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FINITE PARTITIONS OF THE SET OF
ATOMS AS SUPPORTS

Benjamin Baker Bruce^a

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Rose-Hulman Institute of Technology

Department of Mathematics

Terre Haute, IN 47803

Email: mathjournal@rose-hulman.edu

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^aSt. Olaf College

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Abstract. The method of permutation models was introduced by Fraenkel in 1922 to prove the independence of the axiom of choice in set theory with atoms. We present a variant of the basic Fraenkel model in which supports are finite partitions of the set of atoms, rather than finite sets of atoms. Among our results are that, in this model, every well-ordered family of well-orderable sets has a choice function and that the union of such a family is well-orderable.

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1 Introduction

Permutation models date back to Fraenkel's 1922 paper [2], where they were used to prove the independence of the axiom of choice in an axiomatic set theory weaker than Zermelo-Fraenkel (ZF) called *set theory with atoms* (ZFA). Over the subsequent decades, such models have remained valuable tools for assessing the relative strength of numerous consequences and weakenings of the axiom of choice (see Howard and Rubin's monograph [4]). The techniques involved in permutation models are of interest as well, many of them combinatorial or group theoretic. It may be surprising, at least to the novice, that questions of logical independence in ZFA can be settled by such methods.

In this paper, we present a nice example of a permutation model that has not appeared in the literature previously. Our example, which we call the *finite partition model*, is a variant of the basic Fraenkel model, replacing sets of atoms with partitions of the set of atoms as supports. Many, but not all, properties of the basic Fraenkel model transfer to the finite partition model. Our main results are that, in the finite partition model, just as in the basic Fraenkel model, every well-ordered family of well-orderable sets has a choice function (Theorem 4.17) and the union of such a family is well-orderable (Theorem 4.18). Some of the lemmas used to prove these theorems are already known, or can be proven straightforwardly from known results. We do, however, include original combinatorial proofs of those lemmas so that the paper is self-contained.

We begin with an overview of set theory with atoms, borrowed largely from Jech [5, Ch. 4] and Halbeisen [3, Ch. 7]. Unlike standard Zermelo-Fraenkel set theory, set theory with atoms admits the existence of a set of objects that are not sets, called *atoms*. An atom has no elements and is different from the empty set, but may be an element of a set. The language of ZFA adds to the language of ZF the constant symbols 0 and A , which denote the empty set and the set of atoms, respectively. Thus, the primitive symbols of ZFA are \in , 0 and A . In ZFA, the statement " a is an atom" will simply mean " $a \in A$," and the statement " X is a set" will be understood as " $X \notin A$." The axioms of ZFA are the same as those of ZF, except for a few additions and modifications. The additional axioms stipulate the meanings of the symbols 0 and A . They are:

- *Empty set* $\neg \exists x(x \in 0)$,
- *Atoms* $\forall z[z \in A \leftrightarrow z \neq 0 \wedge \neg \exists x(x \in z)]$.

The modifications concern the axioms of extensionality and regularity. In ZF, extensionality says, roughly, that two objects are identical if they contain the same elements. Extensionality must therefore be modified to appear in ZFA since, in view of the axioms of 0 and A , the atoms and the empty set all (vacuously) contain the same elements but are pairwise distinct. Regularity, in ZF, says that every object X different from the empty set contains an element disjoint from X . If X were allowed to be an atom, as in ZFA, then regularity would fail. The modified axioms simply restrict quantification to sets:

- *Extensionality* $(\forall \text{ set } X)(\forall \text{ set } Y)[\forall u(u \in X \leftrightarrow u \in Y) \leftrightarrow X = Y]$,

- *Regularity* $(\forall \text{ nonempty set } S)(\exists x \in S)[x \cap S = \emptyset]$.

We note that the set A may be of any size, though we are interested in the case that $A \neq \emptyset$. If ZFA were to include the axiom $A = \emptyset$, then ZFA would be ZF.

The development of ZFA is analogous to the cumulative hierarchy ZF. The ordinals can be defined, in ZF, as transitive sets that are well-ordered. In ZFA, we will further require that no ordinal have an atom among its elements. Then for any set S , define $\mathcal{P}^\alpha(S)$ by

$$\begin{aligned}\mathcal{P}^0(S) &= S, \\ \mathcal{P}^{\alpha+1}(S) &= \mathcal{P}^\alpha(S) \cup \mathcal{P}(\mathcal{P}^\alpha(S)), \\ \mathcal{P}^\lambda(S) &= \bigcup_{\alpha < \lambda} \mathcal{P}^\alpha(S) \quad (\lambda \text{ limit ordinal}),\end{aligned}$$

where $\mathcal{P}(x)$ denotes the power set of x . Now define

$$\mathcal{P}^\infty(S) = \bigcup_{\alpha \in \text{Ord}} \mathcal{P}^\alpha(S),$$

where Ord is the class of all ordinals. Then the universe of ZFA is the class $\mathcal{P}^\infty(A)$.

Definition 1.1. The class $\mathcal{P}^\infty(\emptyset)$ of all pure sets is called the *kernel*.

When building a permutation model, we will always work in the theory ZFA + axiom of choice. The universe $\mathcal{P}^\infty(A)$ satisfies the axiom of choice because, at each stage of its development, *all* possible sets of previously available objects (including choice functions) are added. A permutation model, in this paper, will be a submodel of ZFA in which the axiom of choice does not hold. Every permutation model will include the set A of atoms and all elements of the kernel $\mathcal{P}^\infty(\emptyset)$. Note that the kernel contains all the ordinals and is a model of ZF + axiom of choice.

The rest of the paper is organized as follows: We define permutation models in detail in Section 2. In Section 3, we present some instructive examples involving the basic Fraenkel model. In Section 4, we introduce the new finite partition model and prove several propositions and theorems concerning its properties. We end the paper with Section 5, a brief mention of some potential directions for future research relating to the finite partition model.

2 Permutation models

In this section, we will give the definition of *permutation model*, following Jech [5, Ch. 4]. The crucial idea behind permutation models is that the axioms of ZFA do not distinguish between the atoms. Any permutation of the atoms may be extended to a permutation of the universe: Let π be any permutation of A . For every set x , there is a least ordinal α such that $x \in \mathcal{P}^\alpha(A)$. So, by induction on the ordinals, we can define πx by

$$\pi x = \{\pi y : y \in x\}.$$

It is easily shown that $x \in y$ if and only if $\pi x \in \pi y$ for any two objects x and y in the universe. Thus, π is an ϵ -automorphism of the universe.

Now, let \mathcal{G} be a group of permutations of A . Very roughly, the elements of a permutation model are the objects that are “stable” under the action of certain subgroups of \mathcal{G} . The following two definitions will make this characterization more precise:

Definition 2.1. A set \mathcal{F} of subgroups of \mathcal{G} is a *normal filter* on \mathcal{G} if it satisfies the following:

- i. $\mathcal{G} \in \mathcal{F}$;
- ii. if $H \in \mathcal{F}$ and K is a subgroup of \mathcal{G} such that $H \subseteq K$, then $K \in \mathcal{F}$;
- iii. if $H \in \mathcal{F}$ and $K \in \mathcal{F}$, then $H \cap K \in \mathcal{F}$;
- iv. if $\pi \in \mathcal{G}$ and $H \in \mathcal{F}$, then $\pi H \pi^{-1} \in \mathcal{F}$;
- v. for each $a \in A$, $\{\pi \in \mathcal{G} : \pi a = a\} \in \mathcal{F}$.

