Vertex-Magic

Daisy Cunningham
Advisor : Dr. Crista Coles
Elon University
Elon, NC 27244
dsc288@aol.com

1 Introduction

Graph theory was introduced by Leonhard Euler in 1736 when he began discussing whether or not it was possible to cross all of the bridges in the city of Kaliningrad, Russia only once. The publication of the problem and his proposed solution began what is now known as graph theory. Since then, the topic has been thoroughly studied by mathematicians such as Thomas Pennyngton Kirkman and William Rowan Hamilton. In Euler's original problem, vertices represented locations around the city which were connected by bridges, or the edges of the graph. For more information on the history of graph theory, see [3] and [4].

One of the most famous problems in graph theory is the Four Color Conjecture proposed by Francis Guthrie, a student of Augustus DeMorgan, around 1850 [2]. The main objective of this problem is to color a map in such a way that no two adjacent countries, those sharing a border, are the same color. In this problem, countries are represented with vertices. If two countries share a border, then their corresponding vertices are adjacent. The conclusion showed that any map on a sphere can be colored with four colors.

The vertices of a graph can be labeled in many different ways. Another way to label vertices is with numbers. An interesting vertex labeling with numbers is vertex-magic. Vertex-magic graphs are graphs labeled with numbers in which every vertex and its incident edges add up to the same number. This number is called the magic number. It is still unknown what types of graphs are vertex-magic and which are not. One type of graph that has interesting vertex-magic properties is the cycle graph. The following questions have interesting solutions regarding the labelings of vertex-magic cycle graphs: Can sharp bounds be found for the magic number of a given cycle graph? What are some properties of vertex-magic cycle graphs with both an even and odd number of vertices?

2 Preliminary Information

Before looking at vertex labelings, we must first look at some basic concepts of graph theory.

Definition 2.1. A **graph** is a finite set of vertices and edges where every edge connects two vertices. Any two vertices that share an edge are said to be **adjacent**. Any two edges that are incident to the same vertex are said to be **adjacent**. If an edge connects two vertices, then that edge is considered **incident** to both of these vertices.

Definition 2.2. A graph is **connected** if any two vertices are connected by a sequence of adjacent edges.

One unique type of graph that has very interesting vertex-magic characteristics is the cycle graph.

Definition 2.3. A **cycle graph** is a connected graph where every vertex is adjacent to two other distinct vertices.

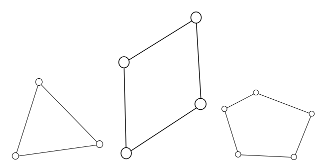

Figure 1: Three examples of cycle graphs.

Definition 2.4. If a graph G with v vertices and e edges is labeled with numbers 1 through $v + e$ such that every vertex and its incident edges adds up to the same number, then G is a vertex-magic graph. We call this number the magic number.

Definition 2.5. If a graph G with v vertices and e edges is labeled with numbers 1 through $v + e$ such that every edge and its two adjacent vertices adds up to the same magic number, then G is an edge-magic graph.

In Figure 2, we have an example of vertex magic graphs. In the left graph, notice that each vertex and its incident edges add up to 10. This graph is also edge-magic because each edge and its adjacent vertices adds up to 11. Every cycle graph that is vertex-magic with a magic number, k, can also be labeled so that it is edge-magic with a magic number k. By maintaining the order of the vertex and edge labelings and rotating them clockwise, an edge-magic cycle graph can be created from a vertex-magic cycle graph.

Figure 2 shows just one way to create a vertex-magic graph with three vertices. Depending upon the number of vertices and edges, a graph can be labeled in different ways with different magic numbers.

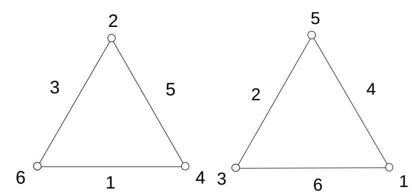

Figure 2: A cycle graph with a magic number of 10. If the labels are rotated clockwise, an edge-magic graph is created with a magic number of 10.

3 A Lower Bound for the Magic Number

In order to find a vertex-magic cycle graph, one cannot possibly try every number as the magic number. Having a range for the magic number would be helpful in finding vertex-magic labelings of a graph. One can determine a range for the magic number based on the number of vertices and edges of the graph.

