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Ping Ngai Chung [a]
Miguel A. Fernandez [b]
Niralee Shah [c]
Luis Sordo Vieira [d]

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Rose-Hulman Institute of Technology
Department of Mathematics
Terre Haute, IN 47803
Email: mathjournal@rose-hulman.edu
http://www.rose-hulman.edu/mathjournal

[a]University of Cambridge
[b]Truman State University
[c]Williams College
[d]University of Kentucky

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Abstract. We prove that an isoperimetric region in $\mathbb{R}^2$ with density e^r must be convex and contain the origin, and provide numerical evidence that circles about the origin are isoperimetric, as predicted by the Log-Convex Density Conjecture.

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1 Introduction

There has been a recent surge of interest in Riemannian manifolds with a positive "density" function that weights volume and area (see recent works by Morgan [M1-4]). In fact, Perelman [P, Section 1.1] used the concept of manifolds with density in his beautiful proof of the Poincaré conjecture (see [M4] for an intuitive explanation). Although one might choose to weight area and perimeter differently, the case where both the area and the perimeter are weighted by the same function is of most interest. A fundamental problem in manifolds with density is the isoperimetric problem, which seeks to enclose given area or volume with the least amount of perimeter. For example, it is well known that in the Euclidean plane, a circle encloses given area with least perimeter (for a historical treatment, see [B]). Due to its many applications in probability theory and in Perelman's proof of the Poincaré conjecture, one of the most important examples is $\mathbb{R}^n$ with Gaussian density e^{-r^2}. In 1975, C. Borell [B1] and V.N. Sudakov and B.S. Tsirel'son [ST] proved that half-spaces are perimeter minimizing for the Gaussian density. On the other hand, in $\mathbb{R}^n$ with density e^{+r^2}, C. Borell ([B2], see [Ro, Introduction and Theorem 5.2]) proved that balls about the origin are perimeter minimizing. Rosales et al. gave a conjecture on when balls about the origin are isoperimetric:

Log-Convex Density Conjecture 1.1. *[Ro, Conjecture 3.12] Consider $\mathbb{R}^n$ with a smooth radial density. If the log of the density is convex, then balls about the origin provide isoperimetric regions of any given volume.*

The borderline case of density e^r, whose log is linear and hence just barely convex, is a pivotal case and the subject of the following study. Although we are not able to prove that the isoperimetric solution is a circle about the origin, we obtain the following partial result:

Theorem 3.19. *In the plane with density e^r, an isoperimetric region is convex and contains the origin in its interior.*

The proof uses symmetrization, the Four-Vertex Theorem, and the equation for geodesic curvature.

We conclude Section 3 with some numerical analysis. Relative to a given density e^ψ there is a generalization of curvature κ_ψ. Just as classically an isoperimetric curve (circle) has constant curvature, in the presence of a density an isoperimetric curve has constant generalized curvature. Corollary 3.23 provides a simple differential equation for curves $r(\theta)$ of constant generalized curvature κ_ψ:

$$\frac{dr}{d\theta} = \mp r\sqrt{\left(\frac{r}{\kappa_\psi(r-1-e^{-r}C)^2}\right)-1}$$

for some constant $C \geq (1-\kappa_\psi)e^{1/(\kappa_\psi-1)}/\kappa_\psi$. Given κ_ψ, C, and initial position $r(0)$, there are two solutions to the differential equation: a nondegenerate solution that looks like one of the curves in Figures 1 and 2, and a degenerate solution that corresponds to a circle about the

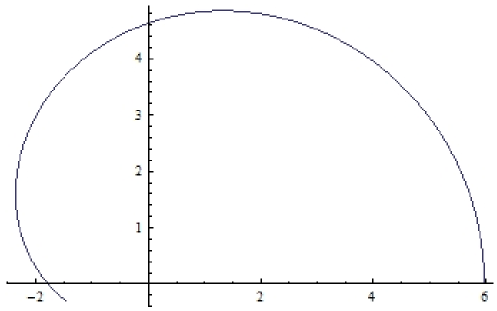

Figure 1: Curve with $r'(\theta) < 0$ ($\kappa_\psi = 1.2$, $C = -2$)

