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Abstract. A graph on n vertices is said to admit a prime labeling if we can label its vertices with the first n natural numbers such that any two adjacent vertices have relatively prime labels. Here we extend the idea of prime labeling to the Gaussian integers, which are the complex numbers whose real and imaginary parts are both integers. We begin by defining an order on the Gaussian integers that lie in the first quadrant. Using this ordering, we show that all trees of order at most 72 admit a prime labeling with the Gaussian integers.

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1 Introduction

A graph on n vertices admits a **prime labeling** if its vertices can be labeled with the first n natural numbers in such a way that any two adjacent vertices have relatively prime labels. Many families of graphs are known to admit prime labelings, such as paths, stars, caterpillars, complete binary trees, spiders, palm trees, fans, flowers, and many more [1, 4]. Entringer conjectured that any tree admits a prime labeling, but this conjecture has not been proven for all trees [1]. In this paper we extend the study of prime labeling to the Gaussian integers, the complex numbers of the form $a + b\mathbf{i}$ where $a, b \in \mathbb{Z}$ and $\mathbf{i}^2 = -1$. In the context of the Gaussian integers, a prime Gaussian integer has no divisors other than itself and the units, and two Gaussian integers are relatively prime if their only common divisors are units.

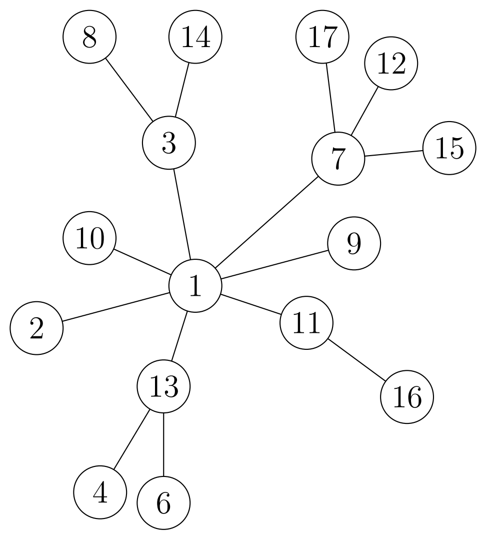

Figure 1: Example prime labeling of a tree on 16 vertices

We begin with necessary background on Gaussian integers, graphs, and prime labeling in Section 2. In order to extend the notion of a prime labeling to Gaussian integers, we must define what we mean by "the first n Gaussian integers." Later in Section 2, we define a spiral ordering on the Gaussian integers that allows us to linearly order the Gaussian integers. This spiral ordering preserves many familiar properties of the natural ordering on $\mathbb{N}$. For example, the spiral ordering alternates parity, and consecutive odd integers in the spiral ordering are relatively prime. We discuss further properties of the spiral ordering in Section 2. In Section 3 we examine trees with few vertices. We use an approach introduced by Pikhurko [3] to show that any tree with at most 72 vertices admits a prime labeling with the Gaussian integers. Finally, the appendix contains reference tables for the spiral ordering and the methods of Section 3.

2 Background and Definitions

2.1 Background on Gaussian Integers

We begin with some relevant background on Gaussian integers to provide a foundation for our work.

The **Gaussian integers**, denoted $\mathbb{Z}[\mathbf{i}]$, are the complex numbers of the form $a+b\mathbf{i}$, where $a, b \in \mathbb{Z}$ and $\mathbf{i}^2 = -1$. A **unit** in the Gaussian integers is one of $\pm 1, \pm\mathbf{i}$. An **associate** of a Gaussian integer α is $u \cdot \alpha$ where u is a Gaussian unit. The **norm** of a Gaussian integer $a+b\mathbf{i}$, denoted by $N(a+b\mathbf{i})$, is given by a^2+b^2. A Gaussian integer is **even** if it is divisible by $1+\mathbf{i}$ and **odd** otherwise. This is because Gaussian integers with even norms are divisible by $1+\mathbf{i}$.

Definition 2.1. A Gaussian integer, ρ, is **prime** if its only divisors are ± 1, $\pm\mathbf{i}$, $\pm\rho$, or $\pm\rho\mathbf{i}$.

Besides this definition of Gaussian primes, we have the following characterization theorem for Gaussian primes. Further information on the Gaussian integers can be found in Rosen's *Elementary Number Theory* [5].

Theorem 2.2. *A Gaussian integer $\alpha \in \mathbb{Z}[\mathbf{i}]$ is prime if and only if either*

- $\alpha = \pm 1 \pm \mathbf{i}$,
- $N(\alpha)$ *is a prime integer congruent to* $1 \pmod 4$, *or*
- $\alpha = p + 0\mathbf{i}$ *or* $\alpha = 0 + p\mathbf{i}$ *where p is a prime in $\mathbb{Z}$ and* $|p| \equiv 3 \pmod 4$.

Definition 2.3. Let α be a Gaussian integer and let β be a Gaussian integer. We say α and β are **relatively prime** or **coprime** if their only common divisors are the units in $\mathbb{Z}[\mathbf{i}]$.

2.2 Background on Graphs

We also need some definitions relating to graphs before we continue to Gaussian prime labeling itself.

Definition 2.4. A **graph** $G = (V, E)$ consists of a finite, nonempty set V of **vertices** and a set E of unordered pairs of distinct vertices called **edges**. If $\{u, v\} \in E$, we say u and v are connected by an edge and write $uv \in E$ for brevity.

A graph is commonly represented as a diagram representing a collection of dots (vertices) connected by line segments (edges).

Definition 2.5. The **degree** of a vertex is the number of edges incident to that vertex.

Definition 2.6. A **tree** is a connected graph that contains no cycles.

Definition 2.7. An **internal node** of a tree is any vertex of degree greater than 1. A **leaf** or **endvertex** of a tree is a vertex of degree 1.

For more information on graph theory, we refer to Trudeau's *Introduction to Graph Theory* [6].

2.3 Prime Labeling with Gaussian Integers

Our goal is to extend the study of the prime labeling of trees to the Gaussian integers. Although there are several standard total orderings of the Gaussian integers, none of these are satisfactory for prime labeling. The lexicographic ordering would result in a labeling of any finite tree strictly with Gaussian integers lying on a vertical line. A norm-based ordering will not always alternate in parity, which is highly desirable for labeling. Therefore we must first give an appropriate definition of "the first n Gaussian integers." We propose the following ordering:

Definition 2.8. The **spiral ordering of the Gaussian integers** is a recursively defined ordering of the Gaussian integers. We denote the n^{th} Gaussian integer in the spiral ordering by γ_n. The ordering is defined beginning with $\gamma_1 = 1$ and continuing as:

$$\gamma_{n+1} = \begin{cases} \gamma_n + \mathbf{i}, & \text{if } \operatorname{Re}(\gamma_n) \equiv 1 \pmod 2,\ \operatorname{Re}(\gamma_n) > \operatorname{Im}(\gamma_n) + 1 \\ \gamma_n - 1, & \text{if } \operatorname{Im}(\gamma_n) \equiv 0 \pmod 2,\ \operatorname{Re}(\gamma_n) \le \operatorname{Im}(\gamma_n) + 1,\ \operatorname{Re}(\gamma_n) > 1 \\ \gamma_n + 1, & \text{if } \operatorname{Im}(\gamma_n) \equiv 1 \pmod 2,\ \operatorname{Re}(\gamma_n) < \operatorname{Im}(\gamma_n) + 1 \\ \gamma_n + \mathbf{i}, & \text{if } \operatorname{Im}(\gamma_n) \equiv 0 \pmod 2,\ \operatorname{Re}(\gamma_n) = 1 \\ \gamma_n - \mathbf{i}, & \text{if } \operatorname{Re}(\gamma_n) \equiv 0 \pmod 2,\ \operatorname{Re}(\gamma_n) \ge \operatorname{Im}(\gamma_n) + 1,\ \operatorname{Im}(\gamma_n) > 0 \\ \gamma_n + 1, & \text{if } \operatorname{Re}(\gamma_n) \equiv 0 \pmod 2,\ \operatorname{Im}(\gamma_n) = 0 \end{cases}$$

This is illustrated below.

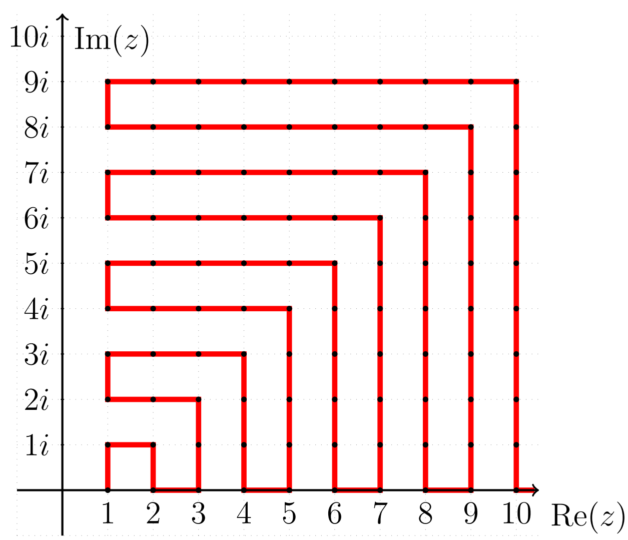

Figure 2: Spiral Ordering of Gaussian Integers

Under this ordering, the first 10 Gaussian integers are

$$1,\ 1+\mathbf{i},\ 2+\mathbf{i},\ 2,\ 3,\ 3+\mathbf{i},\ 3+2\mathbf{i},\ 2+2\mathbf{i},\ 1+2\mathbf{i},\ 1+3\mathbf{i},\ \ldots,$$

and we write $[\gamma_n]$ to denote the set of the first n Gaussian integers in the spiral ordering.

We exclude the imaginary axis to ensure that the spiral ordering excludes associates. Consecutive Gaussian integers in this ordering are separated by a unit and therefore alternate parity, as in the usual ordering of $\mathbb{N}$. However, several properties of the ordinary integers do not hold. For example, let γ_p be a prime Gaussian integer. Then in the set of the first $N(\gamma_p) \cdot k$ Gaussian integers under our ordering, ($k \in \mathbb{N}$), it is not guaranteed that there are exactly k multiples of γ_p (or any other residue class mod γ_p). Furthermore, odd Gaussian integers with indices separated by a power of two are not guaranteed to be relatively prime to each other. A final curious property of this ordering is that a Gaussian prime can be preceded by one of its multiples in this order (see Lemma 3.1).

This definition of the spiral ordering for the Gaussian integers leads to the following definition of prime labeling of trees with Gaussian integers.

Definition 2.9. A **Gaussian prime labeling** of a graph G on n vertices is a labeling of the vertices of G with the first n Gaussian integers in the spiral ordering such that if two vertices are adjacent, their labels are relatively prime. When it is necessary, we view the labeling as a bijection $\ell : V(G) \to [\gamma_n]$.

2.4 Properties of the Spiral Ordering

We prove several lemmas about Gaussian integers that will help us to prove that small trees have a Gaussian prime labeling.

Lemma 2.10. *Let α be a Gaussian integer and u be a unit. Then α and $\alpha + u$ are relatively prime.*

Proof. Suppose that there exists a Gaussian integer λ such that $\lambda|\alpha$ and $\lambda|(\alpha + u)$. This means that λ must also divide $u = (\alpha + u) - \alpha$. But the only Gaussian integers that divide u are the units, so λ must be a unit. Thus α and $\alpha + u$ are relatively prime. □

The following corollary is immediate because consecutive Gaussian integers in the spiral ordering have a difference of one unit.

Corollary 2.11. *Consecutive Gaussian integers in the spiral ordering are relatively prime.*

Lemma 2.12. *Let α be an odd Gaussian integer, let c be a positive integer, and let u be a unit. Then α and $\alpha + u \cdot (1 + \mathbf{i})^c$ are relatively prime.*

Proof. Suppose that α and $\alpha + u \cdot (1 + \mathbf{i})^c$ share a common divisor γ. It follows that γ divides $u \cdot (1 + \mathbf{i})^c = (\alpha + u \cdot (1 + \mathbf{i})^c) - \alpha$. However, the only divisors of $u \cdot (1 + \mathbf{i})^c$ in $\mathbb{Z}[\mathbf{i}]$ are $(1 + \mathbf{i})$ or its associates and the units in $\mathbb{Z}[\mathbf{i}]$. Since α is odd, it is not divisible $(1 + \mathbf{i})$ or its associates because those numbers are all even. Therefore, γ must be a unit. Hence α and $\alpha + u \cdot (1 + \mathbf{i})^c$ are relatively prime. □

Corollary 2.13. *Consecutive odd Gaussian integers in the spiral ordering are relatively prime.*

Proof. Consecutive odd Gaussian integers in the spiral ordering differ by two units. The only possible differences between them are therefore $1+\mathbf{i}$, 2, or one of their associates. Since $2 = -\mathbf{i}(1+\mathbf{i})^2$, all of these differences are of the form $u \cdot (1+\mathbf{i})^c$ so the result follows from Lemma 2.12. □

3 Results for Small Trees

We would like to be able to say whether trees with a certain small number of vertices have a Gaussian prime labeling. In order to do this, we prove three lemmas, which we can use in conjunction with a lemma of Pikhurko [3] to prove that all trees of order at most 72 admit a Gaussian prime labeling. Our technique for doing so is a modification of Pikhurko's work in the regular integers.

A curious fact about the spiral ordering is that a Gaussian prime can occur after one of its multiples in the spiral ordering. For example, $\gamma_6 = 3+\mathbf{i} = (1-\mathbf{i})(1+2\mathbf{i})$ occurs before $\gamma_9 = 1+2\mathbf{i}$. The following lemma says that this is only possible under very specific circumstances.

Lemma 3.1. *Let ρ be a prime Gaussian integer. If there exists a multiple of ρ, μ, preceding ρ in the spiral ordering, then ρ has the form $1+b\mathbf{i}$ for some $b > 1$ and $\mu = (1-\mathbf{i}) \cdot \rho$.*

Proof. Let $\rho = a+b\mathbf{i}$ and assume that $\mu = (a+b\mathbf{i})\cdot(c+d\mathbf{i})$ precedes ρ in the spiral ordering. Note that $N((a+b\mathbf{i})(c+d\mathbf{i})) = (a^2+b^2)(c^2+d^2)$ and that $N(c+d\mathbf{i}) > 1$ because associates are excluded from the spiral ordering. There are now three cases to consider: $a \geq b$, $a < b$ and $N(c+d\mathbf{i}) > 2$, and lastly $a < b$ and $N(c+d\mathbf{i}) = 2$.

