal, $X(-k) = -\overline{X(k)}$, show that $\widehat{X}$ is pure imaginary.*

Exercise 22 *If X is real and even $X(N-k) = X(-k) = X(k)$ then $\widehat{X}$ is real and even. What happens for a real odd signal.*

Exercise 23 *Show that the 2D DFT has the claimed matrix factorization.*

Exercise 24 *If $X = YZ^t$ is factorizable then show that $\widehat{X} = \widehat{Y}\widehat{Z}^t$, specifically*

$$\widehat{X}(k,l) = \widehat{Y}(k)\widehat{Z}(l)$$

Exercise 25 *Determine exactly what the phase shift induced by filtering is.*

References

[1] *S. Allen Broughton*, Mathematics of Image Processing, unpublished lecture notes

[2] Transform Methods in Image Processing
http://www.rose-hulman.edu/~brought/Epubs/mhc/mhctransimage.html