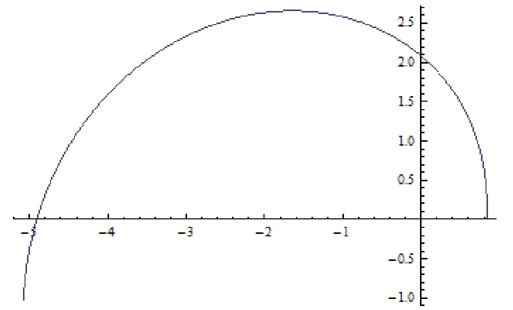

Figure 2: Curve with $r'(\theta) > 0$ ($\kappa_\psi = 1.2$, $C = -2$)

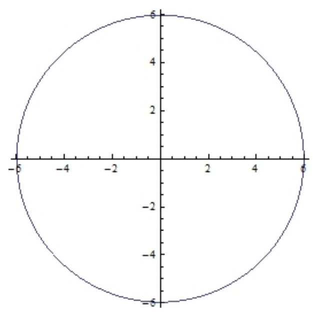

Figure 3: Curve with $r'(\theta) = 0$ ($\kappa_\psi = 1.2$, $C = -2$)

origin that *does not* have constant generalized curvature κ_ψ (Figure 3). When C reaches its minimum, the nondegenerate solution coincides with the degenerate solution and gives the unique constant generalized curvature curve, which is precisely a circle about the origin. To show that discs about the origin are isoperimetric, it suffices to show that a nondegenerate solution is not symmetric with respect to the line through the origin and a maximum point of r when C is strictly greater than its minimum, and therefore cannot be isoperimetric (Proposition 2.5). Thus the only remaining candidate for an isoperimetric curve is a circle about the origin, as desired.

The structure of the paper is as follows:

Section 2 provides basic definitions and known results. Section 3 begins with a regularity result at the origin where the density e^r fails to be smooth. Corollary 3.15 provides an equation for the classical geodesic curvature κ_0 of an isoperimetric curve in terms of r and the constant generalized geodesic curvature κ_ψ (geodesic curvature relative to the density function e^ψ). Using symmetrization, the Four-Vertex Theorem, and the equation for geodesic curvature we prove our main result, Theorem 3.19. The rest of the section provides numerical evidence for the conjecture that circles about the origin are isoperimetric.

2 Background

In this section we introduce some basic definitions and known theorems that will prove crucial to the rest of the paper.

Definition 2.1. A *density* is a positive continuous function that weights both volume and perimeter. For a planar region R with piecewise smooth boundary ∂R, the weighted area is given by $\int_R e^r dA_0$ and the weighted perimeter is given by $\int_{\partial R} e^r dP_0$, where dP_0 and dA_0 are elements of Euclidean perimeter and area. For the rest of this paper, we will omit the word "weighted" and refer to weighted area and weighted perimeter as area and perimeter. An *isoperimetric region*, if it exists, is a region of least perimeter for given area. An *isoperimetric curve* is the boundary of an isoperimetric region. In $\mathbb{R}^2$ with density e^ψ the *generalized curvature* of a curve is defined as $\kappa_\psi = \kappa_0 - \partial\psi/\partial\mathbf{n}$, where κ_0 is the classical geodesic curvature and $\mathbf{n}$ is the unit vector normal to the curve pointing towards the interior of the (closed) curve. Throughout the paper, we write "smooth" for the class C^2.