Case 1: If $a \geq b$, any Gaussian integer preceding ρ in the spiral ordering has real and imaginary parts at most a hence its norm is bounded by $2a^2$. Also we know that $c^2+d^2 \geq 2$, so we can write:

$$2(a^2+b^2) \leq N(\rho \cdot (c+d\mathbf{i})) \leq 2a^2,$$

which implies $2b^2 \leq 0$. This is clearly impossible because the natural ordering on $\mathbb{N}$ is preserved by the spiral ordering. So we conclude that $a < b$ if this multiple exists.

Case 2: If $a < b$, then any Gaussian integer preceding ρ in the spiral ordering has real part at most $b+1$ and imaginary part at most b, and thus has norm bounded by $(b+1)^2+b^2$. Since we have specified that $N(c+d\mathbf{i}) > 2$ we know that $c^2+d^2 \geq 4$ and we can write:

$$\begin{aligned} & 4(a^2+b^2) \leq N(\rho \cdot (c+d\mathbf{i})) \leq (b+1)^2+b^2 \\ \text{so} \quad & 4a^2+2b^2 \leq 2b+1 \end{aligned}$$

By assumption, $a < b$ so $b > 1$. However $2b^2 > 2b+1$ for all $b > 1$, so this is impossible. Thus we conclude that $c+d\mathbf{i}$ must be $1+\mathbf{i}$ or an associate for the multiple of ρ to precede ρ.

Case 3: Here we directly examine the product $\rho \cdot (1 - \mathbf{i})$. We choose the associate $1 - \mathbf{i}$ so that the resulting product will remain in the first quadrant. In order for this product to precede ρ in the ordering, its real part must be at most $b + 1$. We compute the product

$$\begin{aligned}\rho \cdot (1 - \mathbf{i}) &= (a + b\mathbf{i})(1 - \mathbf{i}) \\ &= (a + b) + (b - a)\mathbf{i}.\end{aligned}$$

Since $a + b \leq b + 1$, it follows that $a \leq 1$. But the spiral ordering excludes the imaginary axis so $a = 1$. The claim follows. □

The fact that a prime can be preceded by one of its multiples in the spiral ordering makes labeling small trees more difficult. In prime labeling with the ordinary integers, if we can label every tree on n vertices with the interval of integers $[n] = \{1, 2, \ldots, n\}$ and p is a prime greater than n, we can also label any tree on $n + 1$ vertices with $[n] \cup \{p\}$ by labeling any endvertex with p.

We would like to be able to make a similar argument for the Gaussian integers. For ease of notation, we write $[\gamma_n]$ to denote the set of the first n Gaussian integers in the spiral ordering. Consider a prime γ_p for some $p > n$. If every tree on n vertices admits a Gaussian prime labeling and γ_p does not have the form $1 + b\mathbf{i}$, we can label any tree on $n + 1$ vertices in the same way as with ordinary integers.

If γ_p has the form $1 + b\mathbf{i}$ we must avoid labeling adjacent vertices with γ_p and $(1 - \mathbf{i})\gamma_p$. Fortunately, this is also possible. The key idea of the next lemma is that if we can find a Gaussian prime labeling of a tree of order m where $[\gamma_m]$ includes $(1 - \mathbf{i})\gamma_p$ but does not include γ_p itself, then we can also find a Gaussian prime labeling of a tree of order $m + 1$ using the set $[\gamma_m] \cup \{\gamma_p\}$. Further, we can choose any vertex of this tree to either be labeled with γ_p or 1. This fact will be useful later.

Lemma 3.2. *Let $\gamma_p = 1 + b\mathbf{i}$ be a Gaussian prime and let T be a tree on m vertices with $p - b \leq m \leq p$. Choose any vertex x in T. If $T - x$ admits a Gaussian prime labeling with $[\gamma_{m-1}]$, then T admits a Gaussian prime labeling with $[\gamma_{m-1}] \cup \{\gamma_p\}$ such that x is labeled with either 1 or γ_p.*

Proof. Let T be a tree on m vertices to be labeled with $[\gamma_{m-1}] \cup \{\gamma_p\}$. Choose a vertex x in T and label it with γ_p, and label $T - x$ with $[\gamma_{m-1}]$. Note that the index of $(1 - \mathbf{i})\gamma_p$ is $p - b - 1 \leq m - 1$ [2, Lemma 2.11], so there is a vertex v in $T - x$ labeled with $(1 - \mathbf{i})\gamma_p$. This is the only multiple of γ_p in $[\gamma_{m-1}]$ by Lemma 3.1. If v is not adjacent to x we have exhibited a Gaussian prime labeling of T. For the remainder of the proof, we assume v is adjacent to x and examine the labeling of the tree near these vertices.

If there exists a vertex v_2 labeled with $\alpha = u \cdot (1 + \mathbf{i})^c$ for some unit u and integer $c \geq 1$ not adjacent to x, swap the labels of v_2 and v. Because both α and$(1 - \mathbf{i})\gamma_p$ are adjacent to odd Gaussian integers, both are relatively prime to all odd Gaussian integers besides γ_p, and α is relatively prime to γ_p, this gives a prime labeling with x labeled with γ_p.

If such a vertex v_2 does not exist, then all powers of $1 + \mathbf{i}$ and their associates label vertices adjacent to x. There exists a vertex v_3 labeled with 1. If v_3 is not adjacent to v, swap the labels of v_3 and x. As γ_p is relatively prime to all Gaussian integers in the labeling besides $(1-\mathbf{i})\gamma_p$ and 1 is relatively prime to all Gaussian integers, this gives a prime labeling with x labeled with 1.

Finally, assume all powers of $1+\mathbf{i}$ and their associates label vertices adjacent to x and v_3 is adjacent to v. The vertex v^* labeled with $1+\mathbf{i}$ is adjacent to x. Swap the labels of v_3 and x, then swap the labels of v^* and v. This ensures that γ_p does not label a vertex adjacent to the vertex labeled with $(1-\mathbf{i})\gamma_p$ and so by a similar argument as above this gives a prime labeling with x labeled with 1. □

The next key piece of our result is a Gaussian analogue of Lemma 1 in Pikhurko's paper [3].

Lemma 3.3. *Let γ_p be a prime Gaussian integer with index p in the spiral ordering. If all trees of order $p-1$ admit a Gaussian prime labeling, then all trees of orders $p, p+1$, or $p+2$ also admit a Gaussian prime labeling.*

Proof. Suppose that every tree of order $p-1$ admits a Gaussian prime labeling. Let T be a tree.

First suppose T has order p. If γ_p does not have the form $1+b\mathbf{i}$, then we label an endvertex in T with γ_p. Because γ_p is relatively prime to all Gaussian integers in $[\gamma_{p-1}]$ we can label the remainder of the tree by the assumption and this gives a Gaussian prime labeling. If $\gamma_p = 1+b\mathbf{i}$ for some b, then by Lemma 3.2 there exists a Gaussian prime labeling of the tree.

Next assume T has order $p+1$. Consider the longest path in in T, denoted by $\{v_1,\ v_2,\ \ldots,\ v_k\}$. Since we chose a longest path, v_1 must be a leaf vertex. If γ_p does not have the form $1+b\mathbf{i}$, label v_1 with γ_{p+1} and v_2 with γ_p. By assumption, $T-\{v_1,\ v_2\}$ can be labeled by $[\gamma_{p-1}]$, which gives a Gaussian prime labeling of T.

If γ_p has the form $1+b\mathbf{i}$, label v_1 with γ_{p+1}. By Lemma 3.2 we can now choose $x=v_2$ and obtain a prime labeling of $T-v_1$ with v_2 labeled by either 1 or γ_p, both of which are relatively prime to γ_{p+1}. This gives a Gaussian prime labeling of T.

Finally, assume T has order $p+2$ and again consider the longest path in the tree, denoted by $\{v_1,\ v_2,\ \ldots,\ v_k\}$. First assume γ_p does not have the form $1+b\mathbf{i}$, we consider two cases. If $\deg(v_2)=2$, label v_1 with γ_{p+2}, v_2 with γ_{p+1}, v_3 with γ_p, and $T-\{v_1,\ v_2,\ v_3\}$ with $[\gamma_{p-1}]$. If $\deg(v_2)>2$, there is an endvertex $u\neq v_3$ adjacent to v_2 and we label v_1 with γ_{p+2}, u with γ_{p+1}, v_2 with γ_p, and $T-\{v_1,\ v_2,\ u\}$ with $[\gamma_{p-1}]$. This yields a Gaussian prime labeling.

If $\gamma_p = 1+b\mathbf{i}$ for some b, there are again two cases. If $\deg(v_2)=2$, label $v_1=\gamma_{p+1}$ and $v_2=\gamma_{p+2}$, and by Lemma 3.2 we can choose $x=v_3$ and obtain a prime labeling of the remainder of the tree so that v_3 is labeled with either 1 or γ_p, both of which are relatively prime to γ_{p+1}. If $\deg(v_3)>2$, there is an endvertex $u\neq v_3$ adjacent to v_2 and we label v_1 and u as above and by Lemma 3.2 we can choose $x=v_2$ and obtain a prime labeling of the remainder of the tree so that v_2 is labeled with either 1 or γ_p, both of which are relatively prime to both γ_{p+2} and γ_{p+3}. Thus this yields a Gaussian prime labeling. □

Aside from these three lemmas about Gaussian integers, we require the following lemma of Pikhurko about the structure of particular regions of trees.

Lemma 3.4. (Pikhurko [3, Lemma 2]) *Let T be a tree with at least 4 vertices. Then there exists a vertex u in T such that either*

1. *there is a subset of 3 vertices $A \subseteq T - u$ that can be represented as a union of (one or more) components of $T - u$, or*

2. *u has a neighbor, v, of degree $k + 1 \geq 3$ such that its other k neighbors $v_1, \ldots, v_k$ have degree 2 each and are incident to endvertices $u_1, \ldots, u_k$ correspondingly.*

These two possible cases are illustrated below.

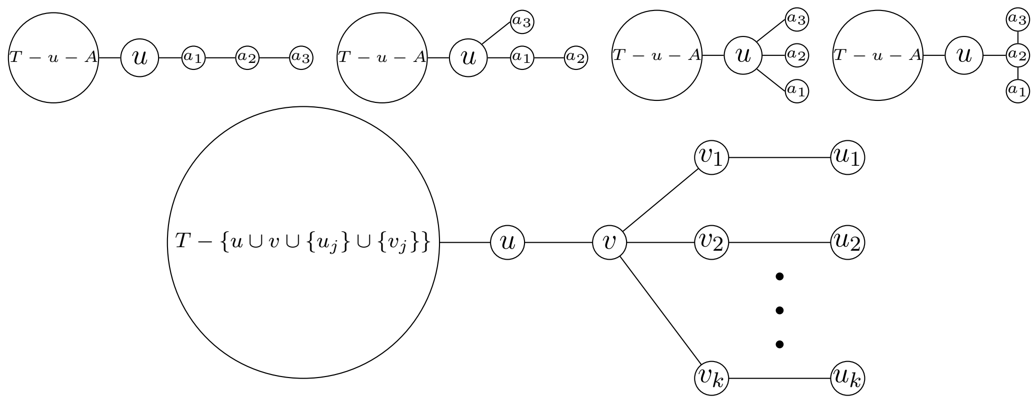

Figure 3: Case 1 (Above) and Case 2 (Below) of Lemma 3.4

There are four possible configurations of vertices in A as illustrated by Figure 3 above. In order for a set of 3 Gaussian integers to label A in a prime labeling, at least one of the three must be relatively prime to the other two and all three must be relatively prime to the Gaussian used to label u.

We call the vertices not specified by this lemma, $T - u - A$ in the first case or $T - \{u \cup v \cup \{u_j\} \cup \{v_j\}\}$ in the second, the **remainder of the tree** or simply the **remainder**. Because the remainder of the tree may be a forest rather than another tree, we need to show that results for arbitrary trees on n vertices also hold for arbitrary forests on n vertices.

Lemma 3.5. *If any tree on n vertices admits a Gaussian prime labeling, then any forest on n vertices admits a Gaussian prime labeling.*

Proof. Suppose any tree on n vertices admits a Gaussian prime labeling. Let a forest on n vertices be composed of q trees denoted $\bigcup_{i=1}^{q} T_i$. Observe that we can add one edge connecting T_k to T_{k+1} for $1 \leq k < q$ to create a tree T^* on n vertices. It follows that we can label T^* by assumption because it is a tree on n vertices. Removing the added edges results in a Gaussian prime labeling for the forest by definition. □

We can now combine these lemmas to label small trees. The general strategy will be to use Lemma 3.3 if it applies, and to outline a labeling using Lemma 3.4 if it does not. When using Lemma 3.4, we will generally attempt to label u with a large Gaussian prime, v with a large even Gaussian integer, A with the three largest composites available, $\{v_j\}$ with odd Gaussian integers, $\{u_j\}$ with even Gaussian integers consecutive to the corresponding v_j. The remainder of the tree will then be a either a tree or forest of smaller order and can be labeled accordingly. For our proof, we will assume without loss of generality that the remainder is a tree, as if it is a forest the result will still hold by Lemma 3.5. When we give a set of Gaussian integers for the labeling of $\{v_j\}$ or $\{u_j\}$, elements of the set are assigned to each v_j, u_j in order with each element used once. This approach yields the following theorem about small trees.

Theorem 3.6. *Let T be a tree with $n \leq 72$ vertices. Then T admits a Gaussian prime labeling under the spiral ordering.*

Proof. Table 1 provides a listing of these first 72 Gaussian integers together with their prime factorizations up to unit multiples.

First, we can label a tree on one vertex trivially. Then for all $2 \leq n \leq 72$ except $n = 14,\ 18,\ 24,\ 34,\ 42,\ 43,\ 44,\ 48,\ 54,\ 55,\ 56,\ 64,\ 65,\ 66,\ 67,\ 68$ and 72, γ_n is either prime or lies no more than two indices after a prime, so Lemma 3.3 applies. For each of these exceptional n values we will present a scheme for labeling the tree based on the structure given by Lemma 3.4.

$n = 14$ vertices: If the first case of Lemma 3.4 applies, we label u with $\gamma_{11} = 2 + 3\mathbf{i}$, which is relatively prime to all other $\gamma_j \in [\gamma_{14}]$. We label the vertices in A with $\{\gamma_{12},\ \gamma_{13},\ \gamma_{14}\} = \{3 + 3\mathbf{i},\ 4 + 3\mathbf{i},\ 4 + 2\mathbf{i}\}$. Since γ_{13} is relatively prime to γ_{12} and γ_{14}, they can label all four configurations of A. Then inductively label the remaining 10 vertices with $[\gamma_{10}]$ and we have a prime labeling.

