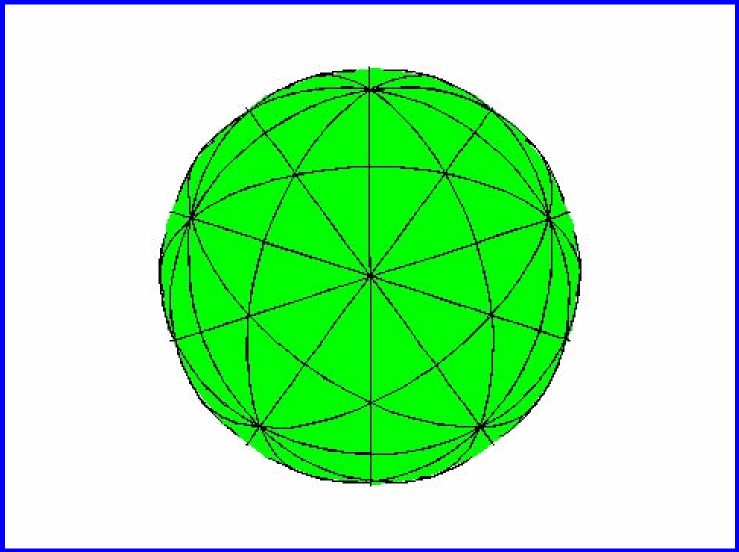
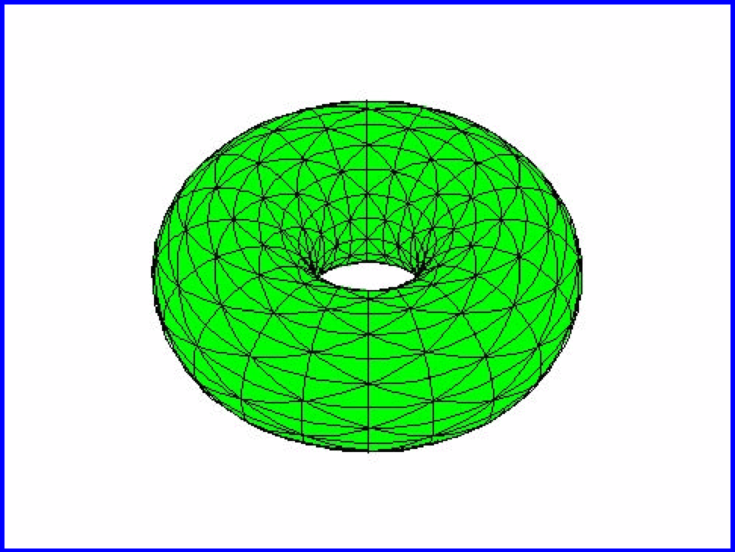
 

Tilings, Finite Groups, and Hyperbolic Geometry at the Rose-Hulman REU

S. Allen Broughton

Rose-Hulman Institute of Technology

Outline

- A Philosophy of Undergraduate Research
- Tilings: Geometry and Group Theory
- Tiling Problems - Student Projects
- Example Problem: Divisible Tilings
- Some results & back to group theory
- Questions