Lemma 3.1. *If G is a vertex-magic graph with v vertices and e edges, then*

$$\frac{(v+e)(v+e+1)}{2v} + \frac{E_{sum}}{v} = k.$$

Proof. Let V_{sum} be the sum of all vertex labels and E_{sum} be the sum of all edge labels of graph G. Since each edge is incident to two vertices, each edge label will be counted towards its two adjacent vertices' magic numbers. Therefore,

$$V_{sum} + 2E_{sum} = vk.$$

Since edges and vertices are labeled 1 through $v+e$,

$$V_{sum} + E_{sum} = 1 + 2 + ... + (v+e) = \frac{(v+e)(v+e+1)}{2}.$$

Therefore,

$$\frac{(v+e)(v+e+1)}{2} + E_{sum} = vk,$$
$$\frac{(v+e)(v+e+1)}{2v} + \frac{E_{sum}}{v} = k.$$

□

Theorem 3.1. *Let G be a graph with v vertices and e edges. If G is a vertex-magic graph, then the magic number, k, is bounded such that*

$$\frac{e(e+1) + (v+e+1)(v+e)}{2v} \leq k \leq e + \frac{e(e+1) + (v+e+1)(v+e)}{2v}.$$

Proof. From Lemma 3.1, we can conclude that

$$E_{sum} = vk - \frac{(v+e+1)(v+e)}{2}.$$

The minimum E_{sum} occurs when numbers 1 through e are assigned to the edges. Therefore,

$$\frac{e(e+1)}{2} \leq E_{sum}.$$

The maximum E_{sum} occurs when numbers $v+1$ through $v+e$ are assigned to the edges. Hence,

$$\begin{aligned} E_{sum} &\leq \sum_{i=1}^{e} v + i, \\ &= \sum_{i=1}^{e} v + \sum_{i=1}^{e} i, \\ &= ve + \frac{e(e+1)}{2}. \end{aligned}$$

Hence,

$$\frac{e(e+1)}{2} \leq E_{sum} \leq ve + \frac{e(e+1)}{2},$$
$$\frac{e(e+1)}{2} \leq vk - \frac{(v+e+1)(v+e)}{2} \leq ve + \frac{e(e+1)}{2},$$
$$\frac{e(e+1)+(v+e+1)(v+e)}{2v} \leq k \leq e + \frac{e(e+1)+(v+e+1)(v+e)}{2v}.$$

□

In this paper, we will be looking at vertex-magic labelings of cycle graphs. We can now use Theorem 3.1 in order to find a range for the magic number for all cycle graphs.

Corollary 3.1. *Let G be a cycle graph with v vertices. If G is a vertex-magic graph, then the magic number, k, is bounded such that*

$$\frac{5}{2}v + \frac{3}{2} \leq k \text{ where } v \text{ is odd,}$$

and

$$\frac{5}{2}v + 2 \leq k \text{ where } v \text{ is even.}$$

Proof. For every cycle graph, $v = e$. By substituting v in for e in Theorem 3.1, we obtain,

$$\begin{aligned} k &\geq \frac{v(v+1)+(2v+1)(2v)}{2v}, \\ &= 2v+1+\frac{v+1}{2}, \\ &= \frac{5}{2}v+\frac{3}{2}. \end{aligned}$$

When v is odd, $v+1$ is divisible by 2, so the minimum magic number for a cycle graph with an odd number of vertices is $\frac{5}{2}v+\frac{3}{2}$. For an example of how to apply this bound to a vertex-magic problem, see example 3.1.

The formula, $k = 2v+1+\frac{v+1}{2}$, is possible only if v is odd, and a better bound can be found if v is even.

The minimum vertex labeling for a cycle graph occurs when numbers 1 through v are assigned to the edges. From Lemma 3.1, $2v+1+\frac{E_{sum}}{v} = k$. This implies that E_{sum} must be divisible by v. What will the edge labelings be in order to have a minimum vertex labeling for an even cycle graph where E_{sum} is divisible by v?

If v is even, can edges be labeled with numbers 1 through v? If this labeling is used, then $k = 2v+1+\frac{v+1}{2}$. We know that $v+1$ must be divisible by 2, however if v is even, then $v+1$ cannot be divisible by 2. Therefore, the numbers 1 through v cannot be used for the edges to find a minimum labeling.

Although the edge labeling of 1 through v cannot be used for a minimum labeling, consider adding $\frac{v}{2}$ to one of the labels. See examples 3.2 and 3.3 in order to see how E_{sum} can be adjusted so that the magic number is an integer. Adding $\frac{v}{2}$ to E_{sum} if v is even will create a valid edge labeling. Note that this new E_{sum} is divisible by v.

$$\begin{aligned} E_{sum} &= 1+2+3+...+v+\frac{v}{2}, \\ &= \frac{v(v+1)}{2}+\frac{v}{2}. \end{aligned}$$

As before,

$$
\begin{aligned}
k &= 2v + 1 + \frac{E_{sum}}{v}, \\
&\geq 2v + 1 + \frac{\frac{v(v+1)+v}{2}}{v}, \\
&= 2v + 1 + \frac{v+2}{2}, \\
&= \frac{5}{2}v + 2.
\end{aligned}
$$

□

Example 3.1. Let G be a graph with $v = 15$ vertices. If G is vertex-magic what would the minimum magic number be?