If the second case of Lemma 3.4 applies, we have two further cases. If $k \leq 4$, we label u with $\gamma_{11} = 2 + 3\mathbf{i}$ for the same reason as in the first case, v with $\gamma_{14} = 4 + 2\mathbf{i}$, each v_j that appears with $\{\gamma_{13}, \gamma_{13-2j,\ 2\leq j\leq k}\}$[1], and each adjacent u_j with $\{\gamma_{14-2j,\ 1\leq j\leq k}\}$. The remainder of the tree consisting of $14-2k-2$ vertices is then labeled with $[\gamma_{14-2k-2}]$. It is easy to verify that each v_j is relatively prime to v and the corresponding u_j (as v_j and u_j are consecutive Gaussian integers for all j).

If $k > 4$, we label u with $\gamma_3 = 2 + \mathbf{i}$, v with $\gamma_4 = 2$, each v_j with $\{\gamma_{15-2j,\ 1\leq j\leq k-1}, \gamma_1\}$, and each adjacent u_j with $\{\gamma_{16-2j,\ 1\leq j\leq k-1}, \gamma_2\}$. The remainder of the tree consisting of $14-2k-2$ vertices is then inductively labeled with $[\gamma_{14-2k-2}]$. Again it is easy to verify that each v_j is relatively prime to v since v is a power of $1 + \mathbf{i}$ and that each v_j is relatively prime to u_j since they are consecutive Gaussian integers. So this is also a prime labeling.

Therefore all trees on $n = 14$ vertices have a Gaussian prime labeling. We illustrate these possibilities in the figures on the following page.

[1] This is a shorthand notation indicating that v_1 is labeled with γ_{13} and $v_2, v_3, \ldots, v_k$ are labeled with $\gamma_9, \gamma_7, \ldots, \gamma_{13-2k}$ respectively.

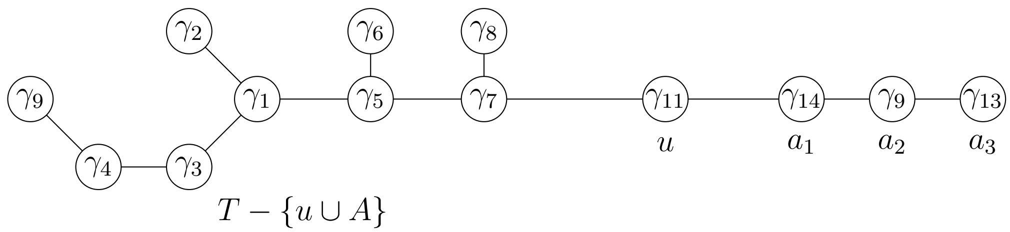

Figure 4: $n = 14$, Example Case 1 Labeling

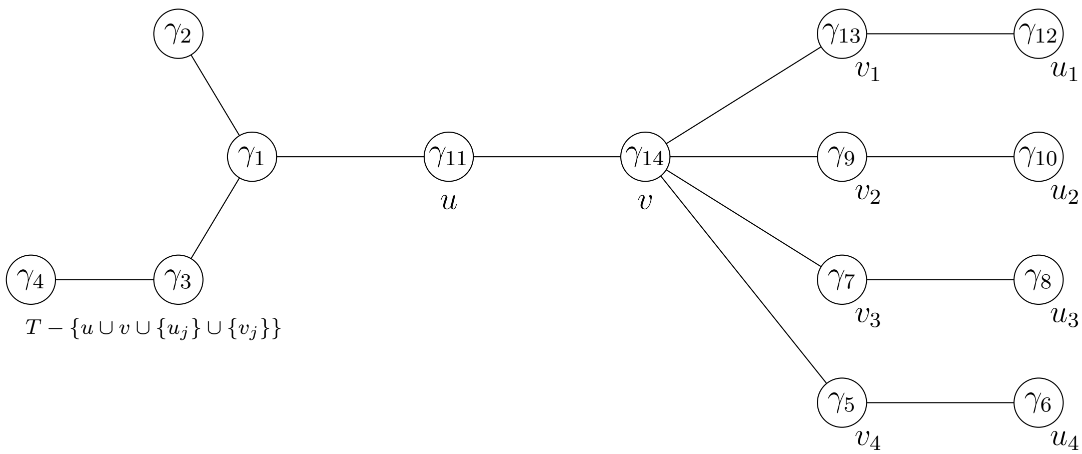

Figure 5: $n = 14$, Example Case 2 Labeling, $k = 4$

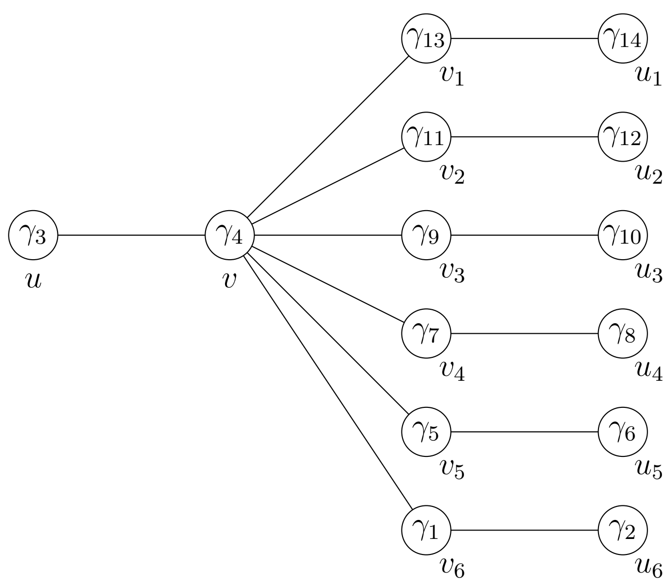

Figure 6: $n = 14$, Example Case 2 Labeling, $k = 6$

$n = 18$ vertices: If the first case of Lemma 3.4 applies, we label u with $\gamma_{15} = 4 + \mathbf{i}$ as it is relatively prime to all other $\gamma_j \in [\gamma_{18}]$. We then label A with $\{\gamma_{16},\ \gamma_{17},\ \gamma_{18}\} = \{4,\ 5,\ 5+\mathbf{i}\}$. This set can label all four configurations of A because 5 is relatively prime to 4 and $5 + \mathbf{i}$. We then inductively label the remaining 14 vertices with $[\gamma_{14}]$ and so we have a Gaussian prime labeling.

If the second case of Lemma 3.4 applies, we still label u with $\gamma_{15} = 4 + \mathbf{i}$. We label v with $\gamma_{16} = 4 = -(1+\mathbf{i})^4$ as it is relatively prime to all odd Gaussian integers. We label the v_j that appear with $\{\gamma_{17}, \gamma_{17-2j,\ 2\leq j\leq k}\}$ and the adjacent u_j with $\{\gamma_{18}, \gamma_{18-2j,\ 2\leq j\leq k}\}$. Since adjacent v_j and u_j have consecutive labels, they are relatively prime. We then inductively label the remaining $18 - 2k - 2$ vertices with $[\gamma_{18-2k-2}]$ and so we have a Gaussian prime labeling.

Therefore all trees on $n = 18$ vertices have a Gaussian prime labeling.

$n = 24$ vertices: If the first case of Lemma 3.4 applies, we label u with $\gamma_{21} = 5 + 4\mathbf{i}$ which is relatively prime to all other $\gamma_j \in [\gamma_{24}]$. We then label A with $\{\gamma_{22},\ \gamma_{23},\ \gamma_{24}\} = \{4+4\mathbf{i},\ 3+4\mathbf{i},\ 2+4\mathbf{i}\}$. This set can label all configurations of A because $3+4\mathbf{i}$ is relatively prime to $4 + 4\mathbf{i}$ and $2 + 4\mathbf{i}$. We then inductively label the remaining 20 vertices with $[\gamma_{20}]$ and so we have a Gaussian prime labeling.

If the second case of Lemma 3.4 applies, we still label u with $\gamma_{21} = 5+4\mathbf{i}$. We label v with $\gamma_{22} = 4 + 4\mathbf{i} = -(1+\mathbf{i})^5$ as it is relatively prime to all odd Gaussian integers. We label the v_j that appear with $\{\gamma_{23}, \gamma_{23-2j,\ 2\leq j\leq k}\}$ and the adjacent u_j with $\{\gamma_{24}, \gamma_{24-2j,\ 2\leq j\leq k}\}$. Since adjacent v_j and u_j have consecutive labels, they are relatively prime. We then inductively label the remaining $24 - 2k - 2$ vertices with $[\gamma_{24-2k-2}]$ and so we have a Gaussian prime labeling.

Therefore all trees on $n = 24$ vertices have a Gaussian prime labeling.

$n = 34$ vertices: If the first case of Lemma 3.4 applies, we label u with $\gamma_{31} = 6 + 5\mathbf{i}$ which is relatively prime to all other $\gamma_j \in [\gamma_{34}]$. We then label A with $\{\gamma_{32},\ \gamma_{33},\ \gamma_{34}\} = \{6+4\mathbf{i},\ 6+3\mathbf{i},\ 6+2\mathbf{i}\}$. This set can label all configurations of A because $6+3\mathbf{i}$ is relatively prime to $6 + 4\mathbf{i}$ and $6 + 2\mathbf{i}$. We inductively label the remaining 30 vertices with $[\gamma_{30}]$ and so we have a Gaussian prime labeling.

If the second case of Lemma 3.4 applies, we label u with $\gamma_{31} = 6+5\mathbf{i}$ for the same reason. For the rest of the labeling there are two cases. If $k \leq 5$, we label v with $\gamma_{32} = 6+4\mathbf{i}$, which is relatively prime to all odd Gaussian integers between γ_{23} and γ_{33}. We label the v_j that appear with $\{\gamma_{33}, \gamma_{33-2j, 2\leq j\leq k}\}$ and the adjacent u_j that appear with $\{\gamma_{34}, \gamma_{34-2j, 2\leq j\leq k}\}$. Since adjacent v_j and u_j have consecutive labels they are relatively prime. We then inductively label the remaining $34-2k-2$ vertices with $[\gamma_{34-2k-2}]$ and so have a Gaussian prime labeling.

If $k > 5$, we label v with $\gamma_{22} = 4 + 4\mathbf{i} = -(1 + \mathbf{i})^5$ which is relatively prime to all odd Gaussian integers. We then label the v_j that appear with $\{\gamma_{33}, \gamma_{33-2j,\ 2\leq j\leq k}\}$ and the adjacent u_j that appear with $\{\gamma_{34}, \gamma_{34-2j,\ 2\leq j\leq 5}, \gamma_{32}, \gamma_{34-2j,\ 7\leq j\leq k}\}$. Again since adjacent v_j and u_j have consecutive labels they are relatively prime. Finally we inductively label the remaining $34 - 2k - 2$ vertices with $[\gamma_{34-2k-2}]$ and so have a Gaussian prime labeling.

Therefore all trees on $n = 34$ vertices have a Gaussian prime labeling.

$n = 42$ vertices: If the first case of Lemma 3.4 applies, we label u with $\gamma_{39} = 7 + 2\mathbf{i}$ which is relatively prime to all other $\gamma_j \in [\gamma_{42}]$. We then label A with $\{\gamma_{40},\ \gamma_{41},\ \gamma_{42}\} = \{7+3\mathbf{i},\ 7+4\mathbf{i},\ 7+5\mathbf{i}\}$. This set can label all configurations of A because $7+4\mathbf{i}$ is relatively prime to $7+3\mathbf{i}$ and $7+5\mathbf{i}$. We then inductively label the remaining 38 vertices with $[\gamma_{38}]$ and so we have a Gaussian prime labeling.

If the second case of Lemma 3.4 applies, we still label u with $\gamma_{39} = 7 + 2\mathbf{i}$. We label v with $\gamma_{42} = 7 + 5\mathbf{i}$ as it is relatively prime to all odd Gaussian integers in $[\gamma_{42}]$. We label the v_j that appear with $\{\gamma_{41},\ \gamma_{41-2j,\ 2\leq j\leq k}\}$ and the adjacent u_j with $\{\gamma_{42-2j,\ 1\leq j\leq k}\}$. Since adjacent v_j and u_j have consecutive labels, they are relatively prime. We then inductively label the remaining $42 - 2k - 2$ vertices with $[\gamma_{42-2k-2}]$ and so we have a Gaussian prime labeling.

Therefore all trees on $n = 42$ vertices have a Gaussian prime labeling.

$n = 43$ vertices: In this case we lack a prime Gaussian integer within four indices of γ_{43}. This means that in the first case of Lemma 3.4 if we label u with the last prime and A with three composites near γ_{43} we will not be left with an interval to label the remainder of the tree. To deal with this, we will apply Lemma 3.4 again to show that we can label the remainder of the tree. The second case strategy remains the same.

First suppose the first case applies to T. Then we label u with $\gamma_{39} = 7 + 2\mathbf{i}$ which is relatively prime to all Gaussian integers in $[\gamma_{43}]$. We then label A with $\{\gamma_{38},\ \gamma_{40},\ \gamma_{41}\} = \{7 + 3\mathbf{i}, 7 + 4\mathbf{i}, 7 + \mathbf{i}\}$. We can label all configurations of A with this set because $7 + 4\mathbf{i}$ is relatively prime to $7+3\mathbf{i}$ and $7+\mathbf{i}$. Now we need to show that we can label $T^* = T-\{u\cup A\}$, the remaining 39 vertices, with the set $S = [\gamma_{37}] \cup \{\gamma_{42}, \gamma_{43}\}$.

If the second case applies to T, we treat u as part of the remainder set of nodes and do not assign a label yet. We label v with $\gamma_{39} = 7+2\mathbf{i}$, which is relatively prime to all Gaussian integers in $[\gamma_{43}]$. We label the v_j that appear with $\{\gamma_{43},\ \gamma_{41},\ \gamma_{43-2j,\ 3\leq j\leq k}\}$ and the adjacent u_j that appear with $\{\gamma_{44-2j,\ 1\leq j\leq k}\}$. Since adjacent v_j and u_j have consecutive labels, they are relatively prime. We then inductively label the remaining $43 - 2k - 1$ vertices with $[\gamma_{43-2k-1}]$ to give a Gaussian prime labeling.

We now apply Lemma 3.4 to T^* and let u^*, v^*, A^*, v_j^*, and u_j^* be the vertices in T^* described by the lemma. If the first case applies to T^*, we label u^* with $\gamma_{37} = 7$, which is relatively prime to all Gaussian integers in S. We then label A^* with $\{\gamma_{36},\ \gamma_{42},\ \gamma_{43}\} = \{6,\ 7 + 5\mathbf{i},\ 7 + 6\mathbf{i}\}$. We can label all configurations of A^* with this set because $7 + 6\mathbf{i}$ is relatively prime to 6 and $7 + 5\mathbf{i}$. We then inductively label the remaining 35 vertices with $[\gamma_{35}]$, resulting in a Gaussian prime labeling.

