

# Voronoi Tessellations, Delaunay Tessellations and Flat Surfaces - Sabbatical Report Part II

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# sabbatical at IU - 1

## My sabbatical at Indiana University

- Sabbatical at Indiana University Spring 2008 - Thank you Rose-Hulman.
- Sabbatical partly to learn new areas of mathematics related to surfaces.
- Also to come up with problems suitable for undergraduate research.

# sabbatical at IU - 2

## Two reports

- First talk (October): gave an introduction to flat surfaces via the historical motivation of billiard trajectories.
- Second talk (today): discuss tessellations on flat surfaces and present problems.

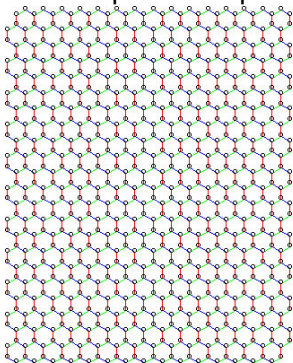
# parts of today's talk

There are four main parts of the talk:

- motivation and discussion of Voronoi tessellations
- discussion of Delaunay tessellations
- recall definition of a flat surface
- Delaunay tessellation of flat surfaces

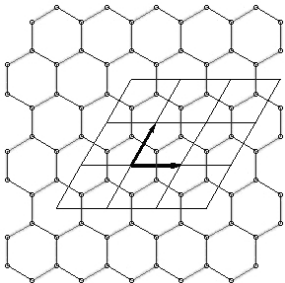
# graphene example - 1

- cartoon picture of planar graphene crystal



# graphene example - 2 - unit cell

- What is a unit cell?
- cartoon picture of unit cell of graphene



## more on unit cells

- Atoms of a crystal form a lattice in 2D or 3D. One lattice for each species of atom.
- There is an infinite group of translational symmetries.
- There may also be rotational point symmetries.
- The unit cell is a polygon that tiles the plane or three space by translation.
- The unit cell contains at least one example of each species.
- The unit cell is often a parallelogram or a parallelepiped.

# graphene example - 3 - Voronoi diagram

- Voronoi diagrams (tessellations) can be used refine the unit cell.
- There is only one atom in each Voronoi cell.
- Rotational point symmetries are captured by the Voronoi cells.
- Show Voronoi diagram for graphene.

# Voronoi tessellations (diagrams) - 1

Here is an informal definition of Voronoi tessellation or diagram.

## Definition

Let  $\Sigma \subset V$  be a discrete set of points in the plane or 3-space (denoted  $V$ ). Let  $d(x, y)$  denote the distance between two points. For  $p \in \Sigma$  define an open Voronoi cell (of maximal dimension) as follows:

$$F_p^\circ = \{x \in V : d(x, p) < d(x, q) \text{ for all } q \in \Sigma - \{p\}\}$$

i.e.,  $x$  is closer to  $p$  than any other point of  $\Sigma$ . The closure of each Voronoi cells  $F_p = \overline{F_p^\circ}$  is a (possibly infinite) polyhedron. The lower dimensional faces of the polygons or polyhedra are the lower dimensional cells of the Voronoi tessellation.

## Voronoi tessellations (diagrams) - 2

Some facts about 2D Voronoi tessellations.

- points in open 2-cells (faces) are closest to exactly one point of  $\Sigma$
- points in open 1-cells (edges) are closest to exactly two points of  $\Sigma$
- points in open 0-cells (vertices) are closest to three or more points of  $\Sigma$
- two closed Voronoi 2-cells meet in a single point, an edge or not all. Therefore, the Voronoi diagram gives a tessellation by (possibly infinite) polygons or polyhedra.

# Voronoi tessellations (diagrams) - 3

- For more (basic) information on Voronoi diagrams we go to Wikipedia and the references there.
- [http://en.wikipedia.org/wiki/Voronoi\\_diagram](http://en.wikipedia.org/wiki/Voronoi_diagram)
- <http://hirak99.googlepages.com/voronoi>

# Voronoi tessellations (diagrams) - 4

- The definition of Voronoi diagram only involves the distance so the concept of Voronoi diagram can be used whenever there is a decent metric. For example:
- the sphere
- hyperbolic space
- surfaces and manifolds with Riemannian metrics
- discrete sets such as in coding theory
- Voronoi diagrams are hard to compute in dimensions higher than three.