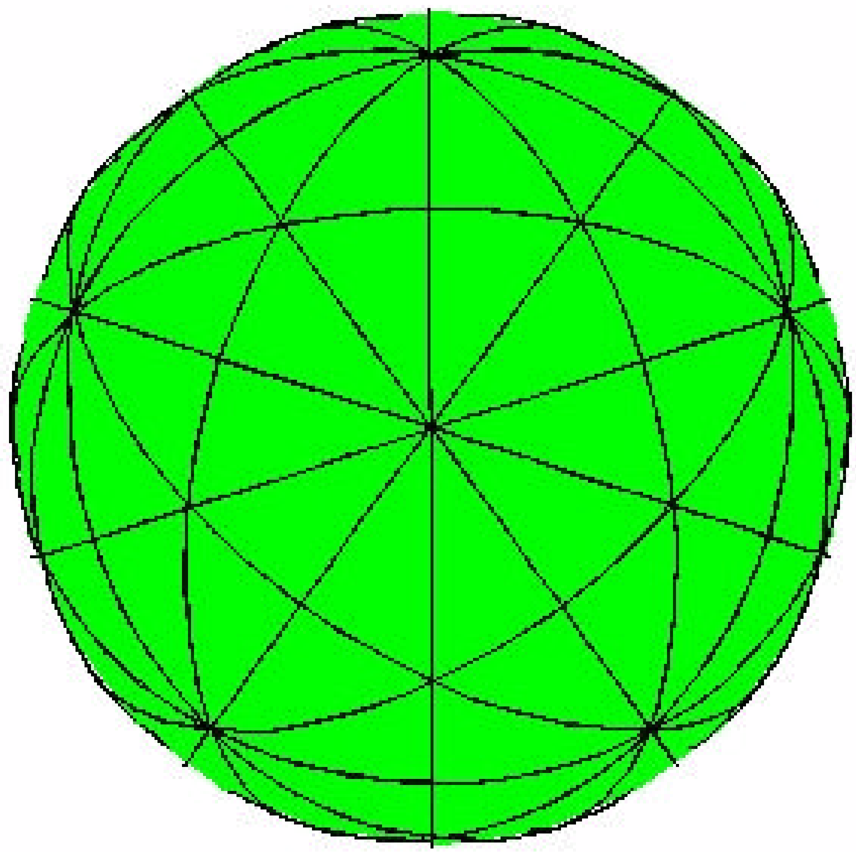
A Philosophy of Undergraduate Research

- doable, interesting problems
- student - student & student -faculty collaboration
- computer experimentation (Magma, Maple)
- student presentations and writing

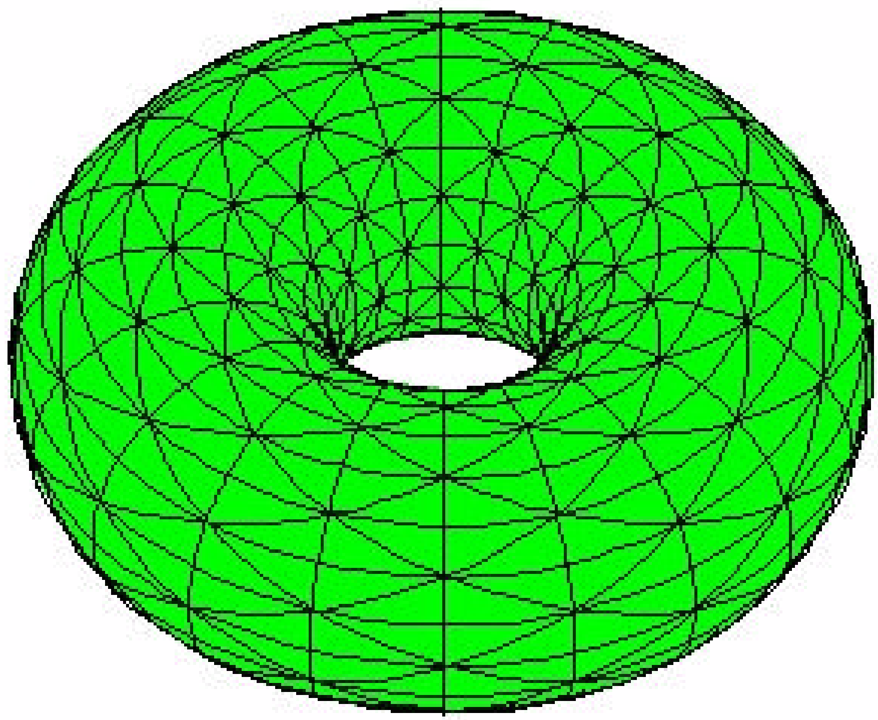
Tilings: Geometry and Group Theory

- show (picture of ball)
- tilings: definition by example
- tilings: master tile
- Euclidean and hyperbolic plane examples
- tilings: the tiling group
- group relations & Riemann Hurwitz equations
- Tiling theorem