If the second case applies to T^*, we again label u^* with $\gamma_{37} = 7$. We label v^* with $\gamma_{42} = 7+5\mathbf{i} = (1-\mathbf{i})(1+6\mathbf{i})$, which is relatively prime to all odd Gaussian integers in S. We label the v_j^* that appear with $\{\gamma_{43},\ \gamma_{39-2j,\ 2\leq j\leq k}\}$ and the adjacent u_j^* with $\{\gamma_{38-2j,\ 1\leq j\leq k}\}$. Each adjacent v_j^*, u_j^* pair is a pair of consecutive Gaussian integers except for $v_1^* = 7 + 6\mathbf{i}$ and $u_1^* = 6$, but these are also relatively prime. Finally we inductively label the remaining

$39 - 2k - 2$ vertices by $[\gamma_{39-2k-2}]$, giving a Gaussian prime labeling.

Therefore all trees on $n = 43$ vertices have a Gaussian prime labeling.

$n = 44$ vertices: As in the previous case, we will need to apply Lemma 3.4 twice to prove this case because there is no Gaussian prime within the final four Gaussian integers in $[\gamma_{44}]$. We use the same notation as in the $n = 43$ case.

If the first case of Lemma 3.4 applies to T, then we label u with $\gamma_{39} = 7 + 2\mathbf{i}$, which is relatively prime to all Gaussian integers in $[\gamma_{44}]$. We then label A with $\{\gamma_{40},\ \gamma_{43},\ \gamma_{44}\} = \{7+3\mathbf{i},\ 7+6\mathbf{i},\ 6+6\mathbf{i}\}$. Because $7+6\mathbf{i}$ is relatively prime to $7+3\mathbf{i}$ and $6+6\mathbf{i}$, we can label all configurations of A with this set. We then need to show that we can label $T^* = T - \{u \cup A\}$, the remaining 40 vertices, with $S = [\gamma_{38}] \cup \{\gamma_{41},\ \gamma_{42}\}$.

If the second case of Lemma 3.4 applies to T, then we label u with $\gamma_{39} = 7 + 2\mathbf{i}$, which is relatively prime to all Gaussian integers in $[\gamma_{44}]$. We label v with $\gamma_{42} = 7+5\mathbf{i} = (1-\mathbf{i})(1+6\mathbf{i})$, which is relatively prime to all odd Gaussian integers in $[\gamma_{44}]$. Then we label the v_j that appear with $\{\gamma_{43},\ \gamma_{41},\ \gamma_{43-2j,\ 3\leq j\leq k}\}$ and the adjacent u_j with $\{\gamma_{44},\ \gamma_{44-2j,\ 2\leq j\leq k}\}$. Because adjacent v_j and u_j have consecutive labels, they are relatively prime. We then inductively label the remaining $44 - 2k - 2$ vertices with $[\gamma_{44-2k-2}]$ to give a Gaussian prime labeling.

We now apply Lemma 3.4 to T^*. If the first case applies to T^*, we label u^* with $\gamma_{37} = 7$, which is relatively prime to all Gaussian integers in S. We label A^* with $\{\gamma_{38},\ \gamma_{41},\ \gamma_{42}\} = \{7+\mathbf{i},\ 7+4\mathbf{i},\ 7+5\mathbf{i}\}$. Because $7+4\mathbf{i}$ is relatively prime to $7+\mathbf{i}$ and $7+5\mathbf{i}$ we can label all configurations of A^* with this set. We then inductively label the remaining 36 vertices with $[\gamma_{36}]$ to give a Gaussian prime labeling.

If the second case applies to T^*, we again label u^* with $\gamma_{37} = 7$. We label v^* with $\gamma_{42} = 7 + 5\mathbf{i} = (1 - \mathbf{i})(1 + 6\mathbf{i})$ as it is relatively prime to all odd Gaussian integers in S. Then we label the v_j^* that appear with $\{\gamma_{41},\ \gamma_{39-2j,\ 2\leq j\leq k}\}$ and the adjacent u_j^* with $\{\gamma_{40-2j,\ 1\leq j\leq k}\}$. Because adjacent v_j^* and u_j^* have consecutive labels except for $v_1^* = 7 + 4\mathbf{i}$ and $u_1^* = 7 + \mathbf{i}$ (which are relatively prime), they are relatively prime. We then inductively label the remaining $40 - 2k - 2$ vertices with $[\gamma_{40-2k-2}]$ and so we have a Gaussian prime labeling.

Therefore all trees on $n = 44$ vertices have a Gaussian prime labeling.

$n = 48$ vertices: If the first case of Lemma 3.4 applies, we label u with $\gamma_{45} = 5 + 6\mathbf{i}$ as it is relatively prime to all other $\gamma_j \in [\gamma_{48}]$. We then label A with $\{\gamma_{46},\ \gamma_{47},\ \gamma_{48}\} = \{4 + 6\mathbf{i},\ 3 + 6\mathbf{i},\ 2 + 6\mathbf{i}\}$. This set can label all four configurations of A because $3 + 6\mathbf{i}$ is relatively prime to $4 + 6\mathbf{i}$ and $2 + 6\mathbf{i}$. We then inductively label the remaining 44 vertices with $[\gamma_{44}]$ and so we have a Gaussian prime labeling.

If the second case of Lemma 3.4 applies, we label u with $\gamma_{45} = 5+6\mathbf{i}$ for the same reason. We label v with $\gamma_{42} = 7 + 5\mathbf{i} = (1 - \mathbf{i})(1 + 6\mathbf{i})$ as it is relatively prime to all odd Gaussian integers in $[\gamma_{48}]$. We label the v_j that appear with $\{\gamma_{47}, \gamma_{47-2j,\ 2\leq j\leq k}\}$ and the adjacent u_j with $\{\gamma_{50-2j,\ 1\leq j\leq k}\}$. Since adjacent v_j and u_j have consecutive labels except for $v_2 = 7 + 6\mathbf{i}$ and $u_2 = 4 + 6\mathbf{i}$ (which are relatively prime), they are relatively prime. We then inductively label the remaining $48 - 2k - 2$ vertices with $[\gamma_{48-2k-2}]$ and so we have a Gaussian prime labeling.

Therefore all trees on $n = 48$ vertices have a Gaussian prime labeling.

$n = 54$ vertices: If the first case of Lemma 3.4 applies to T, we label u with $\gamma_{51} = 2 + 7\mathbf{i}$, which is relatively prime to all Gaussian integers in $[\gamma_{54}]$. We then label A with $\{\gamma_{52},\ \gamma_{53},\ \gamma_{54}\} = \{3 + 7\mathbf{i},\ 4 + 7\mathbf{i},\ 5 + 7\mathbf{i}\}$. Because $4 + 7\mathbf{i}$ is relatively prime to $3 + 7\mathbf{i}$ and $5 + 7\mathbf{i}$ this set can label all configurations of A. We then inductively label the remaining 50 vertices with $[\gamma_{50}]$ and so we have a Gaussian prime labeling.

If the second case of Lemma 3.4 applies, we have two further cases. The first case, for $k \leq 8$, is straightfoward. We again label u with $\gamma_{51} = 2 + 7\mathbf{i}$. We label v with $\gamma_{54} = 5 + 7\mathbf{i}$, which is relatively prime to all odd Gaussian integers between γ_{37} and γ_{53} in the spiral ordering. We label the v_j that appear with $\{\gamma_{53},\ \gamma_{53-2j,\ 2\leq j\leq k}\}$ and the adjacent u_j with $\{\gamma_{54-2j,\ 1\leq j\leq k}\}$. Finally we inductively label the remaining $54-2k-2$ vertices with $[\gamma_{54-2k-2}]$. Since adjacent v_j and u_j have consecutive labels they are relatively prime and we have a Gaussian prime labeling.

The second case is a little more complicated. For $k > 8$, we have no label for v that will allow us to label v_j and u_j similarly to the previous cases until $k = 16$ when we can use $\gamma_{22} = 4 + 4\mathbf{i} = -(1 + \mathbf{i})^5$. To resolve this we claim that we can label v with γ_{22} for all $k > 8$, and replace the spot in the remainder of the tree that γ_{22} would have labeled by γ_{54}. We can do this because the only odd in the tree that is not relatively prime to γ_{54} is γ_{35}, which is always assigned to v_9 for $k > 8$ and so cannot be adjacent to γ_{54}. After this, label v_j and u_j as in the case where $k \leq 8$ except that when the formula for u_j gives γ_{22}, use γ_{54} instead. This will give a Gaussian prime labeling.

Therefore all trees on $n = 54$ vertices have a Gaussian prime labeling.

$n = 55$ vertices: As in the case of $n = 43$ or $n = 44$ vertices, we will need to apply Lemma 3.4 multiple times to show that trees on $n = 55$ vertices admit a Gaussian prime labeling. Initially, we need to label a tree T on 55 vertices with $[\gamma_{55}]$.

If the first case of Lemma 3.4 applies to T, we label u with $\gamma_{51} = 2 + 7\mathbf{i}$, which is relatively prime to all other Gaussian integers in $[\gamma_{55}]$. We then label A with $\{\gamma_{50},\ \gamma_{52},\ \gamma_{53}\} = \{1+7\mathbf{i},\ 3+7\mathbf{i},\ 4+7\mathbf{i}\}$. Because $4+7\mathbf{i}$ is relatively prime to $1+7\mathbf{i}$ and $3+7\mathbf{i}$, this set can label all configurations of A. For this to be a Gaussian prime labeling, we now need to show that we can label $T^* = T - \{u \cup A\}$, the remaining 51 vertices, with the set $S = [\gamma_{49}] \cup \{\gamma_{54},\ \gamma_{55}\}$.

If the second case of Lemma 3.4 applies to T, we treat u as part of the remainder of the tree and do not specify a label for it yet. We label v with $\gamma_{51} = 2 + 7\mathbf{i}$, which is relatively prime to all other Gaussian integers in $[\gamma_{55}]$. We then label the v_j that appear with $\{\gamma_{55},\ \gamma_{53},\ \gamma_{53-2j,\ 2\leq j\leq k}\}$ and the adjacent u_j with $\{\gamma_{56-2j,\ 1\leq j\leq k}\}$. Because the labels of adjacent v_j and u_j are consecutive Gaussian integers, they are relatively prime. Finally we inductively label the remaining $55 - 2k - 1$ vertices with $[\gamma_{55-2k-1}]$ and so we have a Gaussian prime labeling.

We now apply Lemma 3.4 again to T^*. If the first case of Lemma 3.4 applies to a tree on 51 vertices, we label u^* with $\gamma_{49} = 1 + 6\mathbf{i}$, which is relatively prime to all other Gaussian integers in S except γ_{42}. We then label A^* with $\{\gamma_{48},\ \gamma_{54},\ \gamma_{55}\} = \{2+6\mathbf{i},\ 5+7\mathbf{i},\ 6+7\mathbf{i}\}$. Because $6+7\mathbf{i}$ is relatively prime to $2+6\mathbf{i}$ and $5+7\mathbf{i}$, this set can label all configurations of A^*. We inductively label the remaining 47 vertices with $[\gamma_{47}]$ and by Lemma 3.2 with $x = u^*$, we know this has a Gaussian prime labeling and so we have a Gaussian prime labeling of T^* with $S = [\gamma_{49}] \cup \{\gamma_{54},\ \gamma_{55}\}$.

If the second case of Lemma 3.4 applies to T^*, we again do not specify a label for u^* yet. We label v^* with $\gamma_{49} = 1+6\mathbf{i}$, which is relatively prime to all other Gaussian integers in S except γ_{42}. We then label the v_j^* that appear with $\{\gamma_{55},\ \gamma_{51-2j},\ 2 \le j \le k\}$ and the adjacent u_j^* with $\{\gamma_{54},\ \gamma_{52-2j},\ 2 \le j \le k\}$. Because the labels of v_j^* and u_j^* are consecutive Gaussian integerse, they are relatively prime. Finally we inductively label the remaining $51-2k-1$ vertices with the set $[\gamma_{51-2k-1}]$ which by Lemma 3.2 with $x = v^*$ we know admits a Gaussian prime labeling with v^* labeled either with γ_{49} or 1. In either case, this is a Gaussian prime labeling of T^* with $S = [\gamma_{49}] \cup \{\gamma_{54},\ \gamma_{55}\}$.

Therefore all trees on $n = 55$ vertices have a Gaussian prime labeling.

$n = 56$ vertices: As in the case of $n = 55$ vertices we will need to apply Lemma 3.4 two times to show trees on $n = 56$ vertices have a Gaussian prime labeling. We start by labeling a tree T on 56 vertices with the interval $[\gamma_{56}]$.

If the first case of Lemma 3.4 applies to T, we label u with $\gamma_{51} = 2+7\mathbf{i}$, which is relatively prime to all Gaussian integers in $[\gamma_{56}]$. We label A with $\{\gamma_{50},\ \gamma_{55},\ \gamma_{56}\} = \{1+7\mathbf{i},\ 6+7\mathbf{i},\ 7+7\mathbf{i}\}$. Because $6+7\mathbf{i}$ is relatively prime to $1+7\mathbf{i}$ and $7+7\mathbf{i}$, this set can label all configurations of A. We now need to show that we can label $T^* = T - \{u \cup A\}$, the remaining 52 vertices, with the set $S = [\gamma_{49}] \cup \{\gamma_{52},\ \gamma_{53},\ \gamma_{54}\}$.

If the second case of Lemma 3.4 applies to T, we again label u with $\gamma_{51} = 2+7\mathbf{i}$. Then we suppose $k \le 8$ and label v with $\gamma_{56} = 7+7\mathbf{i}$ and label v_j with $\{\gamma_{55},\ \gamma_{53},\ \gamma_{55-2j,\ 3\le j\le k}\}$ and u_j with $\{\gamma_{56-2j,\ 1\le j\le k}\}$. We inductively label the remaining $56-2k$ vertices with $[\gamma_{56-2k}]$. Because $\gamma_{56} = 7+7\mathbf{i}$ is relatively prime to all odd Gaussian integers between γ_{38} and $\gamma_5 5$ and the Gaussian integers labeling adjacent v_j and u_j are consecutive, this is a prime labeling.

If $k > 8$, then we label v with $\gamma_{22} = 4+4\mathbf{i} = -(1+\mathbf{i})^5$ and label v_j and u_j similar to the $k \le 8$ case. We inductively label the remaining $56-2k-2$ vertices with $[\gamma_{56-2k-2}]$. When γ_{22} would be assigned to a vertex other than v we replace it with γ_{56} similar to the $n = 54$ case. This is valid because γ_{56} will never label a neighbor of $\gamma_{37} = 7$ for $k > 8$ in this scheme. Thus we have a Gaussian prime labeling.