Convention 2.2. The symbols \mathcal{G} and \mathcal{F} will always denote, respectively, a fixed group of permutations of A and a fixed normal filter on \mathcal{G} .

Definition 2.3. For each x , let $\text{sym}(x) = \{\pi \in \mathcal{G} : \pi x = x\}$. Then x is said to be *symmetric* if $\text{sym}(x) \in \mathcal{F}$.

This now allows us to define a *permutation model* as the class

$$\mathcal{V} = \{x : x \text{ is symmetric and } x \subseteq \mathcal{V}\}$$

of all hereditarily symmetric objects. (Note that the definition of \mathcal{V} is a recursion on the rank of x . An object x is hereditarily symmetric if x is symmetric and all elements of x are symmetric and all of their elements are symmetric, etc.) For proof that \mathcal{V} is a model of ZFA, the reader is referred to Jech’s book [5, Thm. 4.1]. As mentioned above, all elements of the kernel are in \mathcal{V} and $A \in \mathcal{V}$: Since 0 is symmetric, it is easily shown, by induction on the ordinals, that all pure sets are hereditarily symmetric. The set A is symmetric since $\text{sym}(A) = \mathcal{G} \in \mathcal{F}$. Further, $\text{sym}(a) \in \mathcal{F}$ for each $a \in A$ by definition of \mathcal{F} . Thus, A is hereditarily symmetric.

The permutation models in this paper will be of a special form. To state that form precisely, we need the following definition:

Definition 2.4. For each x , let $\text{fix}(x) = \{\pi \in \mathcal{G} : \pi y = y \text{ for all } y \in x\}$. We say that x is a *support* of y if $\text{fix}(x) \subseteq \text{sym}(y)$.

In our permutation models, the normal filter will always be determined by the group \mathcal{G} and by a set of supports. Here, let S denote the chosen set of supports. Then the corresponding normal filter is

$$\mathcal{F} = \{H : H \text{ is a subgroup of } \mathcal{G}, H \supseteq \text{fix}(x) \text{ for some } x \in S\}. \quad (1)$$

Note that not every set of supports will generate a normal filter. We will always choose S so that a normal filter is produced, but this must be verified in each case. Let us make two additional remarks: First, by this construction, x is symmetric if and only if there exists $y \in S$ such that $\text{fix}(y) \subseteq \text{sym}(x)$. To mean “ x has a support $y \in S$,” we will usually just write “ x has a support y ,” and it will be understood that y is of the correct type. Second, when building permutation models in this paper, we will not mention the normal filter explicitly; only the set of supports.

To conclude this section, we list a few tricks (found in Jech’s book [5, Ch. 4]) which we will use regularly.

Lemma 2.5.

- (a) For all $\pi \in \mathcal{G}$ and all x , $\text{sym}(\pi x) = \pi \text{sym}(x) \pi^{-1}$ and $\text{fix}(\pi x) = \pi \text{fix}(x) \pi^{-1}$.
- (b) If x is a support of y , then πx is a support of πy for all $\pi \in \mathcal{G}$.
- (c) A set x is well-orderable in \mathcal{V} if and only if $\text{fix}(x) \in \mathcal{F}$.

Proof.

- (a) Let $\sigma \in \text{sym}(\pi x)$. Then $\sigma \pi x = \pi x$, so that $\pi^{-1} \sigma \pi x = x$. Hence, $\pi^{-1} \sigma \pi \in \text{sym}(x)$. It follows that $\sigma \in \pi \text{sym}(x) \pi^{-1}$. Conversely, let $\tau \in \pi \text{sym}(x) \pi^{-1}$. Writing $\tau = \pi \gamma \pi^{-1}$, where $\gamma \in \text{sym}(x)$, we have $\tau \pi x = \pi \gamma \pi^{-1} \pi x = \pi \gamma x = \pi x$, so that $\tau \in \text{sym}(\pi x)$. The other identity is proved similarly.
- (b) Since x is a support of y , we have $\text{fix}(x) \subseteq \text{sym}(y)$. So by (a), we get $\text{fix}(\pi x) = \pi \text{fix}(x) \pi^{-1} \subseteq \pi \text{sym}(y) \pi^{-1} = \text{sym}(\pi y)$. Hence, πx is a support of πy .
- (c) We know that x is well-orderable in \mathcal{V} if and only if there exists in \mathcal{V} a one-to-one mapping $f : x \rightarrow \alpha$ for some ordinal α . It is easily shown, by induction on the ordinals, that each element of the kernel is fixed by \mathcal{G} . Suppose there is such a function. Then $f(y)$ is in the kernel for each $y \in x$. So, for all $\pi \in \mathcal{G}$ and $y \in x$, we have $\pi f(y) = f(y)$, so that $\pi \langle y, f(y) \rangle = \langle \pi y, f(y) \rangle$. Since f is one-to-one, this implies that $\pi f = f$ only if $\pi y = y$ for all $y \in x$. That is, $\text{fix}(x) \supseteq \text{sym}(f) \in \mathcal{F}$, so that $\text{fix}(x) \in \mathcal{F}$. Conversely, suppose $\text{fix}(x) \in \mathcal{F}$, and use the axiom of choice in the full universe $\mathcal{P}^\infty(A)$ (the class of all atoms and sets, hereditarily symmetric or not) to find a one-to-one mapping $f : x \rightarrow \alpha$ for some ordinal α . We claim that $f \in \mathcal{V}$. For all $\pi \in \text{fix}(x)$ and $y \in x$, we have $\pi \langle y, f(y) \rangle = \langle y, \pi f(y) \rangle$. This implies that $\pi f(y) = f(y)$ for all y , so that $\pi \in \text{sym}(f)$. Hence, $\text{sym}(f) \supseteq \text{fix}(x) \in \mathcal{F}$, so that f is symmetric. Since x and α are hereditarily symmetric, f is also hereditarily symmetric.

□

3 The basic Fraenkel model

In this section, we present a few well-known examples involving the basic Fraenkel model and one of its variants. This is done to acquaint the reader with some common techniques in permutation models and to state a few results to which we can compare the new model introduced in Section 4.

Assume A is countably infinite, let \mathcal{G} be the group of all permutations of A , and let the supports be finite subsets of A . The corresponding permutation model is called the *basic Fraenkel model* and will be denoted by \mathcal{V}_F .