Proposition 2.2. *[Ro, Theorem 2.6] Consider the plane endowed with a non-decreasing radial density $f(r)$ such that $f(r) \to \infty$ as $r \to \infty$. Then isoperimetric regions exist for all given areas.*

Corollary 2.3. *In $\mathbb{R}^2$ with density e^r, isoperimetric regions exist for all given areas.*

Proposition 2.4. *[C, Proposition 3.4] An isoperimetric curve in the plane with density e^ψ has constant generalized curvature κ_ψ.*

Proposition 2.5. *[Da, Lemma 2.1] In $\mathbb{R}^2$ with smooth radial density $e^{\psi(r)}$ except perhaps at the origin, a constant-generalized-curvature curve is symmetric under reflection across every line that passes through the origin and through a critical point of r.*

Proposition 2.6. *[Da, Proposition 2.5] A (connected) constant-generalized-curvature curve in a planar domain with smooth radial density has finitely many critical points of r unless it is a circle about the origin.*

The following beautiful theorem in classical differential geometry was proved by Kneser in 1912:

Proposition 2.7. *[O, Four-Vertex Theorem] For a simple, closed, smooth curve, the curvature function κ has at least four extrema.*

For a treatment of the Four-Vertex Theorem see [O].

3 The Plane with Density e^r

In this section, we prove that an isoperimetric region in the plane with density e^r is convex and contains the origin in its interior (Theorem 3.19). We conclude with some numerical evidence that it must be a disc about the origin. (For earlier numerical evidence, see Li et al. [L] Proposition 6.8; Kolesnikov et al. [KZ] Section 5 does similar analysis for $\mathbb{R}^2$ with density $e^{-r^\alpha}, \alpha \geq 1$).

Proposition 3.1. *[M5, Corollary 3.7, Section 3.10] For $n < 7$, an isoperimetric hypersurface in a C^3 n-dimensional Riemannian manifold with a smooth density is smooth.*

Proposition 3.2. *In $\mathbb{R}^2$ with centrally symmetric density e^ψ that is smooth with $|\nabla\psi|$ bounded except possibly at the origin, an isoperimetric curve is smooth.*

Proof. Where the density is smooth, the curve is smooth by Proposition 3.1 . Suppose there is an isoperimetric curve that passes through the origin O that is not smooth at O. Since $|\nabla\psi|$ is bounded, the curve has bounded classical curvature as it approaches O and hence a limiting tangent exists on each side. If the tangents are the same on both sides, then the curve is C^1. Since an isoperimetric curve has constant generalized curvature (Proposition 2.4) and the density is centrally symmetric, the curve approaches O with the same classical curvature on both sides. Hence it is C^2 at the origin. If the tangents do not coincide, then they form an angle $\theta \neq \pi$. For sufficiently small $\epsilon > 0$, let X, Y be two points on the curve with (classical) distance ϵ from O. Then the angle $\angle XOY \to \theta$ as $\epsilon \to 0$. Moreover, the line segment XY has length $2\epsilon\sin(\theta/2)$ and the area of the triangle XOY is $\epsilon^2\sin(\theta/2)\cos(\theta/2)$. Hence if we replace XO and YO with the line segment XY, $\Delta P = 2\epsilon(\sin(\theta/2) - 1)$ and $\Delta A = \pm\epsilon^2\sin(\theta/2)\cos(\theta/2)$. Since $\theta \neq \pi$, $|\Delta P/\Delta A| \to \infty$ as $\epsilon \to 0$.

Since density is continuous by definition, ψ is bounded in a neighborhood U of the origin. Take ϵ small enough so that the triangle XOY lies in U. There exists ψ_{min} and ψ_{max} such that $\psi_{max} \geq \psi(x) \geq \psi_{min}$ for all x in $\overline{U}$. Hence,

$$\left|\frac{\Delta P_\psi}{\Delta A_\psi}\right| \geq \left|\frac{\Delta P}{\Delta A}\right| \frac{e^{\psi_{min}}}{e^{\psi_{max}}} \geq \left|\frac{\Delta P}{\Delta A}\right| e^{\psi_{min}-\psi_{max}} \to \infty \qquad \text{as } \epsilon \to 0.$$

Therefore $dP_\psi/dA_\psi = +\infty$. Since at smooth points of the curve dP_ψ/dA_ψ is finite (the generalized curvature), eliminating a bit of area at the origin and replacing it elsewhere would reduce perimeter, a contradiction.