We now apply Lemma 3.4 to T^*. If the first case of Lemma 3.4 applies to T^*, we label u^* with $\gamma_{49} = 1+6\mathbf{i}$ and A^* with $\{\gamma_{52},\ \gamma_{53},\ \gamma_{54}\} = \{3+7\mathbf{i},\ 4+7\mathbf{i},\ 5+7\mathbf{i}\}$. Clearly $4+7\mathbf{i}$ is relatively prime to its consecutive neighbors so this set can label all configurations of A^*. We inductively label the remaining 48 vertices with $[\gamma_{48}]$ which we know is possible by Lemma 3.2 taking $x = u$. Thus we have a Gaussian prime labeling of T^* with $S = [\gamma_{49}] \cup \{\gamma_{52},\ \gamma_{53},\ \gamma_{54}\}$.

If the second case applies to T^*, we start by labeling u^* with $\gamma_{49} = 1+6\mathbf{i}$. Then we suppose $k \le 9$ and label v^* with γ_{54}, v_j^* with $\{\gamma_{53},\ \gamma_{51-2j,\ 2\le j\le k}\}$, and u_j^* with $\{\gamma_{52},\ \gamma_{52-2j,\ 2\le j\le k}\}$. Because $\gamma_{54} = 5+7\mathbf{i}$ is relatively prime to all odd Gaussian integers between γ_{36} and γ_{53} and adjacent v_j^* and u_j^* are labeled with consecutive Gaussian integers this is a valid labeling.

If $k > 9$, then we label v^* with γ_{22} and label v_j^* and u_j^* similar to the $k \leq 9$ case. When γ_{22} would be assigned to a vertex other than v^* we replace it with γ_{54} which is valid because γ_{56} will never label a neighbor of γ_{35} in this scheme. We inductively label the remaining $52 - 2k - 2$ vertices with $[\gamma_{48-2k+2}]$ which we know is possible by Lemma 3.2 with $x = u^*$. Thus we have a Gaussian prime labeling of T^* with $S = [\gamma_{49}] \cup \{\gamma_{52},\ \gamma_{53},\ \gamma_{54}\}$.

Therefore all trees on $n = 56$ vertices have a Gaussian prime labeling.

$n = 64$ vertices: If the first case of Lemma 3.4 applies to T, we label u with $\gamma_{61} = 8 + 3\mathbf{i}$ which is relatively prime to all Gaussian integers in $[\gamma_{64}]$. We label A with $\{\gamma_{62},\ \gamma_{63},\ \gamma_{64}\} = \{8 + 2\mathbf{i},\ 8 + \mathbf{i},\ 8\}$. We know $8 + \mathbf{i}$ is relatively prime to its consecutive neighbors so this set can label all configurations of A. We inductively label the remaining 60 vertices with $[\gamma_{60}]$.

If the second case of Lemma 3.4 applies to T, we label u with $\gamma_{61} = 8 + 3\mathbf{i}$ again, v with $\gamma_{64} = 8 = \mathbf{i}(1 + \mathbf{i})^6$, v_j with $\{\gamma_{63},\ \gamma_{63-2j},\ 2 \leq j \leq k\}$, and u_j with $\{\gamma_{64-2j},\ 1 \leq j \leq k\}$. Here γ_{61} is relatively prime to all Gaussian integers in $[\gamma_{64}]$, γ_{64} is relatively prime to all odd Gaussian integers, and adjacent v_j and u_j are labeled with consecutive Gaussian integers. We then inductively label the remaining $64 - 2k - 2$ vertices with $[\gamma_{64-2k-2}]$, which gives a Gaussian prime labeling.

Therefore all trees on $n = 64$ vertices have a Gaussian prime labeling.

$n = 65$ vertices: We will need to apply Lemma 3.4 twice on trees with 65 vertices to demonstrate a Gaussian prime labeling. We start with the problem of labeling a tree T on 65 vertices with the interval $[\gamma_{65}]$.

If the first case of Lemma 3.4 applies to T, we label u with $\gamma_{61} = 8+3\mathbf{i}$, which is relatively prime to all Gaussian integers in $[\gamma_{65}]$. We label A with $\{\gamma_{60},\ \gamma_{62},\ \gamma_{63}\} = \{8+4\mathbf{i},\ 8+2\mathbf{i},\ 8+\mathbf{i}\}$. Because $8 + \mathbf{i}$ is relatively prime to $8 + 4\mathbf{i}$ and $8 + 2\mathbf{i}$, this set can label all configurations of A. Now we must show we can label $T^* = T - \{u \cup A\}$, the remaining 61 vertices, with $S = [\gamma_{59}] \cup \{\gamma_{64},\ \gamma_{65}\}$.

If the second case of Lemma 3.4 applies to T, we label u with $\gamma_{61} = 8 + 3\mathbf{i}$, v with $\gamma_{64} = 8 = \mathbf{i}(1 + \mathbf{i})^6$, v_j with $\{\gamma_{65},\ \gamma_{63},\ \gamma_{65-2j,\ 3\leq j\leq k}\}$, and u_j with $\{\gamma_{64-2j,\ 1\leq j\leq k}\}$. We then inductively label the remaining $65 - 2k - 2$ vertices with $[\gamma_{65-2k-2}]$. Here γ_{61} is relatively prime to all Gaussian integers in $[\gamma_{65}]$, γ_{64} is relatively prime to all odd Gaussian integers, and adjacent v_j and u_j are labeled with consecutive Gaussian integers except for the pairs γ_{65}, γ_{62} and γ_{63}, γ_{60} which are still relatively prime, so this is a Gaussian prime labeling.

We now apply Lemma 3.4 to T^*. If the first case of Lemma 3.4 applies to T^*, we label u^* with $\gamma_{59} = 8 + 5\mathbf{i}$ and A^* with $\{\gamma_{58},\ \gamma_{64},\ \gamma_{65}\} = \{8 + 6\mathbf{i},\ 8,\ 9\}$. We know 9 is relatively prime to both 8 and $8+6\mathbf{i}$ so this set will label all configurations of A^*. We inductively label the remaining 57 vertices with $[\gamma_{57}]$ and so this is a Gaussian prime labeling of T^* vertices with S.

If the second case applies to T^*, we label u^* with $\gamma_{59} = 8+5\mathbf{i}$, v^* with $\gamma_{64} = 8 = \mathbf{i}(1+\mathbf{i})^6$, v_j^* with $\{\gamma_{65},\ \gamma_{61-2j,\ 2\leq j\leq k}\}$, and u_j^* with $\{\gamma_{60-2j,\ 1\leq j\leq k}\}$. We then inductively label the remaining $61 - 2k - 2$ vertices with $[\gamma_{61-2k}]$. Here γ_{59} is relatively prime to all Gaussian integers in S, γ_{64} is relatively prime to all odd Gaussian integers, and adjacent v_j^* and u_j^* are labeled with consecutive Gaussian integers except for γ_{65} and γ_{58} which are also relatively prime. Therefore this is a Gaussian prime labeling of T^* with S.

Therefore all trees on $n = 65$ vertices have a Gaussian prime labeling.

$n = 66$ vertices: We will also need to apply Lemma 3.4 twice on trees with 66 vertices to demonstrate a Gaussian prime labeling. We start with the problem of labeling a tree T on 66 vertices with the interval $[\gamma_{66}]$.

If the first case of Lemma 3.4 applies to T, we label u with $\gamma_{61} = 8+3\mathbf{i}$, which is relatively prime to all Gaussian integers in $[\gamma_{66}]$. We label A with $\{\gamma_{62},\ \gamma_{65},\ \gamma_{66}\} = \{8+2\mathbf{i},\ 9,\ 9+\mathbf{i}\}$. We know 9 is relatively prime to $8+2\mathbf{i}$ and $9+\mathbf{i}$, so this set can label all configurations of A. Now we must show we can label $T^* = T - \{u \cup A\}$, the remaining 62 vertices, with $S = [\gamma_{60}] \cup \{\gamma_{63},\ \gamma_{64}\}$.

If the second case of Lemma 3.4 applies to T, we label u with $\gamma_{61} = 8+3\mathbf{i}$, v with $\gamma_{64} = 8 = \mathbf{i}(1+\mathbf{i})^6$, v_j with $\{\gamma_{65},\ \gamma_{63},\ \gamma_{65-2j,\ 3\le j\le k}\}$, and u_j with $\{\gamma_{66},\ \gamma_{66-2j,\ 2\le j\le k}\}$. We inductively label the remaining $66-2k-2$ vertices with $[\gamma_{66-2k-2}]$. Here γ_{61} is relatively prime to all Gaussian integers in $[\gamma_{66}]$, γ_{64} is relatively prime to all odd Gaussian integers, and adjacent v_j and u_j are labeled with consecutive Gaussian integers, so this is a Gaussian prime labeling.

We now apply Lemma 3.4 to T^*. If the first case of Lemma 3.4 applies to T^*, we label u^* with $\gamma_{59} = 8+5\mathbf{i}$, which is relatively prime to all Gaussian integers in S. We label A^* with $\{\gamma_{60},\ \gamma_{63},\ \gamma_{64}\} = \{8+4\mathbf{i},\ 8+\mathbf{i},\ 8\}$. We know $8+\mathbf{i}$ is relatively prime to both 8 and $8+4\mathbf{i}$ so this set will label all configurations of A^*. We inductively label the remaining 58 vertices with $[\gamma_{58}]$ and so this is a Gaussian prime labeling of T^* with $S = [\gamma_{60}] \cup \{\gamma_{63},\ \gamma_{64}\}$.

If the second case applies to T^*, we label u^* with $\gamma_{59} = 8+5\mathbf{i}$, v^* with $\gamma_{64} = 8 = \mathbf{i}(1+\mathbf{i})^6$, v_j^* with $\{\gamma_{63},\ \gamma_{61-2j,\ 2\le j\le k}\}$, and u_j^* with $\{\gamma_{62-2j,\ 1\le j\le k}\}$. We inductively label the remaining $62-2k-2$ vertices with $[\gamma_{60-2k}]$. Here γ_{59} is relatively prime to all Gaussian integers in S, γ_{64} is relatively prime to all odd Gaussian integers, and adjacent v_j^* and u_j^* are labeled with consecutive Gaussian integers except for γ_{63} and γ_{60} which are relatively prime. Therefore this is a Gaussian prime labeling of T^* with $S = [\gamma_{60}] \cup \{\gamma_{63},\ \gamma_{64}\}$.

Therefore all trees on $n = 66$ vertices have a Gaussian prime labeling.

$n = 67$ vertices: We need to apply Lemma 3.4 four times to show that trees on 67 vertices have a Gaussian prime labeling. We begin with the problem of labeling a tree T on 67 vertices with $[\gamma_{67}]$.

If the first case of Lemma 3.4 applies to T, we label u with $\gamma_{61} = 8+3\mathbf{i}$, which is relatively prime to all Gaussian integers in $[\gamma_{67}]$. We label A with $\{\gamma_{62},\ \gamma_{66},\ \gamma_{67}\} = \{8+2\mathbf{i},\ 9+\mathbf{i},\ 9+2\mathbf{i}\}$. We know $9+2\mathbf{i}$ is relatively prime to $8+2\mathbf{i}$ and $9+\mathbf{i}$, so this set can label all configurations of A. Now we must show we can label $T^* = T - \{u \cup A\}$, the remaining 63 vertices, with $S_1 = [\gamma_{60}] \cup \{\gamma_{63},\ \gamma_{64},\ \gamma_{65}\}$.

If the second case of Lemma 3.4 applies to T, we suppose $k > 2$ and label u with $\gamma_{61} = 8+3\mathbf{i}$, v with $\gamma_{64} = 8$, v_j with $\{\gamma_{67},\ \gamma_{65},\ \gamma_{63},\ \gamma_{67-2j,\ 4\le j\le k}\}$, and u_j with $\{\gamma_{66},\ \gamma_{66-2j,\ 2\le j\le k}\}$. We inductively label the remaining $67-2k-2$ vertices with $[\gamma_{67-2k-2}]$. Here γ_{61} is relatively prime to all Gaussian integers in $[\gamma_{67}]$, γ_{64} is relatively prime to all odd Gaussian integers, and adjacent v_j and u_j are labeled with consecutive Gaussian integers except for the pairs γ_{65}, γ_{62} and γ_{63}, γ_{60} which are also relatively prime, so this is a Gaussian prime labeling.

If $k = 2$, we label u with $\gamma_{61} = 8 + 3\mathbf{i}$, v with $\gamma_{66} = 9 + \mathbf{i}$, v_j with $\{\gamma_{65},\ \gamma_{63}\}$, and u_j with $\{\gamma_{60},\ \gamma_{62}\}$. γ_{61} is relatively prime to all Gaussian integers in $[\gamma_{67}]$, γ_{66} is relatively prime to γ_{65} and γ_{63}, and γ_{65} is relatively prime to γ_{60}. Now we need to show that we can label $T' = T - \{u \cup v \cup \{v_j\} \cup \{u_j\}\}$, the remaining 61 vertices, with $S_2 = [\gamma_{59}] \cup \{\gamma_{64},\ \gamma_{67}\}$.

We now apply Lemma 3.4 to T'. If the first case applies to T', we label u' with $\gamma_{59} = 8+5\mathbf{i}$ and A' with $\{\gamma_{58},\ \gamma_{64},\ \gamma_{67}\} = \{8+6\mathbf{i},\ 8,\ 9+2\mathbf{i}\}$. We know $9+2\mathbf{i}$ is relatively prime to $8+2\mathbf{i}$ and $9 + \mathbf{i}$, so this set can label all configurations of A'. We inductively label the remaining 57 vertices with $[\gamma_{57}]$ and so have a Gaussian prime labeling of T' with S_2.

If the second case applies to T', we label u' with $\gamma_{59} = 8 + 5\mathbf{i}$, v' with $\gamma_{64} = 8$, v'_j with $\{\gamma_{67},\ \gamma_{61-2j,\ 2\leq j\leq k}\}$, and u'_j with $\{\gamma_{60-2j,\ 1\leq j\leq k}\}$. γ_{59} is relatively prime to all Gaussian integers in S_1, γ_{64} is relatively prime to all odd Gaussian integers, and adjacent v'_j and u'_j are labeled with consecutive Gaussian integers except for γ_{67} and γ_{58} which are also relatively prime. We inductively label the remaining $61 - 2k - 2$ vertices with $[\gamma_{61-2k-2}]$ and so we have a Gaussian prime labeling of T' with S_2.

Now we apply Lemma 3.4 to T^*. If the first case of Lemma 3.4 applies to T^*, we label u^* with $\gamma_{59} = 8 + 7\mathbf{i}$, which is relatively prime to all Gaussian integers in S_1. We label A^* with $\{\gamma_{58},\ \gamma_{60},\ \gamma_{65}\} = \{8 + 6\mathbf{i},\ 8 + 4\mathbf{i},\ 9\}$. We know 9 is relatively prime to both $8 + 6\mathbf{i}$ and $8 + 4\mathbf{i}$ so this set will label all configurations of A^*. Now we must show we can label $T'' = T^* - \{u^* \cup A^*\}$, the remaining 59 vertices, with $S_3 = [\gamma_{57}] \cup \{\gamma_{63},\ \gamma_{64}\}$.