Example 3.1. *The Well-Ordering Principle is false in \mathcal{V}_F . [5, §4.3]*

Proof. We will show that the set A cannot be well-ordered. By Lemma 2.5(c), A is well-orderable in \mathcal{V}_F only if $\text{fix}(A) \in \mathcal{F}$. Further, $\text{fix}(A) \in \mathcal{F}$ only if there exists a finite subset E of A such that $\text{fix}(E) \subseteq \text{fix}(A)$. For any such E , however, we can find $a, b \in A \setminus E$ and $\pi \in \text{fix}(E)$ such that $\pi a = b$. That is, we can find $\pi \in \text{fix}(E)$ such that $\pi \notin \text{fix}(A)$. Thus, A is not well-orderable in \mathcal{V}_F . \square

Example 3.2. *The statement “Every family of pairs has a choice function” is false in \mathcal{V}_F . [5, Exercise 4.3]*

Proof. We will show that the family $S = \{\{a, b\} : a, b \in A\}$ has no choice function in \mathcal{V}_F . Suppose there is such a function c . Then c has a support E . Since E is finite, we can find $a, b \in A \setminus E$ and $\pi \in \text{fix}(E)$ such that $\pi a = b$ and $\pi b = a$. Assume that $\langle \{a, b\}, a \rangle \in c$ (otherwise $\langle \{a, b\}, b \rangle \in c$, and the argument is similar). Then since $\pi \in \text{fix}(E) \subseteq \text{sym}(c)$, we have $\langle \{a, b\}, b \rangle = \pi \langle \{a, b\}, a \rangle \in \pi c = c$. But the fact that $\langle \{a, b\}, a \rangle \in c$ and $\langle \{a, b\}, b \rangle \in c$ shows that c is not a function, a contradiction. \square

Corollary 3.3. *The Ordering Principle is false in \mathcal{V}_F .*

Proof. If A could be linearly ordered, then $c(\{a, b\}) = \min(\{a, b\})$ would be a choice function on S . \square

Definition 3.4. A set is *amorphous* if it is not the union of two infinite, disjoint sets.

If the axiom of choice holds, then all amorphous sets are finite (see Lévy’s paper [6]). In models of ZFA without the axiom of choice, such as the basic Fraenkel model, we can find infinite amorphous sets:

Example 3.5. *The set A is infinite, but amorphous in \mathcal{V}_F . [5, Exercise 4.7]*

Proof. To show that A is amorphous, assume for contradiction that there exist infinite, disjoint sets X and Y such that $A = X \cup Y$. Then the set X has a support E . Since E is finite, the sets $X \setminus E$ and $Y \setminus E$ are nonempty. So, we can find $a \in X \setminus E$, $b \in Y \setminus E$ and $\pi \in \text{fix}(E)$ such that $\pi a = b$. Since $\text{fix}(E) \subseteq \text{sym}(X)$, we have $b = \pi a \in \pi X = X$. This contradicts the fact that X and Y are disjoint. \square

Consider the following variant of the basic Fraenkel model: A is uncountable, \mathcal{G} is the group of all permutations of A , and the supports are countable subsets of A . We will call the corresponding permutation model \mathcal{V}_{F^+} .

Example 3.6. *The statement “Every well-ordered family of sets has a choice function” is true in \mathcal{V}_{F^+} . [4, $\aleph_{12}(\aleph_1)$]*

Proof. Let W be a well-ordered family of sets. By Lemma 2.5(c), $\text{fix}(W) \in \mathcal{F}$. Therefore, there exists a countable subset E_0 of A such that $\text{fix}(E_0) \subseteq \text{fix}(W)$. Without loss of generality, assume that $E_0 = 0$ (the set $A \setminus E_0$ looks like A). Now, let S be any countable subset of A . First, we will show that every set in W contains an element that is fixed by the group $\text{fix}(S)$. Let $X \in W$ and let $x \in X$. We know that x has a support E ; assume E is infinite. Since E and S are isomorphic to \mathbb{N} , we can find a permutation $\pi \in \mathcal{G}$ such that $\pi E = S$. By Lemma 2.5(b), S is a support of πx . Further, $\pi x \in X$, since X is fixed by $\text{fix}(E_0) = \mathcal{G}$. Now, let $W_S = \{X_S : X \in W\}$, where $X_S = \{x \in X : \text{fix}(S) \subseteq \text{sym}(x)\}$. We have shown that $0 \neq X_S \subseteq X$ for each $X \in W$. Therefore, a choice function on W_S will yield a choice function on W . The axiom of choice is true in the full universe, so let c be a choice function on W_S . We will show that, in fact, c is in the permutation model; i.e. that c is hereditarily symmetric. For each $\langle X_S, x \rangle \in c$ and $\pi \in \text{fix}(S)$, we have $\pi \langle X_S, x \rangle = \langle X_S, x \rangle \in c$. This shows that c is symmetric, supported by S . Since X_S and x are elements of \mathcal{V}_{F^+} and are therefore hereditarily symmetric, it follows that c is hereditarily symmetric. \square

4 The finite partition model

In this section, we introduce the finite partition model and present our main results.

Assume A is countably infinite, let \mathcal{G} be the group of all permutations of A , and let the supports be finite partitions of A (Lemma 4.1 will show that the supports generate a normal filter). We call the corresponding permutation model the *finite partition model*, denoted by \mathcal{V}_p .

Lemma 4.1. *Let S be the set of all finite partitions of A . Then the set*

$$\mathcal{F} = \{H : H \text{ is a subgroup of } \mathcal{G}, H \supseteq \text{fix}(P) \text{ for some } P \in S\}$$

is a normal filter on \mathcal{G} .

Proof. Provided \mathcal{F} is of the form (1) and the set S is nonempty, clauses i and ii of Definition 2.1 are trivial. Since that is the case here, we verify clauses iii-v:

- iii. Let $H, K \in \mathcal{F}$. Then there exist $P, Q \in S$ such that $\text{fix}(P) \subseteq H$ and $\text{fix}(Q) \subseteq K$. Let $P \wedge Q$ denote the coarsest common refinement of P and Q , given by

$$P \wedge Q = \{p \cap q : p \in P, q \in Q, p \cap q \neq 0\}.$$

Since $P \wedge Q$ is a common refinement of P and Q , it is clear that $\text{fix}(P \wedge Q) \subseteq \text{fix}(P)$ and $\text{fix}(P \wedge Q) \subseteq \text{fix}(Q)$. Therefore, $\text{fix}(P \wedge Q) \subseteq \text{fix}(P) \cap \text{fix}(Q) \subseteq H \cap K$. Since $P \wedge Q \in S$, this implies that $H \cap K \in \mathcal{F}$.

- iv. Let $\pi \in \mathcal{G}$ and $H \in \mathcal{F}$. Then there exists $P \in \mathcal{S}$ such that $\text{fix}(P) \subseteq H$. By Lemma 2.5(a), $\pi \text{fix}(P)\pi^{-1} = \text{fix}(\pi P)$. Since $\pi \text{fix}(P)\pi^{-1} \subseteq \pi H\pi^{-1}$, it is enough to show that $\pi P \in \mathcal{S}$. Clearly, πP is finite, since P is finite. To see that πP is a partition of A , let $a \in A$. Since $\pi^{-1}a \in p$ for some $p \in P$, we have $a \in \pi p \in \pi P$. Let $\pi p' \in \pi P$ be such that $a \in \pi p'$. Then $\pi^{-1}a \in p'$, so that $p' \cap p \neq \emptyset$. Consequently, $p' = p$ and $\pi p' = \pi p$.
- v. For each $a \in A$, $\{\pi \in \mathcal{G} : \pi a = a\} = \text{fix}(\{\{a\}, A \setminus \{a\}\}) \in \mathcal{F}$.