The smallest E_{sum} comes from labeling the edges with numbers 1 through 15. Referring to Corollary 3.1, the magic number for this particular graph would be $\frac{5}{2}(15) + \frac{3}{2} = 39$.

Example 3.2. Let G be a cycle graph with $v = 4$ vertices. What are some combinations of vertex labelings that could be used in creating a vertex-magic graph with a minimum k?

Begin by making $E_{sum} = 1 + 2 + 3 + 4$. Since $v = 4$, and 4 does not divide $E_{sum} = 10$, then E_{sum} must be adjusted such that E_{sum} is divisible by v. The edge labels cannot be shifted down because the smallest numbers are already on the edges. The closest multiple of 4 after 10 is 12. Therefore, E_{sum} must be shifted from 10 to 12 by adding 2. There are numerous ways of doing this. By adding 2 to 4, E_{sum} becomes $1 + 2 + 3 + 6$. Since the sum of the edges equals 12, $\{1, 2, 3, 6\}$ is a valid edge labeling for finding a possible minimum k.

Example 3.3. Let G be a cycle graph with $v = 6$ vertices. What is a valid E_{sum} that can be used to make G vertex-magic with a minimum k?

By using the same technique, the edge labelings can be altered for $v = 6$. Let $E_{sum} = 1 + 2 + 3 + 4 + 5 + 6$. Since this sum of 21 is not divisible by 6, we must add 3 to obtain 24 which is divisible by 6. Therefore, the minimum magic number is 24 when $v = 6$.

Knowing whether a graph has an even or odd number of vertices allows us to know the minimum bound on the magic number. It also determines what the possible edge labelings are in order to create a minimum vertex-magic graph.

4 An Upper Bound for the Magic Number

We have just seen how to determine the lower bound of the magic number of any cycle graph. There also exists an upper bound for any vertex-magic cycle graph. Again, the bounds depend on whether the graph has an even or odd number of vertices.

Corollary 4.1. *Let G be a graph with v vertices. If G is a vertex-magic graph, then the magic number, k, is bounded such that*

$$k \leq \frac{7}{2}v + \frac{3}{2} \text{ where } v \text{ is odd,}$$

and

$$k \leq \frac{7}{2}v + 1 \text{ where } v \text{ is even.}$$

Proof. The upper bound would occur on any graph when the largest numbers, $v+1$ through $2v$, are placed on the edges. Therefore,

$$\begin{aligned} E_{sum} &= \sum_{i=1}^{v} v + i, \\ &= v^2 + \frac{v(v+1)}{2}, \\ &= \frac{v(3v+1)}{2}. \end{aligned}$$

Also,

$$\begin{aligned} k &= 2v + 1 + \frac{E_{sum}}{v}, \\ &\leq 2v + 1 + \frac{v(3v+1)}{2v}, \\ &= 3v + 1 + \frac{v+1}{2}, \\ &= \frac{7}{2}v + \frac{3}{2}. \end{aligned}$$

As in the case of the lower bound, 2 must divide $v+1$ and v must divide E_{sum}. For any cycle graph with an odd number of vertices, this holds true when the edges are the largest numbers. So, the maximum possible k for any cycle graph with an odd number of vertices is $\frac{7}{2}v + \frac{3}{2}$. However, a cycle graph with an even number of vertices is more difficult.

When looking at the upper bound for k of a cycle graph with an even number of vertices, assume that the numbers $v+1$ through $2v$ are placed on the edges. Since $v+1$ is not divisible by 2, one must alter the edge labelings. To find the lower bound for the magic number when v is even, the number E_{sum} was increased, but in this case, the numbers cannot be increased because no label can be greater than $2v$. So, in order to find an E_{sum} divisible by v, the original E_{sum} must be decreased, see examples 4.1 and 4.2.