If the second case applies to T^*, we label u^* with $\gamma_{59} = 8 + 5\mathbf{i}$, v^* with $\gamma_{64} = 8$, v^*_j with $\{\gamma_{63},\ \gamma_{65},\ \gamma_{63-2j,\ 3\leq j\leq k}\}$, and u^*_j with $\{\gamma_{62-2j,\ 1\leq j\leq k}\}$. We inductively label the remaining $63 - 2k - 2$ vertices with $[\gamma_{63-2k-2}]$. Here γ_{59} is relatively prime to all Gaussian integers in S_2, γ_{64} is relatively prime to all odd Gaussian integers, and adjacent v_j and u_j are labeled with consecutive Gaussian integers except the pairs γ_{63}, γ_{60} and γ_{65}, γ_{58} which are relatively prime. Therefore this is a Gaussian prime labeling of T^* with $S_1 = [\gamma_{60}] \cup \{\gamma_{63},\ \gamma_{64},\ \gamma_{65}\}$.

Finally, we apply Lemma 3.4 to T''. If the first case of Lemma 3.4 applies to T'', we label u'' with $\gamma_{57} = 8 + 7\mathbf{i}$, which is relatively prime to all Gaussian integers in S_3. We label A'' with $\{\gamma_{56},\ \gamma_{63},\ \gamma_{64}\} = \{7 + 7\mathbf{i},\ 8 + \mathbf{i},\ 8\}$. We know $8 + \mathbf{i}$ is relatively prime to both 8 and $7 + 7\mathbf{i}$ so this set will label all configurations of A''. We inductively label the remaining 55 vertices with $[\gamma_{55}]$ and so this is a Gaussian prime labeling of T'' with $S_3 = [\gamma_{57}] \cup \{\gamma_{63},\ \gamma_{64}\}$.

If the second case applies to T'', we label u'' with $\gamma_{57} = 8 + 7\mathbf{i}$, v'' with $\gamma_{64} = 8$, v''_j with $\{\gamma_{63},\ \gamma_{59-2j,\ 2\leq j\leq k}\}$, and u''_j with $\{\gamma_{58-2j,\ 1\leq j\leq k}\}$. We inductively label the remaining $59-2k-2$ vertices with $[\gamma_{59-2k-2}]$. Here γ_{57} is relatively prime to all Gaussian integers in S_3, γ_{64} is relatively prime to all odd Gaussian integers, and adjacent v''_j and u''_j are labeled with consecutive Gaussian integers except for γ_{63} and γ_{56} which are relatively prime. Therefore this is a Gaussian prime labeling of T'' with $S_3 = [\gamma_{57}] \cup \{\gamma_{63},\ \gamma_{64}\}$.

Therefore all trees on $n = 67$ vertices have a Gaussian prime labeling.

$n = 68$ vertices: Similar to the $n = 67$ case, we need to apply Lemma 3.4 four times to show that trees on 68 vertices have a Gaussian prime labeling. We begin with the problem of labeling a tree T on 68 vertices with $[\gamma_{68}]$.

If the first case of Lemma 3.4 applies to T, we label u with $\gamma_{61} = 8+3\mathbf{i}$, which is relatively prime to all Gaussian integers in $[\gamma_{68}]$. We label A with $\{\gamma_{66},\ \gamma_{67},\ \gamma_{68}\} = \{9+\mathbf{i},\ 9+2\mathbf{i},\ 9+3\mathbf{i}\}$. We know $9+2\mathbf{i}$ is relatively prime to its neighbors $9+\mathbf{i}$ and $9+3\mathbf{i}$, so this set can label all configurations of A. Now we must show we can label $T^* = T - \{u \cup A\}$, the remaining 64 vertices, with $S_1[\gamma_{60}] \cup \{\gamma_{62},\ \gamma_{63},\ \gamma_{64},\ \gamma_{65}\}$.

If the second case of Lemma 3.4 applies to T, we suppose $k > 2$ and label u with $\gamma_{61} = 8+3\mathbf{i}$, v with $\gamma_{64} = 8 = \mathbf{i}(1+\mathbf{i})^6$, v_j with $\{\gamma_{67},\ \gamma_{65},\ \gamma_{63},\ \gamma_{67-2j,\ 4\leq j\leq k}\}$, and u_j with $\{\gamma_{68},\ \gamma_{66},\ \gamma_{68-2j,\ 3\leq j\leq k}\}$. We inductively label the remaining $67-2k-2$ vertices with $[\gamma_{67-2k-2}]$. Here γ_{61} is relatively prime to all Gaussian integers in $[\gamma_{68}]$, γ_{64} is relatively prime to all odd Gaussian integers, and adjacent v_j and u_j are labeled with consecutive Gaussian integers, so this is a Gaussian prime labeling.

If $k = 2$, we label u with $\gamma_{61} = 8+3\mathbf{i}$, v with $\gamma_{66} = 9+\mathbf{i}$, v_j with $\{\gamma_{67},\ \gamma_{65}\}$, and u_j with $\{\gamma_{68},\ \gamma_{60}\}$. γ_{61} is relatively prime to all Gaussian integers in $[\gamma_{68}]$, γ_{66} is relatively prime to γ_{67} and γ_{65}, and γ_{65} is relatively prime to γ_{60}. Now we need to show that we can label $T' = T - \{u \cup v \cup \{v_j\} \cup \{u_j\}\}$, the remaining 62 vertices, with $S_2 = [\gamma_{59}] \cup \{\gamma_{62},\ \gamma_{63},\ \gamma_{64}\}$.

We apply Lemma 3.4 to T'. If the first case applies to T', we label u' with $\gamma_{59} = 8+5\mathbf{i}$, which is relatively prime to all Gaussian integers in S_2. We then label A' with $\{\gamma_{62},\ \gamma_{63},\ \gamma_{64}\} = \{8+2\mathbf{i},\ 8+\mathbf{i},\ 8\}$. We know $8+\mathbf{i}$ is relatively prime to $8+2\mathbf{i}$ and 8, so this set can label all configurations of A'. We inductively label the remaining 58 vertices with $[\gamma_{58}]$ and so have a Gaussian prime labeling of T' with S_2.

If the second case applies to T', we label u' with $\gamma_{59} = 8+5\mathbf{i}$, v' with $\gamma_{64} = 8 = \mathbf{i}(1+\mathbf{i})^6$, v'_j with $\{\gamma_{63},\ \gamma_{61-2j,\ 2\leq j\leq k}\}$, and u'_j with $\{\gamma_{62},\ \gamma_{62-2j,\ 2\leq j\leq k}\}$. γ_{59} is relatively prime to all Gaussian integers in S_1, γ_{64} is relatively prime to all odd Gaussian integers, and adjacent v'_j and u'_j are labeled with consecutive Gaussian integers. We inductively label the remaining $62-2k-2$ vertices with $[\gamma_{62-2k-2}]$ and so we have a Gaussian prime labeling of T' with S_2.

We now apply Lemma 3.4 to T^*. If the first case of Lemma 3.4 applies to T^*, we label u^* with $\gamma_{59} = 8+5\mathbf{i}$, which is relatively prime to all Gaussian integers in S_1. We then label A^* with $\{\gamma_{60},\ \gamma_{62},\ \gamma_{63}\} = \{8+4\mathbf{i},\ 8+2\mathbf{i},\ 8+\mathbf{i}\}$. We know $8+\mathbf{i}$ is relatively prime to both $8+4\mathbf{i}$ and $8+2\mathbf{i}$ so this set will label all configurations of A^*. For this to be a Gaussian prime labeling of T^* with $S_1 = [\gamma_{60}] \cup \{\gamma_{62},\ \gamma_{63},\ \gamma_{64},\ \gamma_{65}\}$ we need to show that we can label $T'' = T^* - \{u^* \cup A^*\}$, the remaining 60 vertices, with $S_3 = [\gamma_{58}] \cup \{\gamma_{64},\ \gamma_{65}\}$.

If the second case applies to T^*, we label u^* with $\gamma_{59} = 8+5\mathbf{i}$, v^* with $\gamma_{64} = 8$, v^*_j with $\{\gamma_{65},\ \gamma_{63},\ \gamma_{63-2j,\ 3\leq j\leq k}\}$, and u^*_j with $\{\gamma_{64-2j,\ 1\leq j\leq k}\}$. We inductively label the remaining $63-2k-2$ vertices with $[\gamma_{63-2k-2}]$. Here γ_{59} is relatively prime to all Gaussian integers in S_1 γ_{64} is relatively prime to all odd Gaussian integers, and adjacent v^*_j and u^*_j are labeled with consecutive Gaussian integers except the pairs γ_{65}, γ_{62} and γ_{63}, γ_{60} which are relatively prime. Therefore this is a Gaussian prime labeling of T^* with $S_1 = [\gamma_{60}] \cup \{\gamma_{62},\ \gamma_{63},\ \gamma_{64},\ \gamma_{65}\}$.

Finall, we apply Lemma 3.4 to T''. If the first case of Lemma 3.4 applies to T'', we label u'' with $\gamma_{57} = 8+7\mathbf{i}$, which is relatively prime to all Gaussian integers in S_3. We label A'' with $\{\gamma_{58},\ \gamma_{64},\ \gamma_{65}\} = \{8+6\mathbf{i},\ 8,\ 9\}$. We know 9 is relatively prime to both $8+6\mathbf{i}$ and 8 so this set will label all configurations of A''. We inductively label the remaining 56 vertices with $[\gamma_{56}]$ and so this is a Gaussian prime labeling of T'' with $S_3 = [\gamma_{58}] \cup \{\gamma_{64},\ \gamma_{65}\}$.

If the second case applies to T'', we label u'' with $\gamma_{57} = 8 + 7\mathbf{i}$, v'' with $\gamma_{64} = 8$, v''_j with $\{\gamma_{65},\ \gamma_{59-2j,\ 2\leq j\leq k}\}$, and u''_j with $\{\gamma_{60-2j,\ 1\leq j\leq k}\}$. We inductively label the remaining $60-2k-2$ vertices with $[\gamma_{60-2k-2}]$. Here γ_{57} is relatively prime to all Gaussian integers in S_3, γ_{64} is relatively prime to all odd Gaussian integers, and adjacent v''_j and u''_j are labeled with consecutive Gaussian integers except for γ_{65} and γ_{58} which are relatively prime. Therefore this is a Gaussian prime labeling of T'' with $S_3 = [\gamma_{58}] \cup \{\gamma_{64},\ \gamma_{65}\}$.

Therefore all trees on $n = 68$ vertices have a Gaussian prime labeling.

$n = 72$ vertices: If the first case of Lemma 3.4 applies to T, we label u with $\gamma_{69} = 9 + 4\mathbf{i}$ and A with $\{\gamma_{70},\ \gamma_{71},\ \gamma_{72}\} = \{9 + 5\mathbf{i},\ 9 + 6\mathbf{i},\ 9 + 7\mathbf{i}\}$. We know $9 + 4\mathbf{i}$ is relatively prime to all Gaussian integers in $[\gamma_{72}]$ and $9 + 6\mathbf{i}$ is relatively prime to both $9 + 5\mathbf{i}$ and $9 + 7\mathbf{i}$, so this will label all configurations of A. We inductively label the remaining 68 vertices with $[\gamma_{68}]$ and so this is a Gaussian prime labeling of T.

If the second case of Lemma 3.4 applies to T, we have two cases. In either cases we label u with $\gamma_{69} = 9 + 4\mathbf{i}$ as $9 + 4\mathbf{i}$ is relatively prime to all Gaussian integers in $[\gamma_7 2]$. If $k \leq 3$ then we label v with $\gamma_{72} = 9 + 7\mathbf{i}$, each v_j that appears with $\{\gamma_{71},\ \gamma_{67},\ \gamma_{65}\}$, the adjacent u_j with $\{\gamma_{70},\ \gamma_{68},\ \gamma_{66}\}$, and the remaining $72 - 2k - 2$ vertices with $[\gamma_{72-2k-2}]$. Because $9 + 7\mathbf{i}$ is relatively prime to each possible label of v_j and adjacent v_j and u_j are labeled with consecutive Gaussian integers this is a Gaussian prime labeling.

If $k > 3$ we label v with $\gamma_{64} = 8$, each v_j that appears with $\{\gamma_{71},\ \gamma_{71-2j,\ 2\leq j\leq k}\}$, the adjacent u_j with $\{\gamma_{72},\ \gamma_{68},\ \gamma_{70},\ \gamma_{66},\ \gamma_{72-2j,\ 5\leq j\leq k}\}$, and the remaining $72 - 2k - 2$ vertices with $[\gamma_{72-2k-2}]$. Because $\gamma_{64} = 8$ is relatively prime to all odd Gaussian integers and each adjacent v_j and u_j are labeled with consecutive Gaussian integers except for the pairs γ_{65}, γ_{70}, and γ_{63}, γ_{66} which are also relatively prime, this is a Gaussian prime labeling.

Therefore all trees on $n = 72$ vertices have a Gaussian prime labeling.

The above cases and Lemma 3.3 imply that all trees on $n \leq 72$ vertices have a Gaussian prime labeling under the spiral ordering. $\square$

References

[1] Joseph A. Gallian. A dynamic survey of graph labeling. *Electron. J. Combin.*, 21:Dynamic Survey 6, 43 pp. (electronic), 2014.

[2] S. Klee, H. Lehmann, and A. Park. Prime labeling of families of trees with gaussian integers. submitted. Preprint available: `http://fac-staff.seattleu.edu/klees/web/GaussianLabelings.pdf`, 2015.

[3] Oleg Pikhurko. Every tree with at most 34 vertices is prime. *Util. Math.*, 62:185–190, 2002.

[4] Leanne Robertson and Ben Small. On Newman's conjecture and prime trees. *Integers*, 9:A10, 117–128, 2009.

[5] K.H. Rosen. *Elementary Number Theory and Its Applications.* Addison-Wesley, 2011.

[6] R.J. Trudeau. *Introduction to Graph Theory.* Dover Books on Mathematics. Dover Publications, 2013.

Appendix

In this section we provide reference tables for our labeling. In Table 1 we provide the index in the spiral ordering, primality, and prime factorization up to associates of the first 72 Gaussian integers ordered by index. In Tables 2, 3, and 4 we provide a reference for the labeling of each case of the proof of Theorem 3.6 not covered by Lemma 3.3. In each row of Table 2 we give the number of vertices to be labeled, the set S to label with, and the labels of u, A, and the remainder of the tree T^*. In each row of Tables 3 and 4 we give the number of vertices to be labeled, the set S to label with, and the labels of u, v, $\{v_j\}$, and $\{u_j\}$. We omit giving the label for the remainder of the tree as it is the interval of the remaining Gaussian integers for all cases except trees on 67 and 68 vertices. In these cases the label for the remainder of the tree is the set in the following row of the table.