□

The rest of this paper will address the question, “What is happening in the finite partition model?” It will be beneficial to keep in mind the basic Fraenkel model. For example, the finite partition model includes the basic Fraenkel model as a submodel:

Proposition 4.2. $\mathcal{V}_F \subset \mathcal{V}_p$.

Proof. Let \mathcal{F}_F denote the normal filter of \mathcal{V}_F , generated by finite subsets of A . Let \mathcal{F}_p denote the normal filter of \mathcal{V}_p , generated by finite partitions of A . If x is symmetric with respect to \mathcal{F}_F , then there exists a finite subset E of A such that $\text{fix}(E) \subseteq \text{sym}(x)$. Let $P = \{\{a\}\}_{a \in E} \cup \{A \setminus E\}$. Then P is a finite partition of A , and $\text{fix}(P) = \text{fix}(E)$. Consequently, $\text{fix}(P) \subseteq \text{sym}(x)$, so that x is symmetric with respect to \mathcal{F}_p . This shows that any object that is hereditarily symmetric with respect to \mathcal{F}_F is hereditarily symmetric with respect to \mathcal{F}_p ; i.e. $y \in \mathcal{V}_F$ implies $y \in \mathcal{V}_p$ for all y . □

Due to Proposition 4.2 and the intensional similarity of \mathcal{V}_F and \mathcal{V}_p (same A and same \mathcal{G}), we might expect that most statements which are true in \mathcal{V}_F will also be true in \mathcal{V}_p . Though little is known yet about \mathcal{V}_p , our results are consistent with this intuition. Proposition 4.3 shows that Example 3.5 does not transfer to \mathcal{V}_p . Further, \mathcal{V}_F satisfies the statement “Every set is either well-orderable or has an infinite amorphous subset” ($\mathcal{N}1$ in Howard and Rubin’s text [4]), which is consistent with Examples 3.1 and 3.5, while Propositions 4.3 and 4.4 show that \mathcal{V}_p does not. However, Proposition 4.4 and all subsequent propositions and theorems are true in both models.

Proposition 4.3. *The set A has no infinite amorphous subset in \mathcal{V}_p .*

Proof. Let S be an infinite subset of A , and let X and Y be infinite, disjoint subsets of S such that $S = X \cup Y$. Then $\{X, Y, A \setminus S\}$ is a finite partition of A , and $\text{fix}(\{X, Y, A \setminus S\}) \subseteq \text{sym}(X)$ and $\text{fix}(\{X, Y, A \setminus S\}) \subseteq \text{sym}(Y)$. Hence, $X, Y \in \mathcal{V}_p$. □

Proposition 4.4. *The Well-Ordering Principle is false in \mathcal{V}_p .*

Proof. We will show that the set A cannot be well-ordered. Let P be any finite partition of A . Then there exists a block p in P such that $|p| \geq 2$. Let $a, b \in p$ and $\pi \in \text{fix}(P)$ be such that $\pi a = b$. Since $\pi \notin \text{fix}(A)$, we have that $\text{fix}(P) \not\subseteq \text{fix}(A)$. Hence, $\text{fix}(A) \notin \mathcal{F}$. But A can be well-ordered in \mathcal{V}_p only if $\text{fix}(A) \in \mathcal{F}$. □

Proposition 4.5. *The statement “Every family of pairs has a choice function” is false in \mathcal{V}_p .*

Proof. We will show that the set $S = \{\{a, b\} : a, b \in A\}$ has no choice function in \mathcal{V}_p . Suppose there is such a function c , and let P be a support of c . Let $p \in P$ be such that $|p| \geq 2$. Then we can find $a, b \in p$ and $\pi \in \text{fix}(P)$ such that $\pi a = b$ and $\pi b = a$. Assume that $\langle \{a, b\}, a \rangle \in c$ (the other case is similar). Then we also have $\langle \{a, b\}, b \rangle = \pi \langle \{a, b\}, a \rangle \in \pi c = c$, which shows that c is not a function. \square

Corollary 4.6. *The Ordering Principle is false in \mathcal{V}_p .*

Definition 4.7. A set is *Dedekind finite* if it has no countably infinite subset.

Like amorphous sets, all Dedekind finite sets are finite, provided the axiom of choice holds (due to Lévy [6]). In the finite partition model, we find the following:

Proposition 4.8. *The set A is infinite, but Dedekind finite in \mathcal{V}_p .*

Proof. Assume for contradiction that A is Dedekind infinite. Then there exists a one-to-one function $f : \mathbb{N} \rightarrow A$. Let P be a support of f , and let $\pi \in \text{fix}(P)$ be such that π moves every atom in each non-singleton block of P . Since P contains only finitely many singletons, π fixes only finitely many atoms. Now, let $n \in \mathbb{N}$. Since n is in the kernel, we have $\pi n = n$. This implies that $\pi(f(n)) = f(n)$, for otherwise $\pi n = n$ while $f(\pi n) = \pi(f(n)) \neq f(n)$. But f is one-to-one, and thus, π fixes ω many values of f in A , a contradiction. \square

Proposition 4.9. *The set $\mathcal{P}(A)$ is infinite, but Dedekind finite in \mathcal{V}_p .*

To prove Proposition 4.9, we will need the following lemma:

Lemma 4.10. *Let P be a finite partition of A . Then $E \subset A$ is supported by P if and only if E is the union of a subset of P . Therefore, only finitely many subsets of A are supported by P .*

Proof. Let E be the union of a subset of P , and assume $E \neq \emptyset$. Then for each $a \in E$, there exists a block $p \in P$ such that $a \in p \subset E$. Let $\pi \in \text{fix}(P)$. Then we have $\pi a \in \pi p = p \subset E$. Thus, P supports E . Conversely, suppose E is not the union of a subset of P . Then there exists a block $p \in P$ such that $p \cap E$ and $p \setminus E$ are nonempty. For $a \in p \cap E$ and $b \in p \setminus E$, let $\pi \in \text{fix}(P)$ be such that $\pi a = b$. Then $\pi a \notin E$, which shows that P does not support E . \square

Proof of Proposition 4.9. Suppose for contradiction that $\mathcal{P}(A)$ is Dedekind infinite. Then there exists a one-to-one function $f : \mathbb{N} \rightarrow \mathcal{P}(A)$. Let P be a support of f . Since each $n \in \mathbb{N}$ is in the kernel and f is one-to-one, each $\pi \in \text{fix}(P)$ fixes the ω many values of f in $\mathcal{P}(A)$. This shows that P supports infinitely many subsets of A , contradicting Lemma 4.10. \square

Proposition 4.11. *The statement “Every well-ordered family of pairs has a choice function” is true in \mathcal{V}_p .*

To prove Proposition 4.11, we will need a definition and a few lemmas:

Definition 4.12. Let P and Q be partitions of a set S . Then P and Q are said to be *independent* if $p \cap q$ is infinite for all $p \in P$ and $q \in Q$.