For any cycle graph with an even number of vertices, the number $\frac{v}{2}$ must be subtracted from the original sum. Therefore,

$$\begin{aligned} E_{sum} &= \frac{v(3v+1)}{2} - \frac{v}{2}, \\ &= \frac{v(3v)}{2}. \end{aligned}$$

Thus,

$$\begin{aligned} k &= 2v + 1 + \frac{E_{sum}}{v}, \\ &\leq 2v + 1 + \frac{3v}{2}, \\ &= \frac{7}{2}v + 1. \end{aligned}$$

□

Example 4.1. Let G be a cycle graph with $v = 4$ vertices. What are some possible combinations of vertex labelings such that G has a maximum k?

Label the graph such that $E_{sum} = 8+7+6+5 = 26$. The largest number less than 26 that is divisible by 4 is 24. Therefore 2, or $\frac{v}{2}$, must be subtracted from E_{sum} making the magic number 15. One possible edge labeling is {8,7,6,3}. By changing the 5 to a 3, E_{sum} becomes 8+7+6+3 which is divisible by v. Therefore, the edge labelings could possibly be used to label this graph with the maximum k.

Example 4.2. Let G be a graph where $v = 6$. What is the maximum E_{sum} for G such that G is vertex-magic?

If G is labeled with the largest numbers on the edges, G has an E_{sum} of 21. The next E_{sum} less than 21 that divides 6 is 18. Thus, 3 must be subtracted from the original E_{sum} to make the new E_{sum} equal to 18. This edge labeling has a magic number of 22.

Knowledge of the number of vertices on a cycle graph determines the upper and lower bounds for the magic number of that graph. These bounds are very useful when trying to create a vertex-magic cycle graph. By knowing the upper and lower bounds, we can conclude that if a graph has an odd number of vertices, then there exist $v + 1$ possible magic numbers. If a graph has an even number of vertices, then there exist v possible magic numbers.

5 Maximum and Minimum Magic Number for Odd Cycle Graphs

Now that we have seen bounds for the magic number, let us look at some concrete algorithms for vertex-magic labelings. Labeling the edges with numbers $v + 1$ through $2v$ will produce a vertex-magic graph with the maximum magic number for odd cycle graphs. On the other hand, assigning the numbers 1 through v to the edges and the numbers $v + 1$ through $2v$ to the vertices, will produce a vertex-magic graph with the minimum magic number.

Theorem 5.1. *Let G be a cycle graph with v vertices where v is odd. There exists a vertex-magic labeling with the numbers 1 to v located on the vertices and a magic number of $\frac{7}{2}v + \frac{3}{2}$, the upper bound for the magic number.*

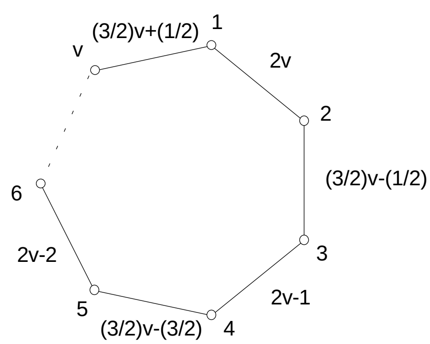

Figure 3: Vertices of a general cycle graph labeled with the smallest numbers.

Proof. Label the vertices with the consecutive numbers 1 through v in a clockwise manner. We will assume in this paper that no two edges of the cycle graph cross. An edge is to the right of a vertex if it is adjacent and clockwise to that vertex, and an edge is to the left of a vertex if it is adjacent and counterclockwise to that vertex. Starting with the edge to the right of the vertex labeled 1, go clockwise around the polygon twice labeling every other edge with consecutive numbers $v + 1$ through $2v$ in descending order starting with $2v$ (Figure 3). Any vertex of a cycle graph can be categorized into one of the following two categories:

Description of Case	Vertex Label	Left Edge Label	Right Edge Label
Every other vertex starting with 1	$2i+1$ $i = 0, ..., \frac{v-1}{2}$	$\frac{3v}{2} + \frac{1}{2} - i$	$2v - i$
Every other vertex starting with 2	$2i+2$ $i = 0, ..., \frac{v-1}{2} - 1$	$2v - i$	$\frac{3v}{2} - \frac{1}{2} - i$

Table 1

A vertex in the first case will have a magic number of

$$2i + 1 + \frac{3v}{2} + \frac{1}{2} - i + 2v - i = \frac{7v}{2} + \frac{3}{2}.$$

A vertex in the second case will have a magic number of

$$2i + 2 + 2v - i + \frac{3v}{2} - \frac{1}{2} - i = \frac{7v}{2} + \frac{3}{2}.$$

Thus, this graph is vertex-magic with a magic number of $\frac{7v}{2} + \frac{3}{2}$. $\square$

Figures 4 and 5 are examples of vertex-magic labeling with the smallest numbers on the vertices.