Index	Gaussian	Prime?	Prime Factorization	Index	Gaussian	Prime?	Prime Factorization
1	1	N	1	37	7	Y	7
2	$1+\mathbf{i}$	Y	$1+\mathbf{i}$	38	$7+\mathbf{i}$	N	$(1+\mathbf{i})(2+\mathbf{i})^2$
3	$2+\mathbf{i}$	Y	$2+\mathbf{i}$	39	$7+2\mathbf{i}$	Y	$7+2\mathbf{i}$
4	2	N	$(1+\mathbf{i})^2$	40	$7+3\mathbf{i}$	N	$(1+\mathbf{i})(2+5\mathbf{i})$
5	3	Y	3	41	$7+4\mathbf{i}$	N	$(1+2\mathbf{i})(2+3\mathbf{i})$
6	$3+\mathbf{i}$	N	$(1+\mathbf{i})(1+2\mathbf{i})$	42	$7+5\mathbf{i}$	N	$(1+\mathbf{i})(1+6\mathbf{i})$
7	$3+2\mathbf{i}$	Y	$3+2\mathbf{i}$	43	$7+6\mathbf{i}$	N	$(2+\mathbf{i})(4+\mathbf{i})$
8	$2+2\mathbf{i}$	N	$(1+\mathbf{i})^3$	44	$6+6\mathbf{i}$	N	$(1+\mathbf{i})^3(3)$
9	$1+2\mathbf{i}$	Y	$1+2\mathbf{i}$	45	$5+6\mathbf{i}$	Y	$5+6\mathbf{i}$
10	$1+3\mathbf{i}$	N	$(1+\mathbf{i})(2+\mathbf{i})$	46	$4+6\mathbf{i}$	N	$(1+\mathbf{i})^2(2+3\mathbf{i})$
11	$2+3\mathbf{i}$	Y	$2+3\mathbf{i}$	47	$3+6\mathbf{i}$	N	$(3)(1+2\mathbf{i})$
12	$3+3\mathbf{i}$	N	$(1+\mathbf{i})(3)$	48	$2+6\mathbf{i}$	N	$(1+\mathbf{i})^3(2+\mathbf{i})$
13	$4+3\mathbf{i}$	N	$(1+2\mathbf{i})^2$	49	$1+6\mathbf{i}$	Y	$1+6\mathbf{i}$
14	$4+2\mathbf{i}$	N	$(1+\mathbf{i})^2(2+\mathbf{i})$	50	$1+7\mathbf{i}$	N	$(1+\mathbf{i})(1+2\mathbf{i})^2$
15	$4+\mathbf{i}$	Y	$4+\mathbf{i}$	51	$2+7\mathbf{i}$	Y	$2+7\mathbf{i}$
16	4	N	$(1+\mathbf{i})^4$	52	$3+7\mathbf{i}$	N	$(1+\mathbf{i})(5+2\mathbf{i})$
17	5	N	$(1+2\mathbf{i})(2+\mathbf{i})$	53	$4+7\mathbf{i}$	N	$(2+\mathbf{i})(3+2\mathbf{i})$
18	$5+\mathbf{i}$	N	$(1+\mathbf{i})(2+3\mathbf{i})$	54	$5+7\mathbf{i}$	N	$(1+\mathbf{i})(6+\mathbf{i})$
19	$5+2\mathbf{i}$	Y	$5+2\mathbf{i}$	55	$6+7\mathbf{i}$	N	$(1+2\mathbf{i})(1+4\mathbf{i})$
20	$5+3\mathbf{i}$	N	$(1+\mathbf{i})(1+4\mathbf{i})$	56	$7+7\mathbf{i}$	N	$(1+\mathbf{i})(7)$
21	$5+4\mathbf{i}$	Y	$5+4\mathbf{i}$	57	$8+7\mathbf{i}$	Y	$8+7\mathbf{i}$
22	$4+4\mathbf{i}$	N	$(1+\mathbf{i})^5$	58	$8+6\mathbf{i}$	N	$(1+\mathbf{i})^2(1+2\mathbf{i})^2$
23	$3+4\mathbf{i}$	N	$(2+\mathbf{i})^2$	59	$8+5\mathbf{i}$	Y	$8+5\mathbf{i}$
24	$2+4\mathbf{i}$	N	$(1+\mathbf{i})^2(1+2\mathbf{i})$	60	$8+4\mathbf{i}$	N	$(1+\mathbf{i})^4(2+\mathbf{i})$
25	$1+4\mathbf{i}$	Y	$1+4\mathbf{i}$	61	$8+3\mathbf{i}$	Y	$8+3\mathbf{i}$
26	$1+5\mathbf{i}$	N	$(1+\mathbf{i})(3+2\mathbf{i})$	62	$8+2\mathbf{i}$	N	$(1+\mathbf{i})^2(4+\mathbf{i})$
27	$2+5\mathbf{i}$	Y	$2+5\mathbf{i}$	63	$8+\mathbf{i}$	N	$(1+2\mathbf{i})(3+2\mathbf{i})$
28	$3+5\mathbf{i}$	N	$(1+\mathbf{i})(4+\mathbf{i})$	64	8	N	$(1+\mathbf{i})^6$
29	$4+5\mathbf{i}$	Y	$4+5\mathbf{i}$	65	9	N	3^2
30	$5+5\mathbf{i}$	N	$(1+\mathbf{i})(1+2\mathbf{i})(2+\mathbf{i})$	66	$9+\mathbf{i}$	N	$(1+\mathbf{i})(4+5\mathbf{i})$
31	$6+5\mathbf{i}$	Y	$6+5\mathbf{i}$	67	$9+2\mathbf{i}$	N	$(2+\mathbf{i})(1+4\mathbf{i})$
32	$6+4\mathbf{i}$	N	$(1+\mathbf{i})^2(3+2\mathbf{i})$	68	$9+3\mathbf{i}$	N	$(1+\mathbf{i})(3)(1+2\mathbf{i})$
33	$6+3\mathbf{i}$	N	$(3)(2+\mathbf{i})$	69	$9+4\mathbf{i}$	Y	$9+4\mathbf{i}$
34	$6+2\mathbf{i}$	N	$(1+\mathbf{i})^3(1+2\mathbf{i})$	70	$9+5\mathbf{i}$	N	$(1+\mathbf{i})(2+7\mathbf{i})$
35	$6+\mathbf{i}$	Y	$6+\mathbf{i}$	71	$9+6\mathbf{i}$	N	$(3)(3+2\mathbf{i})$
36	6	N	$(1+\mathbf{i})^2(3)$	72	$9+7\mathbf{i}$	N	$(1+\mathbf{i})(2+\mathbf{i})(2+3\mathbf{i})$

Table 1: Table of Gaussian integers

n	S	$\ell(u)$	$\ell(A)$	$\ell(T^*)$
14	$[\gamma_{14}]$	$\gamma_{11} = 2 + 3\mathbf{i}$	$\{\gamma_{12},\ \gamma_{13},\ \gamma_{14}\} = \{3 + 3\mathbf{i},\ 4 + 3\mathbf{i},\ 4 + 2\mathbf{i}\}$	$[\gamma_{10}]$
18	$[\gamma_{18}]$	$\gamma_{15} = 4 + \mathbf{i}$	$\{\gamma_{16},\ \gamma_{17},\ \gamma_{18}\} = \{4,\ 5,\ 5 + \mathbf{i}\}$	$[\gamma_{14}]$
24	$[\gamma_{24}]$	$\gamma_{21} = 5 + 4\mathbf{i}$	$\{\gamma_{22},\ \gamma_{23},\ \gamma_{24}\} = \{4 + 4\mathbf{i},\ 3 + 4\mathbf{i},\ 2 + 4\mathbf{i}\}$	$[\gamma_{20}]$
34	$[\gamma_{34}]$	$\gamma_{31} = 6 + 5\mathbf{i}$	$\{\gamma_{32},\ \gamma_{33},\ \gamma_{34}\} = \{6 + 4\mathbf{i},\ 6 + 3\mathbf{i},\ 6 + 2\mathbf{i}\}$	$[\gamma_{30}]$
42	$[\gamma_{42}]$	$\gamma_{39} = 7 + 2\mathbf{i}$	$\{\gamma_{40},\ \gamma_{41},\ \gamma_{42}\} = \{7 + 3\mathbf{i},\ 7 + 4\mathbf{i},\ 7 + 5\mathbf{i}\}$	$[\gamma_{38}]$
43	$[\gamma_{43}]$	$\gamma_{39} = 7 + 2\mathbf{i}$	$\{\gamma_{38},\ \gamma_{40},\ \gamma_{41}\} = \{7 + 3\mathbf{i}, 7 + 4\mathbf{i}, 7 + \mathbf{i}\}$	$[\gamma_{37}] \cup \{\gamma_{42}, \gamma_{43}\}$
39	$[\gamma_{37}] \cup \{\gamma_{42}, \gamma_{43}\}$	$\gamma_{37} = 7$	$\{\gamma_{36},\ \gamma_{42},\ \gamma_{43}\} = \{6,\ 7 + 5\mathbf{i},\ 7 + 6\mathbf{i}\}$	$[\gamma_{35}]$
44	$[\gamma_{44}]$	$\gamma_{39} = 7 + 2\mathbf{i}$	$\{\gamma_{40},\ \gamma_{43},\ \gamma_{44}\} = \{7 + 3\mathbf{i},\ 7 + 6\mathbf{i},\ 6 + 6\mathbf{i}\}$	$[\gamma_{38}] \cup \{\gamma_{41}, \gamma_{42}\}$
40	$[\gamma_{38}] \cup \{\gamma_{41}, \gamma_{42}\}$	$\gamma_{37} = 7$	$\{\gamma_{38},\ \gamma_{41},\ \gamma_{42}\} = \{7 + \mathbf{i},\ 7 + 4\mathbf{i},\ 7 + 5\mathbf{i}\}$	$[\gamma_{36}]$
48	$[\gamma_{48}]$	$\gamma_{45} = 5 + 6\mathbf{i}$	$\{\gamma_{46},\ \gamma_{47},\ \gamma_{48}\} = \{4 + 6\mathbf{i},\ 3 + 6\mathbf{i},\ 2 + 6\mathbf{i}\}$	$[\gamma_{44}]$
54	$[\gamma_{54}]$	$\gamma_{51} = 2 + 7\mathbf{i}$	$\{\gamma_{52},\ \gamma_{53},\ \gamma_{54}\} = \{3 + 7\mathbf{i},\ 4 + 7\mathbf{i},\ 5 + 7\mathbf{i}\}$	$[\gamma_{50}]$
55	$[\gamma_{55}]$	$\gamma_{51} = 2 + 7\mathbf{i}$	$\{\gamma_{50},\ \gamma_{52},\ \gamma_{53}\} = \{1 + 7\mathbf{i},\ 3 + 7\mathbf{i},\ 4 + 7\mathbf{i}\}$	$[\gamma_{49}] \cup \{\gamma_{54},\ \gamma_{55}\}$
51	$[\gamma_{49}] \cup \{\gamma_{54},\ \gamma_{55}\}$	$\gamma_{49} = 1 + 6\mathbf{i}$	$\{\gamma_{48},\ \gamma_{54},\ \gamma_{55}\} = \{2 + 6\mathbf{i},\ 5 + 7\mathbf{i},\ 6 + 7\mathbf{i}\}$	$[\gamma_{47}]$
56	$[\gamma_{56}]$	$\gamma_{51} = 2 + 7\mathbf{i}$	$\{\gamma_{50},\ \gamma_{55},\ \gamma_{56}\} = \{1 + 7\mathbf{i},\ 6 + 7\mathbf{i},\ 7 + 7\mathbf{i}\}$	$[\gamma_{49}] \cup \{\gamma_{52},\ \gamma_{53},\ \gamma_{54}\}$
52	$[\gamma_{49}] \cup \{\gamma_{52},\ \gamma_{53},\ \gamma_{54}\}$	$\gamma_{49} = 1 + 6\mathbf{i}$	$\{\gamma_{52},\ \gamma_{53},\ \gamma_{54}\} = \{3 + 7\mathbf{i},\ 4 + 7\mathbf{i},\ 5 + 7\mathbf{i}\}$	$[\gamma_{48}]$
64	$[\gamma_{64}]$	$\gamma_{61} = 8 + 3\mathbf{i}$	$\{\gamma_{62},\ \gamma_{63},\ \gamma_{64}\} = \{8 + 2\mathbf{i},\ 8 + \mathbf{i},\ 8\}$	$[\gamma_{60}]$
65	$[\gamma_{65}]$	$\gamma_{61} = 8 + 3\mathbf{i}$	$\{\gamma_{60},\ \gamma_{62},\ \gamma_{63}\} = \{8 + 4\mathbf{i},\ 8 + 2\mathbf{i},\ 8 + \mathbf{i}\}$	$[\gamma_{59}] \cup \{\gamma_{64},\ \gamma_{65}\}$
61	$[\gamma_{59}] \cup \{\gamma_{64},\ \gamma_{65}\}$	$\gamma_{59} = 8 + 5\mathbf{i}$	$\{\gamma_{58},\ \gamma_{64},\ \gamma_{65}\} = \{8 + 6\mathbf{i},\ 8,\ 9\}$	$[\gamma_{57}]$
66	$[\gamma_{66}]$	$\gamma_{61} = 8 + 3\mathbf{i}$	$\{\gamma_{62},\ \gamma_{65},\ \gamma_{66}\} = \{8 + 2\mathbf{i},\ 9,\ 9 + \mathbf{i}\}$	$[\gamma_{60}] \cup \{\gamma_{63},\ \gamma_{64}\}$
62	$[\gamma_{60}] \cup \{\gamma_{63},\ \gamma_{64}\}$	$\gamma_{59} = 8 + 5\mathbf{i}$	$\{\gamma_{60},\ \gamma_{63},\ \gamma_{64}\} = \{8 + 4\mathbf{i},\ 8 + \mathbf{i},\ 8\}$	$[\gamma_{58}]$
67	$[\gamma_{67}]$	$\gamma_{61} = 8 + 3\mathbf{i}$	$\{\gamma_{62},\ \gamma_{66},\ \gamma_{67}\} = \{8 + 2\mathbf{i},\ 9 + \mathbf{i},\ 9 + 2\mathbf{i}\}$	$[\gamma_{60}] \cup \{\gamma_{63},\ \gamma_{64},\ \gamma_{65}\}$
63	$[\gamma_{60}] \cup \{\gamma_{63},\ \gamma_{64},\ \gamma_{65}\}$	$\gamma_{59} = 8 + 5\mathbf{i}$	$\{\gamma_{58},\ \gamma_{60},\ \gamma_{65}\} = \{8 + 6\mathbf{i},\ 8 + 4\mathbf{i},\ 9\}$	$[\gamma_{57}] \cup \{\gamma_{63},\ \gamma_{64}\}$
59	$[\gamma_{57}] \cup \{\gamma_{63},\ \gamma_{64}\}$	$\gamma_{57} = 8 + 7\mathbf{i}$	$\{\gamma_{56},\ \gamma_{63},\ \gamma_{64}\} = \{7 + 7\mathbf{i},\ 8 + \mathbf{i},\ 8\}$	$[\gamma_{55}]$
61	$[\gamma_{59}] \cup \{\gamma_{64},\ \gamma_{67}\}$	$\gamma_{59} = 8 + 5\mathbf{i}$	$\{\gamma_{58},\ \gamma_{64},\ \gamma_{67}\} = \{8 + 6\mathbf{i}, 8, 9 + 2\mathbf{i}\}$	$[\gamma_{57}]$
68	$[\gamma_{68}]$	$\gamma_{61} = 8 + 3\mathbf{i}$	$\{\gamma_{66},\ \gamma_{67},\ \gamma_{68}\} = \{9 + \mathbf{i},\ 9 + 2\mathbf{i},\ 9 + 3\mathbf{i}\}$	$[\gamma_{60}] \cup \{\gamma_{62},\ \gamma_{63},\ \gamma_{64},\ \gamma_{65}\}$
64	$[\gamma_{60}] \cup \{\gamma_{62},\ \gamma_{63},\ \gamma_{64},\ \gamma_{65}\}$	$\gamma_{59} = 8 + 5\mathbf{i}$	$\{\gamma_{60},\ \gamma_{62},\ \gamma_{63}\} = \{8 + 4\mathbf{i},\ 8 + 2\mathbf{i},\ 8 + \mathbf{i}\}$	$[\gamma_{58}] \cup \{\gamma_{64},\ \gamma_{65}\}$
60	$[\gamma_{58}] \cup \{\gamma_{64},\ \gamma_{65}\}$	$\gamma_{57} = 8 + 7\mathbf{i}$	$\{\gamma_{58},\ \gamma_{64},\ \gamma_{65}\} = \{8 + 6\mathbf{i},\ 8,\ 9\}$	$[\gamma_{56}]$
62	$[\gamma_{59}] \cup \{\gamma_{62},\ \gamma_{63},\ \gamma_{64}\}$	$\gamma_{59} = 8 + 5\mathbf{i}$	$\{\gamma_{62},\ \gamma_{63},\ \gamma_{64}\} = \{8 + 2\mathbf{i}, 8 + \mathbf{i}, 8\}$	$[\gamma_{58}]$
72	$[\gamma_{72}]$	$\gamma_{69} = 9 + 4\mathbf{i}$	$\{\gamma_{70},\ \gamma_{71},\ \gamma_{72}\} = \{9 + 5\mathbf{i},\ 9 + 6\mathbf{i},\ 9 + 7\mathbf{i}\}$	$[\gamma_{68}]$