The following lemma can be proven straightforwardly using the un-numbered lemma on the first page of the paper by Dixon, Neumann and Thomas [1]. (Three applications of the lemma suffice: first set $\Gamma_1 = X_1 \cup Y_1$ and $\Gamma_2 = X_2 \cup Y_2$, then $\Gamma_1 = X_1$ and $\Gamma_2 = Y_1$, and finally $\Gamma_1 = X_2$ and $\Gamma_2 = Y_2$.) However, for completeness and for the convenience of the reader, we include an independent combinatorial proof.

Lemma 4.13. Let $P = \{X_1, X_2\}$ and $Q = \{Y_1, Y_2\}$ be independent partitions of a subset S of A . Let H be the group of all permutations of S , and let $\pi \in H$. Then there exists a finite sequence of permutations $\pi_1, \dots, \pi_n \in \text{fix}_H(P) \cup \text{fix}_H(Q)$ such that $\pi = \pi_n \cdots \pi_1$.

Proof. For convenience, let $B_1 = X_1 \cap Y_1$, $B_2 = X_1 \cap Y_2$, $B_3 = X_2 \cap Y_2$ and $B_4 = X_2 \cap Y_1$. First, divide each block B_i into four sets S_{ij} defined by $S_{ij} = \{a \in B_i : \pi a \in B_j\}$. Further, for each S_{ij} , let R_{ij} (“red atoms”) and G_{ij} (“green atoms”) be such that $S_{ij} = R_{ij} \dot{\cup} G_{ij}$, and such that R_{ij} and G_{ij} are infinite if S_{ij} is infinite. The proof will be done in three steps: First, we will find permutations $\pi_1, \dots, \pi_m \in \text{fix}_H(P) \cup \text{fix}_H(Q)$ that move each red atom to its correct block, while keeping each green atom in its original block. That is, π_1, \dots, π_m will be such that, for $a \in S_{ij}$, if $a \in R_{ij}$, then $\pi_m \cdots \pi_1 a \in B_j$; otherwise, $\pi_m \cdots \pi_1 a \in B_i$. Second, we will find permutations $\pi_{m+1}, \dots, \pi_{n-1} \in \text{fix}_H(P) \cup \text{fix}_H(Q)$ that move each green atom to its correct block, while keeping the image of each red atom under $\pi_m \cdots \pi_1$ in the same block. Finally, we will find a permutation $\pi_n \in \text{fix}_H(P) \cup \text{fix}_H(Q)$ that moves the image of each atom under $\pi_{n-1} \cdots \pi_1$, now already in the correct block, to its image under π .

To begin, one observation will be useful: For each i , the set S_{ij} is infinite for at least one j , and similarly for each j , the set S_{ij} is infinite for at least one i . Therefore, the sets $\bigcup_j G_{ij}$ and $\bigcup_j R_{ji}$ are infinite for each i .

Now, let $E_i \subset B_i$ be such that

- $|E_1| = |R_{21}|$ and $B_1 \setminus E_1$ is infinite;
- $|E_2| = |R_{12} \cup R_{13}|$ and $B_2 \setminus E_2$ is infinite;
- $|E_3| = |R_{43}|$ and $B_3 \setminus E_3$ is infinite;
- $|E_4| = |R_{31} \cup R_{34}|$ and $B_4 \setminus E_4$ is infinite.

Since $\bigcup_j G_{1j} \subset B_1 \setminus (R_{12} \cup R_{13})$, $\bigcup_j G_{2j} \subset B_2 \setminus R_{21}$, $\bigcup_j G_{3j} \subset B_3 \setminus (R_{31} \cup R_{34})$ and $\bigcup_j G_{4j} \subset B_4 \setminus R_{43}$, we know that $B_1 \setminus (R_{12} \cup R_{13})$, $B_2 \setminus R_{21}$, $B_3 \setminus (R_{31} \cup R_{34})$ and $B_4 \setminus R_{43}$ are infinite.

Therefore, we can find a permutation $\pi_1 \in \text{fix}_H(P)$ such that

$$\begin{cases} \pi_1(R_{12} \cup R_{13}) = E_2 \\ \pi_1(B_1 \setminus (R_{12} \cup R_{13})) = B_1 \setminus E_1 \\ \pi_1 R_{21} = E_1 \\ \pi_1(B_2 \setminus R_{21}) = B_2 \setminus E_2 \\ \pi_1(R_{31} \cup R_{34}) = E_4 \\ \pi_1(B_3 \setminus (R_{31} \cup R_{34})) = B_3 \setminus E_3 \\ \pi_1 R_{43} = E_3 \\ \pi_1(B_4 \setminus R_{43}) = B_4 \setminus E_4 \end{cases}$$

Similarly, let $E'_i \subset B_i$ be such that

- $|E'_1| = |\pi_1(R_{31} \cup R_{41} \cup R_{42})|$ and $B_1 \setminus E'_1$ is infinite;
- $|E'_2| = |\pi_1 R_{32}|$ and $B_2 \setminus E'_2$ is infinite;
- $|E'_3| = |\pi_1(R_{13} \cup R_{23} \cup R_{24})|$ and $B_3 \setminus E'_3$ is infinite;
- $|E'_4| = |\pi_1 R_{14}|$ and $B_4 \setminus E'_4$ is infinite.

Since $B_1 \setminus \pi_1 R_{14}$, $B_2 \setminus \pi_1(R_{13} \cup R_{23} \cup R_{24})$, $B_3 \setminus \pi_1 R_{32}$ and $B_4 \setminus \pi_1(R_{31} \cup R_{41} \cup R_{42})$ are infinite, we can find a permutation $\pi_2 \in \text{fix}_H(Q)$ such that

$$\begin{cases} \pi_2(\pi_1 R_{14}) = E'_4 \\ \pi_2(B_1 \setminus \pi_1 R_{14}) = B_1 \setminus E'_1 \\ \pi_2(\pi_1(R_{13} \cup R_{23} \cup R_{24})) = E'_3 \\ \pi_2(B_2 \setminus \pi_1(R_{13} \cup R_{23} \cup R_{24})) = B_2 \setminus E'_2 \\ \pi_2(\pi_1 R_{32}) = E'_2 \\ \pi_2(B_3 \setminus \pi_1 R_{32}) = B_3 \setminus E'_3 \\ \pi_2(\pi_1(R_{31} \cup R_{41} \cup R_{42})) = E'_1 \\ \pi_2(B_4 \setminus \pi_1(R_{31} \cup R_{41} \cup R_{42})) = B_4 \setminus E'_4 \end{cases}$$

By the same method, we can find a permutation $\pi_3 \in \text{fix}_H(P)$ such that π_3 moves $\pi_2 \pi_1 R_{42}$ from B_1 to B_2 and $\pi_2 \pi_1 R_{24}$ from B_3 to B_4 , while keeping every other $\pi_2 \pi_1 R_{ij}$ in the same block. Now we have $\pi_3 \pi_2 \pi_1 R_{ij} \subset B_j$ for each i , as desired in the first step.