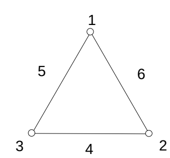

Figure 4: A cycle graph with 3 vertices and a magic number of 12.

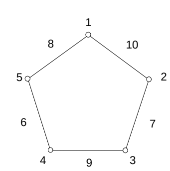

Figure 5: A cycle graph with 5 vertices and a magic number of 19.

Now, assign the largest numbers to the vertices. As expected, the maximum magic number is fixed and depends only on the number of vertices.

Theorem 5.2. *Let G be a cycle graph with v vertices where v is odd. There exists a vertex-magic labeling with the numbers $v+1$ to $2v$ located on the vertices and a magic number of $\frac{5}{2}v + \frac{3}{2}$, the lower bound for the magic number.*

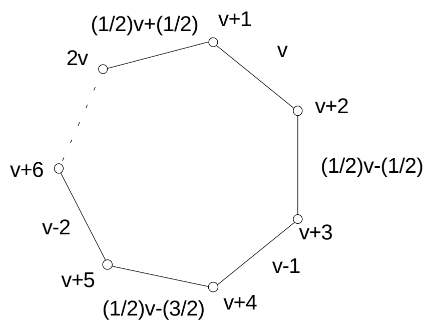

Figure 6: The vertices of a cycle graph labeled with the largest numbers.

Proof. Using the same ideas as in Theorem 5.1, a cycle graph can be labeled with the largest numbers on the vertices. Label the vertices with the consecutive numbers $v+1$ through $2v$ in a clockwise manner. Starting with the edge to the right of vertex $v+1$, go clockwise around the polygon twice labeling every other edge with consecutive numbers 1 through v in descending order starting with v (Figure 6). The vertices of the cycle graph can be categorized into one of the following two categories:

Description of Case	Vertex Label	Left Edge Label	Right Edge Label
Every other vertex starting with $v+1$	$v+1+2i$ $i = 0, ..., \frac{v-1}{2}$	$\frac{v}{2} + \frac{1}{2} - \frac{2i}{2}$	$v - i$
Every other vertex starting with $v+2$	$v+2+2i$ $i = 0, ..., \frac{v-1}{2} - 1$	$v - i$	$\frac{v}{2} - \frac{1}{2} - \frac{2i}{2}$

Table 2

A vertex in the first case will have a magic number of

$$v + 1 + 2i + v - i + \frac{v + 1 - 2i}{2} = \frac{5v}{2} + \frac{3}{2}.$$

A vertex in the second case will have a magic number of

$$v + 2 + 2i + v - i + \frac{v - 1 - 2i}{2} = \frac{5v}{2} + \frac{3}{2}.$$

This cycle graph is vertex-magic and has a magic number of $\frac{5v}{2} + \frac{3}{2}$. □

Some examples of this type of labeling are shown in Figures 7 and 8.

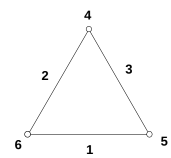

Figure 7: A cycle graph with 3 vertices and a magic number of 9.

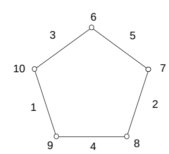

Figure 8: A cycle graph with 5 vertices and a magic number of 14.

We have created algorithms for producing vertex-magic odd cycle graphs with minimum and maximum magic numbers. Since there exist odd cycle graphs where the magic number is equal to the bounds, we conclude that the bounds for odd cycle graphs are sharp. The bounds for even cycle graphs are conjectured to be sharp, however finding an algorithm that makes these bounds sharp is more difficult.

6 Minimum Magic Number for Even Cycle Graphs

A vertex-magic cycle graph with an even number of vertices can be found for the minimum k. Wallis, in [4], gives the following construction for a vertex-labeling for even cycle graphs that gives the minimum magic number.

Conjecture 6.1. *Let G be a cycle graph with v vertices where v is even. There exists a vertex-magic labeling for G with the minimum magic number $k = \frac{5}{2}v+2$.*

If we can find edge labelings that create a vertex-magic graph, then by adding the two incident edges of a vertex and subtracting that sum from the magic number, the vertex labels can be easily obtained. The edges of a cycle graph with an even number of vertices can be labeled as follows:

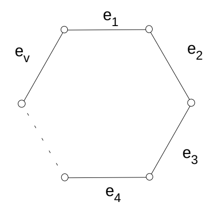

Figure 9: The edges of a cycle graph.