# Delaunay - 1

## Definition

Let  $\Sigma = \{p_i : i \in I\} \subset V$  be a discrete set of points in the plane or 3-space (denoted  $V$ ), and let  $\{P_i : i \in I\}$  be the set of closed Voronoi polygons. The indexing satisfies  $p_i \in P_i$ . The Delaunay tessellation is the dual graph of the Voronoi diagram, namely.

- The vertices of the Delaunay tessellation are the points of  $\Sigma$
- An edge of the Delaunay tessellation joins two points  $p_i \in P_i$  and  $p_j \in P_j$  if  $P_i$  and  $P_j$  meet in a Voronoi edge.
- The two cells are the bounded polygons that remain after cutting out the edges and vertices.

# Delaunay - 2

Here are some examples of Delaunay tessellations.

## Example

- Example 1: go back to graphene sheet
- Example 2: use applet below
- <http://www.cs.cornell.edu/Info/People/chew/Delaunay.html>

# Delaunay - 3

Here are some properties of Delaunay tessellations.

- The polygons of a Delaunay tessellations are cyclic, namely they lie on circle.
- The center of the circle is a Voronoi vertex. Go back to previous applet.

# flat surfaces - definition

Here is an informal definition of a flat surface.

## Definition

Let  $P_1, \dots, P_n$  be a sequence of polygons such that every side of every polygon is matched with exactly one side (same edge length) of another polygon. The match may be to another side of the same polygon. The compact space obtained by gluing the polygons together via the matching is called a *flat surface*.

# flat surfaces - examples

Here are some examples of flat surfaces.

## Example

- any of the platonic surfaces (show models)
- the surface formed for the development of a convex billiard table whose corner angles are rational multiples of  $\pi$  (last talk)
- identify the opposite sides of a regular polygon with an even number of sides

# flat surfaces - geometry - 1

A flat surface has three types of points:

- interior points of polygons
- hinge points where two polygons meet along an edge
- cone points at the vertices of polygons
- The first two types of points are regular points on the surface. A neighbourhood of a hinge point can be made to look like a flat piece of the plane by flattening.
- The cone points are usually considered to be singular. They cannot be flattened unless the total angle is  $2\pi$ .

# flat surfaces - geometry - 2

The local geometry is determined as follows:

- Each regular point has a neighbourhood with the regular flat plane geometry, flattening a hinge as needed.
- Cone points need a measure of non-regularity, called the cone angle.
- If  $\alpha_1, \dots, \alpha_s$  are the angles at a cone point  $v_j$ , then the (total) cone angle at  $v_j$  is

$$\theta_j = \sum_{i=1}^s \alpha_i.$$

- A cone point is regular if and only the cone angle equals  $2\pi$  (lies flat).

# flat surfaces - geometry - 3

Here are some examples of cone angles.

## Example

- A cube has 8 cone points with cone angle  $3\pi/2$ .
- An icosahedron has 12 cone points with cone angle  $5\pi/3$ .
- The flat torus has no singular cone points (last talk).

# Voronoi Tessellation of a flat surface

- Just use the definition using the distance function on a flat surface.
- Show example for a cube.
- What is the Voronoi tiling for an icosahedron?

## Proposition

*Every flat surface has a canonical Delaunay tessellation, and hence may be constructed as an identification space of polygons.*

## Remark

*For the regular platonic solids the Delaunay tessellation is the original tessellation. When projected to the sphere, the Voronoi tessellation forms the dual platonic solid.*

Here are some problems I don't know the answer to.

- A quasiplatonic is a surface of genus 1 or greater that has a regular tiling by regular polygons. This generalizes the definition of a platonic surface. Are there any quasiplatonic solids that can be embedded in 3-space, via the Delaunay tessellation?
- For a quasi-platonic surface how may the Voronoi tessellation be interpreted?
- You can vary the structure of a flat surface by moving the cone points and changing the cone angles. How does the Delaunay tessellation change?

# Any Questions?