Using the fact that the set $\pi_3 \pi_2 \pi_1 \cup_j R_{ji}$ is infinite for each i , the second stage follows by a method similar to that of the first stage. This will give us permutations $\pi_4, \pi_5, \pi_6 \in \text{fix}_H(P) \cup \text{fix}_H(Q)$ such that $\pi_6 \cdots \pi_1 S_{ij} = \pi_6 \cdots \pi_1 (R_{ij} \cup G_{ij}) \subset B_j$ for each i . In other words, if $\pi a \in B_j$, then $\pi_6 \cdots \pi_1 a \in B_j$ for all $a \in S$. This makes the final step clear: We can find a permutation $\pi_7 \in \text{fix}_H(P) \cup \text{fix}_H(Q)$ such that $\pi_7 \cdots \pi_1 a = \pi a$ for all a . Hence, $\pi = \pi_7 \cdots \pi_1$. \square

The following lemma is a version of Mostowski's intersection lemma (88 in Mostowski's paper [7]). Part (a) is a known result, implied, for example, by IIA4 in Pincus' paper [8] and by the aforementioned lemma of Dixon et al. We offer an independent proof of this special case. With only minor adjustments, part (b) is then obtained from (a).

Lemma 4.14.

- (a) *Let E_1 and E_2 be finite subsets of an infinite subset S of A . Let H be the group of all permutations of S , and let $\pi \in \text{fix}_H(E_1 \cap E_2)$. Then there exists a finite sequence of permutations $\pi_1, \dots, \pi_n \in \text{fix}_H(E_1) \cup \text{fix}_H(E_2)$ such that $\pi = \pi_n \cdots \pi_1$.*
- (b) *Let P and Q be finite partitions of a subset S of A , each consisting of singletons and one infinite block. Let $P_1 = \{p \in P : |p| = 1\}$ and $Q_1 = \{q \in Q : |q| = 1\}$. Let H be the group of all permutations of S , and let $\pi \in \text{fix}_H(P_1 \cap Q_1)$. Then there exists a finite sequence of permutations $\pi_1, \dots, \pi_n \in \text{fix}_H(P_1) \cup \text{fix}_H(Q_1)$ such that $\pi = \pi_n \cdots \pi_1$.*

Proof.

- (a) Let $B_1 = E_1 \setminus E_2$, $B_2 = E_2 \setminus E_1$ and $B_3 = S \setminus (E_1 \cup E_2)$. Since $\pi \in \text{fix}_H(E_1 \cap E_2)$ and $\bigcup_i B_i = S \setminus (E_1 \cap E_2)$, we know that $\pi \bigcup_i B_i = \bigcup_i B_i$. So, divide each B_i into three sets S_{ij} defined by $S_{ij} = \{a \in B_i : \pi a \in B_j\}$. Two observations will be useful: First, B_3 is infinite, while B_1 and B_2 are finite. Therefore, S_{33} is infinite. Second, since B_1 is finite, the number of elements in B_1 moved outside B_1 by π is equal to the number of elements outside B_1 moved into B_1 by π ; i.e. $|S_{12} \cup S_{13}| = |S_{21} \cup S_{31}|$. For the same reason, $|S_{21} \cup S_{23}| = |S_{12} \cup S_{32}|$.

By the first observation, we can find a subset T of S_{33} such that $|T| = |S_{12}|$. By the second observation, we have $|B_2| \geq |S_{21} \cup S_{23}| = |S_{12} \cup S_{32}| = |S_{32} \cup T|$. So, we can find a subset X of B_2 such that $|X| = |S_{32} \cup T|$. Let $Y \subset B_3$ be such that $|Y| = |S_{21} \cup S_{23}|$. It follows that $|X| = |Y|$ and that $B_3 \setminus (S_{32} \cup T)$ is infinite. Therefore, we can find a permutation $\pi_1 \in \text{fix}_H(E_1)$ such that

$$\begin{cases} \pi_1(S_{21} \cup S_{23}) = Y \\ \pi_1(B_2 \setminus (S_{21} \cup S_{23})) = B_2 \setminus X \\ \pi_1(S_{32} \cup T) = X \\ \pi_1(B_3 \setminus (S_{32} \cup T)) = B_3 \setminus Y \end{cases}$$

Similarly, we can find $X' \subseteq B_1$ and $Y' \subset B_3$ such that $|X'| = |\pi_1(S_{21} \cup S_{31})|$ and $|Y'| = |\pi_1(S_{12} \cup S_{13})|$. Observing that $|X'| = |Y'|$ and that $B_3 \setminus \pi_1(S_{21} \cup S_{31})$ is infinite, we can find a permutation $\pi_2 \in \text{fix}_H(E_2)$ such that

$$\begin{cases} \pi_2(\pi_1(S_{12} \cup S_{13})) = Y' \\ \pi_2(B_1 \setminus \pi_1(S_{12} \cup S_{13})) = B_1 \setminus X' \\ \pi_2(\pi_1(S_{21} \cup S_{31})) = X' \\ \pi_2(B_3 \setminus \pi_1(S_{21} \cup S_{31})) = B_3 \setminus Y' \end{cases}$$

Finally, recalling that $|T| = |S_{12}|$, we can find a permutation $\pi_3 \in \text{fix}_H(E_1)$ such that $\pi_3(\pi_2\pi_1T) = \pi_2\pi_1S_{12} \subset B_3$ and $\pi_3(\pi_2\pi_1S_{12}) = \pi_2\pi_1T \subseteq B_2$.

Now we have $\pi_3\pi_2\pi_1S_{ij} \subseteq B_j$ for all i . In other words, if $\pi a \in B_j$, then $\pi_3\pi_2\pi_1a \in B_j$ for all $a \in S$. Therefore, we can find a permutation $\pi_4 \in \text{fix}_H(E_1) \cup \text{fix}_H(E_2)$ such that $\pi_4 \cdots \pi_1 a = \pi a$ for all $a \in S$. Hence, $\pi = \pi_4 \cdots \pi_1$.