Let $v = 2n$. The value of n can be either even or odd, and the construction of the vertex-magic graph depends on n. If n is even, the following is a construction for how to label the edges:

$$e_i = \begin{cases} \dfrac{i+1}{2}, & i = 1, 3, ..., n+1, \\[2ex] 3n, & i = 2, \\[2ex] \dfrac{2n+i}{2}, & i = 4, 6, ..., n, \\[2ex] \dfrac{2n+i-1}{2}, & i = n+3, n+5, ..., 2n-1, \\[2ex] \dfrac{i+2}{2}, & i = n+2, n+4, ..., 2n. \end{cases}$$

Figures 10 and 11 are two examples of this type of labeling. A cycle graph with $v = 8$ and $n = 4$ should have a minimum k of 22. By using the given edge labelings for an even n, one can find the edges and vertices for this graph.

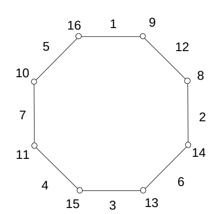

Figure 10: A cycle graph with the minimum magic number of 22.

Another example of this type of labeling occurs when $n = 6$, or $v = 12$. Using Conjecture 6.1, we calculate that the magic number should be 32.

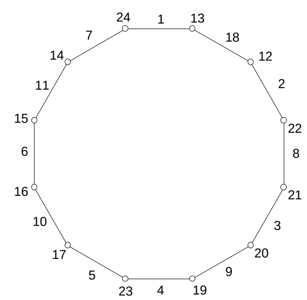

Figure 11: A cycle graph with the minimum magic number of 32.

If n is odd, a similar technique can be used to label the cycle graph.

$$e_i = \begin{cases} \dfrac{i+1}{2}, & i = 1, 3, ..., n, \\ 3n, & i = 2, \\ \dfrac{2n+i+2}{2}, & i = 4, 6, ..., n-1, \\ \dfrac{n+3}{2}, & i = n+1, \\ \dfrac{2n+i}{2}, & i = n+3, n+5..., 2n-2, \\ \dfrac{i+3}{2}, & i = n+2, n+4, ..., 2n-1, \\ n+2, & i = 2n. \end{cases}$$

Figures 12 and 13 show vertex-magic cycles where n is odd. If $v = 6$, then $n = 3$, and the magic number is 17.

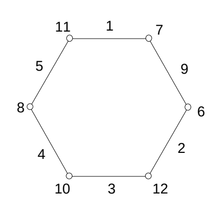

Figure 12: A cycle graph with the minimum magic number of 17.

The following graph shows a vertex-magic labeling when $v = 10$. Thus, $n = 5$, and the magic number is 27.

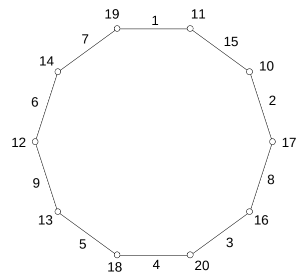

Figure 13: A cycle graph with the minimum magic number of 27.

We have just shown how to produce vertex-magic graphs with the minimum magic number for both even and odd cycles. We have also shown how to create vertex-magic graphs with the maximum magic number for odd cycles.

7 Maximum Magic Number for Even Cycle Graphs

Using the same logic as in Section 6, one can find an algorithm for creating a vertex-magic cycle graph with the maximum magic number.

Theorem 7.1. *Let G be a cycle graph with v vertices where v is even. If Conjecture 6.1 holds, then there exists a vertex-magic labeling for G with the maximum magic number $k = \frac{7}{2}v + 1$.*

One can see that in order to obtain the minimum k in Conjecture 6.1 for an even v, the edge labeling should be $1, 2, 3, ..., v-1, v+\frac{v}{2}$. In order to obtain the maximum k, the edge labels should be $2v$, $2v-1, ..., v+2$, $v+1-\frac{v}{2}$. These values are exactly $2v + 1$ minus the numbers on the edges in the minimum labeling. The following cases produce the edge labelings for a maximum magic vertex graph. Using the maximum k, we can subtract the sum of the two edges in order to label each vertex. Again, let $v = 2n$ and let n be even.

$$
e_i = \begin{cases} 2v - \dfrac{i+1}{2} + 1, & i = 1, 3, ..., n+1, \\ 2v - 3n + 1, & i = 2, \\ 2v - \dfrac{2n+i}{2} + 1, & i = 4, 6, ..., n, \\ 2v - \dfrac{2n+i-1}{2} + 1, & i = n+3, n+5, ..., 2n-1, \\ 2v - \dfrac{i+2}{2} + 1, & i = n+2, n+4, ..., 2n. \end{cases}
$$

Figures 14 and 15 show vertex-magic cycle graphs where n is even.