Table 2: Labeling Table for Case 1 of Theorem 3.6

n	S	k	$\ell(u)$	$\ell(v)$	$\ell(v_j)_{1\leq j\leq k}$	$\ell(u_j)_{1\leq j\leq k}$
14	$[\gamma_{14}]$	≤ 4	γ_{11}	γ_{14}	$\{\gamma_{13}, \gamma_{13-2j},\ 2\leq j\leq k\}$	$\{\gamma_{14-2j},\ 1\leq j\leq k\}$
		> 4	γ_3	γ_4	$\{\gamma_{15-2j},\ 1\leq j\leq k-1, \gamma_1\}$	$\{\gamma_{16-2j},\ 1\leq j\leq k-1, \gamma_2\}$
18	$[\gamma_{18}]$	≥ 2	γ_{15}	γ_{16}	$\{\gamma_{17}, \gamma_{17-2j},\ 2\leq j\leq k\}$	$\{\gamma_{18}, \gamma_{18-2j},\ 2\leq j\leq k\}$
24	$[\gamma_{24}]$	≥ 2	γ_{21}	γ_{22}	$\{\gamma_{23}, \gamma_{23-2j},\ 2\leq j\leq k\}$	$\{\gamma_{24}, \gamma_{24-2j},\ 2\leq j\leq k\}$
34	$[\gamma_{34}]$	≤ 5	γ_{31}	γ_{32}	$\{\gamma_{33}, \gamma_{33-2j, 2\leq j\leq k}\}$	$\{\gamma_{34}, \gamma_{34-2j, 2\leq j\leq k}\}$
		> 5	γ_{31}	γ_{22}	$\{\gamma_{33}, \gamma_{33-2j},\ 2\leq j\leq k\}$	$\{\gamma_{34}, \gamma_{34-2j},\ 2\leq j\leq 5, \gamma_{32}, \gamma_{34-2j},\ 7\leq j\leq k\}$
42	$[\gamma_{42}]$	≥ 2	γ_{39}	γ_{42}	$\{\gamma_{41},\ \gamma_{41-2j},\ 2\leq j\leq k\}$	$\{\gamma_{42-2j},\ 1\leq j\leq k\}$
43	$[\gamma_{43}]$	≥ 2		γ_{39}	$\{\gamma_{43},\ \gamma_{41},\ \gamma_{43-2j},\ 3\leq j\leq k\}$	$\{\gamma_{44-2j},\ 1\leq j\leq k\}$
39	$[\gamma_{37}] \cup \{\gamma_{42}, \gamma_{43}\}$	≥ 2	γ_{37}	γ_{42}	$\{\gamma_{43},\ \gamma_{39-2j},\ 2\leq j\leq k\}$	$\{\gamma_{38-2j},\ 1\leq j\leq k\}$
44	$[\gamma_{44}]$	≥ 2	γ_{39}	γ_{42}	$\{\gamma_{43},\ \gamma_{41},\ \gamma_{43-2j},\ 3\leq j\leq k\}$	$\{\gamma_{44},\ \gamma_{44-2j},\ 2\leq j\leq k\}$
40	$[\gamma_{38}] \cup \{\gamma_{41},\ \gamma_{42}\}$	≥ 2	γ_{37}	γ_{42}	$\{\gamma_{41},\ \gamma_{39-2j},\ 2\leq j\leq k\}$	$\{\gamma_{40-2j},\ 1\leq j\leq k\}$
48	$[\gamma_{48}]$	≥ 2	γ_{45}	γ_{42}	$\{\gamma_{47}, \gamma_{47-2j},\ 2\leq j\leq k\}$	$\{\gamma_{50-2j},\ 1\leq j\leq k\}$
54	$[\gamma_{54}]$	≤ 8	γ_{51}	γ_{54}	$\{\gamma_{53},\ \gamma_{53-2j},\ 2\leq j\leq k\}$	$\{\gamma_{54-2j},\ 1\leq j\leq k\}$
		> 8	γ_{51}	γ_{22}	$\{\gamma_{53},\ \gamma_{53-2j},\ 2\leq j\leq k\}$	$\{\gamma_{54-2j},\ 1\leq j\leq 15,\ \gamma_{54},\ \gamma_{54-2j},\ 17\leq j\leq k\}$
55	$[\gamma_{55}]$	≥ 2		γ_{51}	$\{\gamma_{55},\ \gamma_{53},\ \gamma_{53-2j},\ 2\leq j\leq k\}$	$\{\gamma_{56-2j},\ 1\leq j\leq k\}$
51	$[\gamma_{49}] \cup \{\gamma_{54},\ \gamma_{55}\}$	≥ 2		γ_{49}	$\{\gamma_{55},\ \gamma_{51-2j},\ 2\leq j\leq k\}$	$\{\gamma_{54},\ \gamma_{52-2j},\ 2\leq j\leq k\}$
56	$[\gamma_{56}]$	≤ 8	γ_{51}	γ_{56}	$\{\gamma_{55},\ \gamma_{53},\ \gamma_{55-2j},\ 3\leq j\leq k\}$	$\{\gamma_{56-2j},\ 1\leq j\leq k\}$
		> 8	γ_{51}	γ_{22}	$\{\gamma_{55},\ \gamma_{53},\ \gamma_{55-2j},\ 3\leq j\leq k\}$	$\{\gamma_{56-2j},\ 1\leq j\leq 16,\ \gamma_{56},\ \gamma_{56-2j},\ 18\leq j\leq k\}$
52	$[\gamma_{49}] \cup \{\gamma_{52},\ \gamma_{53},\ \gamma_{54}\}$	≤ 9	γ_{49}	γ_{54}	$\{\gamma_{53},\ \gamma_{51-2j},\ 2\leq j\leq k\}$	$\{\gamma_{52},\ \gamma_{52-2j},\ 2\leq j\leq k\}$
		> 9	γ_{49}	γ_{22}	$\{\gamma_{53},\ \gamma_{51-2j},\ 2\leq j\leq k\}$	$\{\gamma_{52},\ \gamma_{52-2j},\ 2\leq j\leq 14,\ \gamma_{54},\ \gamma_{52-2j},\ 16\leq j\leq k\}$
64	$[\gamma_{64}]$	≥ 2	γ_{61}	γ_{64}	$\{\gamma_{63},\ \gamma_{63-2j},\ 2\leq j\leq k\}$	$\{\gamma_{64-2j},\ 1\leq j\leq k\}$
65	$[\gamma_{65}]$	≥ 2	γ_{61}	γ_{64}	$\{\gamma_{65},\ \gamma_{63},\ \gamma_{65-2j},\ 3\leq j\leq k\}$	$\{\gamma_{64-2j},\ 1\leq j\leq k\}$
61	$[\gamma_{59}] \cup \{\gamma_{64},\ \gamma_{65}\}$	≥ 2	γ_{59}	γ_{64}	$\{\gamma_{65},\ \gamma_{61-2j},\ 2\leq j\leq k\}$	$\{\gamma_{60-2j},\ 1\leq j\leq k\}$

Table 3: Labeling Table for Case 2 of Theorem 3.6, $n \leq 65$

n	S	k	$\ell(u)$	$\ell(v)$	$\ell(v_j)_{1\leq j\leq k}$	$\ell(u_j)_{1\leq j\leq k}$
66	$[\gamma_{66}]$	≥ 2	γ_{61}	γ_{64}	$\{\gamma_{65},\ \gamma_{63},\ \gamma_{65-2j,\ 3\leq j\leq k}\}$	$\{\gamma_{66},\ \gamma_{66-2j,\ 2\leq j\leq k}\}$
62	$[\gamma_{60}]\cup\{\gamma_{63},\ \gamma_{64}\}$	≥ 2	γ_{59}	γ_{64}	$\{\gamma_{63},\ \gamma_{61-2j,\ 2\leq j\leq k}\}$	$\{\gamma_{62-2j,\ 1\leq j\leq k}\}$
67	$[\gamma_{67}]$	$=2$	γ_{61}	γ_{66}	$\{\gamma_{65},\ \gamma_{63}\}$	$\{\gamma_{60},\ \gamma_{62}\}$
		>2	γ_{61}	γ_{64}	$\{\gamma_{67},\ \gamma_{65},\ \gamma_{63},\ \gamma_{67-2j,\ 4\leq j\leq k}\}$	$\{\gamma_{66},\ \gamma_{66-2j,\ 2\leq j\leq k}\}$
63	$[\gamma_{60}]\cup\{\gamma_{63},\ \gamma_{64},\ \gamma_{65}\}$	≥ 2	γ_{59}	γ_{6}	$\{\gamma_{63},\ \gamma_{65},\ \gamma_{63-2j,\ 3\leq j\leq k}\}$	$\{\gamma_{62-2j,\ 1\leq j\leq k}\}$
59	$[\gamma_{57}]\cup\{\gamma_{63},\ \gamma_{64}\}$	≥ 2	γ_{57}	γ_{64}	$\{\gamma_{63},\ \gamma_{59-2j,\ 2\leq j\leq k}\}$	$\{\gamma_{58-2j,\ 1\leq j\leq k}\}$
61	$[\gamma_{59}]\cup\{\gamma_{64},\ \gamma_{67}\}$	≥ 2	γ_{59}	γ_{64}	$\{\gamma_{67},\ \gamma_{61-2j},\ 2\leq j\leq k\}$	$\{\gamma_{60-2j},\ 1\leq j\leq k\}$
68	$[\gamma_{68}]$	$=2$	γ_{61}	γ_{66}	$\{\gamma_{67},\ \gamma_{65}\}$	$\{\gamma_{68},\ \gamma_{60}\}$
		>2	γ_{61}	γ_{64}	$\{\gamma_{67},\ \gamma_{65},\ \gamma_{63},\ \gamma_{67-2j,\ 4\leq j\leq k}\}$	$\{\gamma_{68},\ \gamma_{66},\ \gamma_{68-2j,\ 3\leq j\leq k}\}$
64	$[\gamma_{60}]\cup\{\gamma_{62},\ \gamma_{63},\ \gamma_{64},\ \gamma_{65}\}$	≥ 2	γ_{59}	γ_{64}	$\{\gamma_{65},\ \gamma_{63},\ \gamma_{63-2j,\ 3\leq j\leq k}\}$	$\{\gamma_{64-2j,\ 1\leq j\leq k}\}$
60	$[\gamma_{58}]\cup\{\gamma_{64},\ \gamma_{65}\}$	≥ 2	γ_{57}	γ_{64}	$\{\gamma_{65},\ \gamma_{59-2j,\ 2\leq j\leq k}\}$	$\{\gamma_{60-2j,\ 1\leq j\leq k}\}$
62	$[\gamma_{59}]\cup\{\gamma_{62},\ \gamma_{63},\ \gamma_{64}\}$	≥ 2	γ_{59}	γ_{64}	$\{\gamma_{63},\ \gamma_{61-2j},\ 2\leq j\leq k\}$	$\{\gamma_{62},\ \gamma_{62-2j},\ 2\leq j\leq k\}$
72	$[\gamma_{72}]$	≤ 3	γ_{69}	γ_{72}	$\{\gamma_{71},\ \gamma_{67},\ \gamma_{65}\}$	$\{\gamma_{70},\ \gamma_{68},\ \gamma_{66}\}$
		>3	γ_{69}	γ_{64}	$\{\gamma_{71},\ \gamma_{71-2j,\ 2\leq j\leq k}\}$	$\{\gamma_{72},\ \gamma_{68},\ \gamma_{70},\ \gamma_{66},\ \gamma_{72-2j,\ 5\leq j\leq k}\}$

Table 4: Labeling Table for Case 2 of Theorem 3.6, $66\leq n\leq 72$