- (b) Let $E_1 = \cup P_1$ and $E_2 = \cup Q_1$. Observe that E_1 and E_2 are finite subsets of S and that $\text{fix}_H(P_1) = \text{fix}_H(E_1)$, $\text{fix}_H(Q_1) = \text{fix}_H(E_2)$ and $\text{fix}_H(P_1 \cap Q_1) = \text{fix}_H(E_1 \cap E_2)$. The result follows from (a). □

Lemma 4.15. *Let P be a support of a set $\{x, y\}$, and let Q be a refinement of P such that Q supports x . Let p be an infinite block of P , and let $Q_{p,\infty} = \{q \in Q : q \subseteq p, q \text{ is infinite}\}$. Then $(Q \setminus Q_{p,\infty}) \cup \{\cup Q_{p,\infty}\}$ is a support of x .*

Proof. We proceed by induction on $|Q_{p,\infty}|$. The case that $|Q_{p,\infty}| = 1$ is clear. So, for some $n \in \mathbb{N}$, assume that if Q is a refinement of P such that Q supports x and $|Q_{p,\infty}| = n$, then $(Q \setminus Q_{p,\infty}) \cup \{\cup Q_{p,\infty}\}$ is a support of x . Suppose Q is such that $|Q_{p,\infty}| = n + 1$, and let $q_1, q_2 \in Q_{p,\infty}$. Let $Q' = (Q \setminus \{q_1, q_2\}) \cup \{q_1 \cup q_2\}$. We want to show that Q' is a support of x : Once this is done, we observe that $Q'_{p,\infty} = (Q_{p,\infty} \setminus \{q_1, q_2\}) \cup \{q_1 \cup q_2\}$ is of size n . The induction hypothesis will then allow us to conclude that $(Q' \setminus Q'_{p,\infty}) \cup \{\cup Q'_{p,\infty}\} = (Q \setminus Q_{p,\infty}) \cup \{\cup Q_{p,\infty}\}$ is a support of x . So, to show that Q' is a support of x , let $\pi \in \text{fix}(Q')$. First, write $\pi = \pi_2\pi_1$, where

$$\begin{cases} \pi_1 \upharpoonright_{q_1 \cup q_2} = \pi \upharpoonright_{q_1 \cup q_2} \\ \pi_1 \upharpoonright_{A \setminus (q_1 \cup q_2)} = \text{Id} \upharpoonright_{A \setminus (q_1 \cup q_2)} \end{cases} \quad \text{and} \quad \begin{cases} \pi_2 \upharpoonright_{q_1 \cup q_2} = \text{Id} \upharpoonright_{q_1 \cup q_2} \\ \pi_2 \upharpoonright_{A \setminus (q_1 \cup q_2)} = \pi \upharpoonright_{A \setminus (q_1 \cup q_2)} \end{cases}.$$

We know that $\pi_2 x = x$, since $\pi_2 \in \text{fix}(Q)$. Therefore, if we can show that $\pi_1 x = x$, then we will have $\pi x = x$, as desired.

To show that $\pi_1 x = x$, choose a permutation $\sigma \in \text{fix}(P)$ such that $\sigma \upharpoonright_{A \setminus (q_1 \cup q_2)} = \text{Id} \upharpoonright_{A \setminus (q_1 \cup q_2)}$, and such that $\{q_1, q_2\}$ and $\sigma\{q_1, q_2\}$ are independent partitions of $q_1 \cup q_2$. For example, let B_1, B_2, B_3 and B_4 be infinite, pairwise disjoint sets such that $q_1 = B_1 \cup B_2$ and $q_2 = B_3 \cup B_4$, and let σ be such that $\sigma B_1 = B_1$, $\sigma B_2 = B_4$, $\sigma B_3 = B_3$ and $\sigma B_4 = B_2$. Let H be the group of all permutations of $q_1 \cup q_2$. Then $\pi_1 \upharpoonright_{q_1 \cup q_2} \in H$. So, by Lemma 4.13, there exists a finite sequence of permutations $\tau_1, \dots, \tau_k \in \text{fix}_H(\{q_1, q_2\}) \cup \text{fix}_H(\sigma\{q_1, q_2\})$ such that $\pi_1 \upharpoonright_{q_1 \cup q_2} = \tau_k \cdots \tau_1$. For each τ_i , define γ_i by $\gamma_i \upharpoonright_{q_1 \cup q_2} = \tau_i$ and $\gamma_i \upharpoonright_{A \setminus (q_1 \cup q_2)} = \text{Id} \upharpoonright_{A \setminus (q_1 \cup q_2)}$. Then we have $\pi_1 = \gamma_k \cdots \gamma_1$. Now, since Q is a support of x , we know that σQ is a support of σx . Further, $\sigma \in \text{fix}(P)$, which implies that $\sigma x = x$ or $\sigma x = y$. It follows that σQ is a support of x : If $\sigma' \in \text{fix}(\sigma Q)$ and $\sigma' y = y$, then $\sigma' x = x$, since $\text{fix}(\sigma Q) \subseteq \text{fix}(P)$. Thus, $\gamma_i x = x$ for all i . It follows that $\pi_1 x = \gamma_k \cdots \gamma_1 x = x$, which completes the proof. □

Lemma 4.16. *Let P be a support of a set $\{x, y\}$, and let p be an infinite block of P . Let Q be a refinement of P such that Q supports x , and such that the set $Q_p = \{q \in Q : q \subseteq p\}$ consists of singletons and one infinite block. Then $(Q \setminus Q_p) \cup \{p\}$ is a support of x .*

Proof. Let $Q' = (Q \setminus Q_p) \cup \{p\}$, and let $\pi \in \text{fix}(Q')$. First, write $\pi = \pi_2\pi_1$, where

$$\begin{cases} \pi_1 \upharpoonright_p = \pi \upharpoonright_p \\ \pi_1 \upharpoonright_{A \setminus p} = \text{Id} \upharpoonright_{A \setminus p} \end{cases} \quad \text{and} \quad \begin{cases} \pi_2 \upharpoonright_p = \text{Id} \upharpoonright_p \\ \pi_2 \upharpoonright_{A \setminus p} = \pi \upharpoonright_{A \setminus p} \end{cases}.$$

Now, let $Q_{p,1} = \{q \in Q_p : |q| = 1\}$, and choose a permutation $\sigma \in \text{fix}(P)$ such that $\sigma \upharpoonright_{A \setminus p} = \text{Id} \upharpoonright_{A \setminus p}$ and $Q_{p,1} \cap \sigma Q_{p,1} = \emptyset$. Let H be the group of all permutations of p . Then $\pi_1 \upharpoonright_p \in H = \text{fix}_H(0) = \text{fix}_H(Q_{p,1} \cap \sigma Q_{p,1})$. So, by Lemma 4.14(b), there exists a finite sequence of permutations $\tau_1, \dots, \tau_n \in \text{fix}_H(Q_{p,1}) \cup \text{fix}_H(\sigma Q_{p,1})$ such that $\pi_1 \upharpoonright_p = \tau_n \cdots \tau_1$. For each τ_i , define γ_i by $\gamma_i \upharpoonright_p = \tau_i$ and $\gamma_i \upharpoonright_{A \setminus p} = \text{Id} \upharpoonright_{A \setminus p}$. Then we have $\pi = \pi_2 \gamma_n \cdots \gamma_1$ and $\pi_2, \gamma_n, \dots, \gamma_1 \in \text{fix}(Q) \cup \text{fix}(\sigma Q)$. As in the proof of Lemma 4.15, that Q is a support of x and $\sigma \in \text{fix}(P)$ implies that σQ also supports x . Therefore, $\pi x = \pi_2 \gamma_n \cdots \gamma_1 x = x$, which shows that Q' supports x . \square

Now we are ready to prove Proposition 4.11.