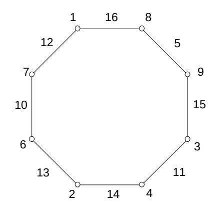

Figure 14: A cycle graph with the maximum magic number of 29.

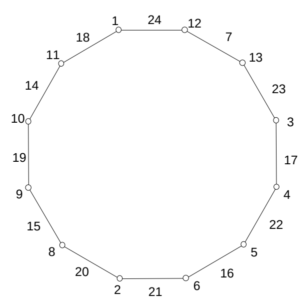

Figure 15: A cycle graph with the maximum magic number of 43.

Now, let n be odd. Subtracting the edge labels of the algorithm in Conjecture 6.1 from $2v$ and adding 1, a new edge labeling with a maximum k can be obtained.

$$e_i = \begin{cases} 2v - \dfrac{i+1}{2} + 1, & i = 1, 3, ..., n, \\ 2v - 3n + 1, & i = 2, \\ 2v - \dfrac{2n+i+2}{2} + 1, & i = 4, 6, ..., n-1, \\ 2v - \dfrac{n+3}{2} + 1, & i = n+1, \\ 2v - \dfrac{2n+i}{2} + 1, & i = n+3, n+5..., 2n-2, \\ 2v - \dfrac{i+3}{2} + 1, & i = n+2, n+4, ..., 2n-1, \\ 2v - (n+2) + 1, & i = 2n. \end{cases}$$

Figures 16 and 17 are examples of cycle graphs when n is odd.

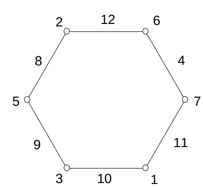

Figure 16: A cycle graph with the maximum magic number of 22.

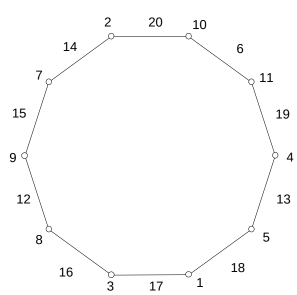

Figure 17: A cycle graph with the maximum magic number of 36.

Different algorithms can be found for other odd cycle graphs such that the magic numbers are within the range for k. Since we have found algorithms for creating vertex-magic graphs such that the magic number is the minimum and maximum bound, we can conclude that the bounds are sharp.

8 Odd and Even Numbers on the Vertices

Another possible labeling for odd vertex magic cycle graphs is to assign the odd numbers to the vertices.

Theorem 8.1. *Let G be a cycle graph with v vertices where v is odd. There exists a vertex-magic labeling for G with the odd numbers from 1 to $2v-1$ located on the vertices and a magic number of $3v+2$.*

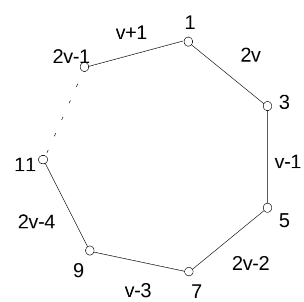

Figure 18: The vertices of a general cycle graph labeled with odd numbers.

Proof. In order to label an odd cycle graph with odd numbers on the vertices, the vertices can be labeled with the consecutive odd numbers 1 through $2v-1$ in a clockwise manner. Starting with the edge to the right of the vertex labeled 1, go clockwise around the polygon twice labeling every other edge with consecutive even numbers 2 through $2v$ in descending order starting with $2v$ (Figure 18). In general, any vertex of this labeling can be categorized into one of the following two categories:

Description of Case	Vertex Label	Left Edge Label	Right Edge Label
Every other vertex starting with 1	$4i+1$ $i=0,...,\frac{v-1}{2}$	$v+1-2i$	$2v-2i$
Every other vertex starting with 3	$4i+3$ $i=0,...,\frac{v-1}{2}-1$	$2v-2i$	$v-1-2i$

Table 3

In the first case, the magic number is $4i+1+v+1-2i+2v-2i$, or $3v+2$. In the second case, the magic number is $4i+3+2v-2i+v-1-2i$, which is also $3v+2$. Therefore, the graph is vertex-magic with a magic number of $3v+2$. □

Some examples of such labelings and their magic numbers are given in Figures 19 and 20.

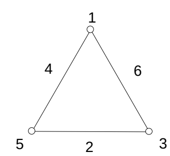

Figure 19: A cycle graph with 3 vertices and a magic number of 11.

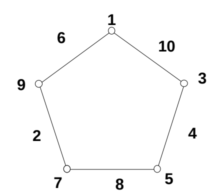

Figure 20: A cycle graph with 5 vertices and a magic number of 17.