Icosahedral-Dodecahedral Tiling



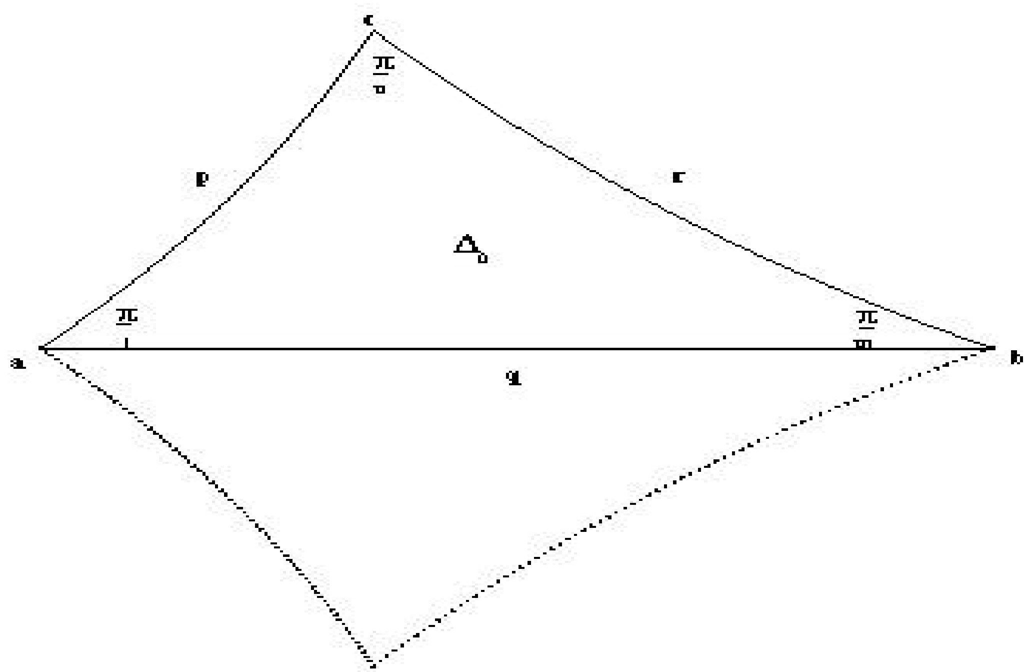
(2,4,4) -tiling of the torus



Tiling: Definition

- Let **S** be a surface of genus ***s*** .
- **Tiling:** Covering by polygons “without gaps and overlaps”
- **Kaleidoscopic:** Symmetric via reflections in edges.
- **Geodesic:** Edges in tilings extend to geodesics in both directions

Tiling: The Master Tile - 1



Tiling: The Master Tile - 2

- mainly interested in tilings by triangles and quadrilaterals
- **reflections in edges:** p, q, r
- **rotations at corners:** a, b, c
- **angles at corners:** $\frac{p}{l}, \frac{p}{m}, \frac{p}{n}$
- terminology: (l,m,n) -triangle, (s,t,u,v) - quadrilateral, etc.,

Tiling: The Master Tile - 3

- terminology: (l,m,n) -triangle, (s,t,u,v) - quadrilateral, etc.
- hyperbolic when $s \geq 2$ or

$$\frac{p}{l} + \frac{p}{m} + \frac{p}{n} < p$$

or

$$1 - \left(\frac{1}{l} + \frac{1}{m} + \frac{1}{n} \right) > 0$$

The Tiling Group

Observe/define:

$$a = pq,\ b = qr, c = rp$$

Tiling Group:

$$G^* = < p, q, r >$$

Orientation Preserving Tiling Group:

$$G = < a, b, c >$$

Group Relations (simple geometric and group theoretic proofs)

$$p^2 = q^2 = r^2 = 1.$$

$$a^l = b^m = c^n = 1,$$

$$abc = 1, (pqqrrp = 1)$$

$$q(a) = qaq^{-1} = qpqq = qp = a^{-1},$$

$$q(b) = qbq^{-1} = qqrq = rq = b^{-1}.$$

Riemann Hurwitz equation (euler characteristic proof)

Let S be a surface of genus s then:

$$\frac{2s-2}{|G|}=1-\frac{1}{l}-\frac{1}{m}-\frac{1}{n}$$

Tiling Theorem

A surface S of genus s has a tiling with tiling group

$$G^* = < p, q, r >$$

if and only if

- the group relations hold
- the Riemann Hurwitz equation holds

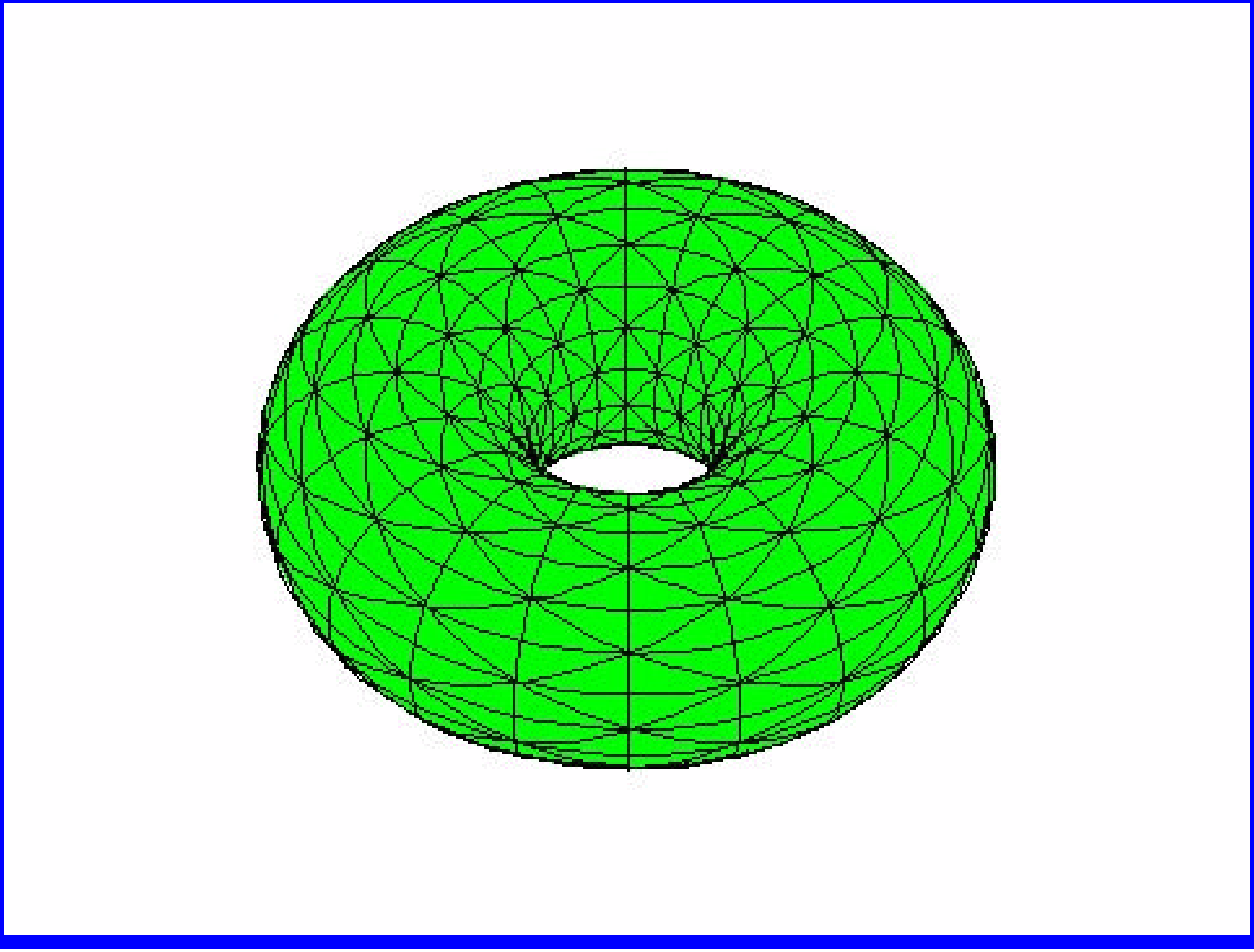
Tiling Problems - Student Projects

- **Tilings of low genus** (Ryan Vinroot, Maria Sloughter, Robert Dirks)
- **Divisible tilings** (Dawn Haney, Lori McKeough, Brandy Smith)
- **Separating reflections** (Jim Belk, Lisa Powell, Jason Deblois, Nick Baeth)
- **Oval intersections** (Dennis Schmidt)
- **Geodesics and Systoles** (Ryan Derby Talbot, Kevin Woods)

Divisible Tilings

- torus - euclidean plane example
- hyperbolic plane example
- hyperbolic plane results
- a group theoretic surprise

Torus example ((2,2,2,2) by (2,4,4))



Euclidean Plane Example ((2,2,2,2) by (2,4,4))

- show picture
- the Euclidean plane is the "unwrapping" of torus "universal cover"

Hyperbolic Plane Example

- show picture
- can't draw tiled surfaces so we work in hyperbolic plane, the universal cover

Divisible Tilings
Students Problem and Results

- Problem: find divisible quadrilaterals
- classification into free and constrained quadrilaterals
- show results
- used Maple to do
  - combinatorial search
  - group theoretic computations in 2x2 complex matrices

A group theoretic surprise

- we have found divisible tilings in hyperbolic plane
- Now find surface of smallest genus with the same divisible tiling
- for (2,3,7) tiling of (3,7,3,7) we have:

$$|G^*| = 2357200374260265501327360000$$

$$s = 1403095460869205655520001$$

A group theoretic surprise - cont'd

$$|G^*| = 2^{21} \cdot 22! \text{ and}$$

$$1 \to Z_2^{21} \to G^* \to \Sigma_{22} \to 1$$

Thank You!
Any Questions???

REU site:

http://www.rose-hulman.edu/Class/ma/HTML/REU/NSF-REU.html

Tiling Site: http://www.tilings.org

Thank You!
Questions???