Proof of Proposition 4.11. Let W be a well-ordered family of pairs. Then $\text{fix}(W) \in \mathcal{F}$, so that W has a support P such that $\text{fix}(P) \subseteq \text{fix}(W)$. Suppose $\langle x, y \rangle \in W$, and let Q be a support of x . Assume, without loss of generality, that all finite blocks of P and Q are singletons and that Q is a refinement of P . We claim that P supports x . Let p_1, \dots, p_n be the infinite blocks of P . First, let $Q_{p_i} = \{q \in Q : q \subseteq p_i\}$, and let $Q_{p_i, \infty} = \{q \in Q_{p_i} : q \text{ is infinite}\}$. By Lemma 4.15, $Q' := (Q \setminus Q_{p_1, \infty}) \cup \{\bigcup Q_{p_1, \infty}\}$ is a support of x . Observe that the set $Q'_{p_1} := \{q \in Q' : q \subseteq p_1\}$ is equal to $(Q_{p_1} \setminus Q_{p_1, \infty}) \cup \{\bigcup Q_{p_1, \infty}\}$; that is, Q'_{p_1} consists of the set $Q_{p_1} \setminus Q_{p_1, \infty}$ of singletons and one infinite block $\bigcup Q_{p_1, \infty}$. Therefore, Lemma 4.16 applies, and we get that $Q_1 := (Q' \setminus Q'_{p_1}) \cup \{p_1\} = (Q \setminus Q_{p_1}) \cup \{p_1\}$ is a support of x . Applying this argument to Q_1 and p_2 , we obtain a new support $Q_2 := (Q \setminus (Q_{p_1} \cup Q_{p_2})) \cup \{p_1, p_2\}$ of x . Continuing this process through p_n , we get that $Q_n := (Q \setminus \bigcup_{i \leq n} Q_{p_i}) \cup \{p_i\}_{i \leq n}$ is a support of x . But $Q_n = P$: The infinite blocks of Q_n are precisely the infinite blocks of P . Each singleton in Q_n is not in $\bigcup_{i \leq n} Q_{p_i}$, and is thus a subset of a singleton in P . Therefore, P is a support of x .

Now, let c be a choice function on W in the full universe, and let $\pi \in \text{fix}(P)$. Then for each $\langle X, x \rangle \in c$ and $\pi \in \text{fix}(P)$, we have $\pi \langle X, x \rangle = \langle X, x \rangle \in c$. Thus, c is symmetric. It follows that c is hereditarily symmetric since X and x are in \mathcal{V}_p . Therefore, $c \in \mathcal{V}_p$. \square

Without much trouble, we can extend Proposition 4.11 to the following:

Theorem 4.17. *The statement “Every well-ordered family of well-orderable sets has a choice function” is true in \mathcal{V}_p .*

To prove Theorem 4.17, we only need to make small extensions of Lemmas 4.15 and 4.16:

- Lemma 4.15 (extended): *Let P be a support of a well-orderable set X , and let Q be a refinement of P such that Q supports each element of X . Let p be an infinite block of P , and let $Q_{p, \infty} = \{q \in Q : q \subseteq p, q \text{ is infinite}\}$. Then $(Q \setminus Q_{p, \infty}) \cup \{\bigcup Q_{p, \infty}\}$ is a support of every element of X .*

Proof. The proof is similar to that of Lemma 4.15: The induction hypothesis is “For some $n \in \mathbb{N}$, assume that if Q is a refinement of P such that Q supports each element of X and $|Q_{p,\infty}| = n$, then $(Q \setminus Q_{p,\infty}) \cup \{\cup Q_{p,\infty}\}$ is a support of every element of X .” For the case that $|Q_{p,\infty}| = n + 1$, let $q_1, q_2 \in Q_{p,\infty}$. Let $Q' = (Q \setminus \{q_1, q_2\}) \cup \{q_1 \cup q_2\}$, and let $\pi \in \text{fix}(Q')$. Choose $\sigma \in \text{fix}(P)$ such that $\{q_1, q_2\}$ and $\sigma\{q_1, q_2\}$ are independent partitions of $q_1 \cup q_2$, and use Lemma 4.13 to write π as $\pi = \pi_2 \gamma_k \cdots \gamma_1$, where $\pi_2, \gamma_k, \dots, \gamma_1 \in \text{fix}(Q) \cup \text{fix}(\sigma Q)$. Now, we want to show that σQ supports each element of X . This is easy: Since Q supports each element of X , we know by Lemma 2.5(b) that σQ supports each element of $\sigma X = X$. Consequently, $\pi x = \pi_2 \gamma_k \cdots \gamma_1 x = x$ for all $x \in X$, which shows that Q' is also a support of every element of X . As in the proof of Lemma 4.15, the rest follows by induction hypothesis. \square

- Lemma 4.16 (extended): *Let P be a support of a well-orderable set X , and let p be an infinite block of P . Let Q be a refinement of P such that Q supports each element of X , and such that $Q_p = \{q \in Q : q \subseteq p\}$ consists of singletons and one infinite block. Then $(Q \setminus Q_p) \cup \{p\}$ is a support of every element of X .*

Proof. Make similar adjustments to the proof of Lemma 4.16. \square

Proof of Theorem 4.17. Let W be a well-ordered family of well-orderable sets, and let $X \in W$. Since W is well-ordered, W has a support P such that $\text{fix}(P) \subseteq \text{fix}(W)$. Similarly, X has a support Q such that $\text{fix}(Q) \subseteq \text{fix}(X)$. Assume, without loss of generality, that all finite blocks of P and Q are singletons and that Q is a refinement of P . Using the extensions of Lemmas 4.15 and 4.16, apply the argument in the proof of Proposition 4.11 to show that P supports each element of X . Any choice function on W will be hereditarily symmetric, supported by P . \square

From the proof of Theorem 4.17, we also obtain the following:

Theorem 4.18. *The statement “The union of any well-ordered family of well-orderable sets is well-orderable” is true in \mathcal{V}_p .*

Proof. Let W be a well-ordered family of well-orderable sets. Since W is well-ordered, W has a support P such that $\text{fix}(P) \subseteq \text{fix}(W)$. By the proof of Theorem 4.17, we know further that $\text{fix}(P) \subseteq \text{fix}(X)$ for all $X \in W$. That is, $\text{fix}(P) \subseteq \text{fix}(\cup W)$. Hence, $\cup W$ is well-orderable. \square

This completes the presentation of our main results.

5 Future research

There are at least two natural, related directions for future research involving the finite partition model.

The first is to continue the project of this paper; that is, classifying \mathcal{V}_p . In Proposition 4.3, we saw that in \mathcal{V}_p , unlike in \mathcal{V}_F , the set of atoms has no amorphous subset. So, one

might extend the work of this paper by, for example, considering whether \mathcal{V}_p has any infinite amorphous sets at all; this is unknown. A useful reference for classifying \mathcal{V}_p is Howard and Rubin's text [4], a comprehensive list of statements that have been proved using the axiom of choice. One might continue the aim of this paper by determining which of those statements hold in the finite partition model.

A second direction for future research could be to explore variants of the finite partition model. For example, assume A is uncountable, let \mathcal{G} be the group of all permutations of A , and let the supports be countable partitions of A . The corresponding permutation model would be to \mathcal{V}_p as \mathcal{V}_{F^+} is to \mathcal{V}_F .

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