Using similar ideas, one can label an odd cycle graph with the even numbers on the vertices. Like the odd numbers on the vertices, the even numbers on the vertices produce a pattern that is dependent upon the number of vertices.

Theorem 8.2. *Let G be a cycle graph with v vertices where v is odd. There exists a vertex-magic labeling for G with the even numbers from 2 to $2v$ located on the vertices and a magic number of $3v+1$.*

Proof. A similar technique to the previous proof can be used in order to label a cycle graph with even numbers on the vertices. The vertices are labeled with the consecutive even numbers 2 through $2v$ in a clockwise manner. Starting with the edge to the right of the vertex labeled 2, go clockwise around the polygon twice labeling every other edge with consecutive odd numbers 1 through $2v-1$ in descending order starting with $2v-1$ (Figure 21). For this labeling, a vertex can be categorized into one of the following two categories:

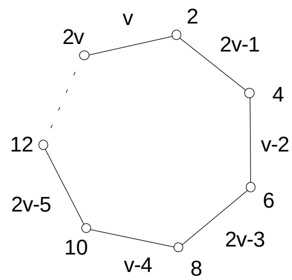

Figure 21: The vertices of a general cycle graph labeled with even numbers.

Description of Case	Vertex Label	Left Edge Label	Right Edge Label
Every other vertex starting with 2	$4i+2$ $i = 0, ..., \frac{v-1}{2}$	$v-2i$	$2v-1-2i$
Every other vertex starting with 4	$4i+4$ $i = 0, ..., \frac{v-1}{2}-1$	$2v-1-2i$	$v-2i-2$

Table 4

In the first case, the magic number is $4i+2+v-2i+2v-1-2i$, or $3v+1$. In the second case, the magic number is $4i+4+2v-1-2i+v-2i-2$, or $3v+1$. Therefore, the graph is vertex-magic with a magic number of $3v+1$. $\square$

Some examples of this type of labeling are given in Figures 22 and 23.

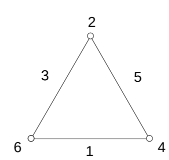

Figure 22: A cycle graph with 3 vertices and a magic number of 10.

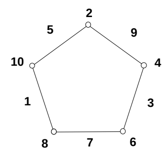

Figure 23: A cycle graph with 5 vertices and a magic number of 16.

Finding the average of the minimum and maximum bounds, we get,

$$\frac{\frac{7}{2}v + \frac{3}{2} + \frac{5}{2}v + \frac{3}{2}}{2} = 3v + \frac{3}{2}.$$

This value is not a possible value for k. However the two middle k values, $3v + \left\lfloor \frac{3}{2} \right\rfloor = 3v + 1$ and $3v + \left\lceil \frac{3}{2} \right\rceil = 3v + 2$ occur when the even and odd numbers are placed on the vertices respectively.

9 Conclusion

This paper addresses labeling graphs in such a way that they are vertex-magic. Exploring bounds for magic numbers is an interesting problem for all graphs. We are able to obtain more accurate bounds if we limit the graphs to be cycle graphs. If a graph has an odd number of vertices, algorithms can be easily found to produce different magic-vertex graphs with the maximum and minimum magic number. Also, every cycle graph with an odd number of vertices can be made into a vertex-magic graph if the odd numbers or even numbers are placed on the vertices. Some interesting problems arise when one begins to look at cycle graphs with an even number of vertices. Bounds for the magic number change, and it becomes harder to make these graphs vertex-magic. We have shown some algorithms for finding vertex-magic cycle graphs with a magic number that lies within the bounds. In [4], Wallis explores a variety of properties of magic graphs, including some of the properties discussed above.

Vertex-magic graphs continue to attract the attention of many graph theorists. Some interesting questions that arise from this paper are: For a given cycle graph, is there a vertex-magic labeling associated with every magic number within the bounds? Can an algorithm be created for finding all possible vertex-magic labelings for a given cycle graph? Can these properties be generalized for any graph?

References

[1] Bollobas, Bela. *Graph Theory: An Introductory Course*, Springer-Verlag, New York, NY, 1979.

[2] Finkbeiner, D. and Lindstrom, W. *A Primer of Discrete Mathematics*, W.H. Freeman and Co., New York, NY, 1987.

[3] Locke, Stephen C. "Graph Theory." Online Posting. http://math.fau.edu/locke/graphthe.htm.

[4] Wallis, W.D., *Magic Graphs*, Birkhauser, New York, NY, 2001.