□

Corollary 3.3. *In $\mathbb{R}^2$ with density e^r, an isoperimetric curve is smooth.*

Proposition 3.4. *[MP, Theorem 3.3] Let f be a density in $\mathbb{R}^n$ which is radially non-decreasing. Then the isoperimetric function $P = P(A)$ is non-decreasing. Moreover, if isoperimetric sets exist for all volumes, then the isoperimetric function is strictly increasing.*

Proposition 3.5. *[MP, Proposition 4.1] An isoperimetric region in $\mathbb{R}^2$ with nondecreasing density $\Psi(r)$ must be bounded.*

Proposition 3.6. *In $\mathbb{R}^2$ with a radially non-decreasing density e^ψ that is smooth with $|\nabla\psi|$ bounded (except possibly at a discrete set of points), each component of an isoperimetric region is a topological disc.*

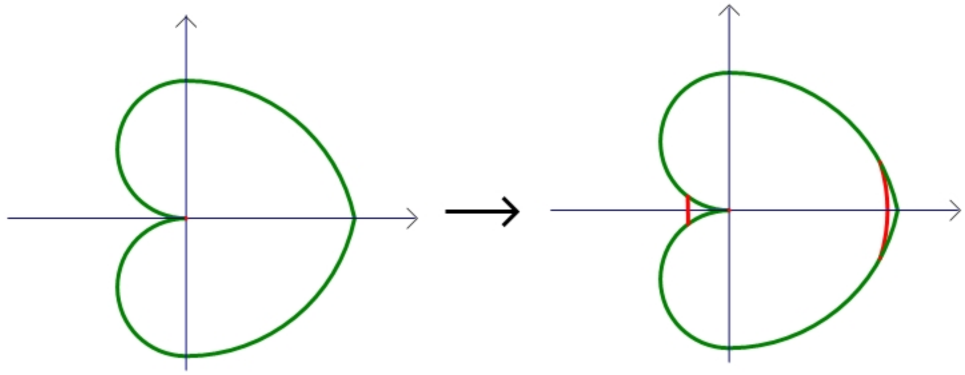

Figure 4: Reducing perimeter and leaving area unchanged.

Proof. Suppose there is a component of an isoperimetric region that is not a topological disc. Since the boundary is smooth by Proposition 3.2 and is bounded by Proposition 3.5, the only possibility is that the region has holes inside. In that case, one can fill in the holes, increasing the area and decreasing the perimeter, contradicting the fact that the isoperimetric function is strictly increasing (Proposition 3.4). □

The next definition establishes spherical symmetrization of a region E. It is most easily visualized as taking a circular annulus of thickness dr, where the intersection of the annulus and the region E is weighted equally. By sliding parts of the region along this annulus, we can make the region E^* symmetric about an axis. For this paper, we choose to symmetrize about the positive x-axis. During this operation, the area of the region E remains the same while the perimeter either decreases or remains the same.

Definition 3.7. Let $S(\rho)$ be the sphere about the origin with radius ρ. Given a radial density, a set $E \subseteq \mathbb{R}^n$ and any $\rho > 0$, define $A_E(\rho)$ to be the area of the section $E \cap S(\rho)$ relative to the density. The *spherical symmetrization* of E is the set $E^* \subseteq \mathbb{R}^n$ such that $A_{E^*} = A_E$ and such that $E^* \cap S(\rho)$ is a spherical cap centered at $(\rho, 0, \cdots, 0)$.

Proposition 3.8. *In $\mathbb{R}^n$ with radial density the spherical symmetrization E^* of an open set E satisfies*

$$|E^*| = |E|, \ P(E^*) \leq P(E).$$

If $P(E^*) = P(E)$ and the boundary of each component of E is smooth, connected, and nontangential at almost all points, then the components of E^* are pairwise congruent to the components of E.

Remark. The region after spherical symmetrization is symmetric with respect to the x-axis.

Proof. Let E be a set of finite volume and finite perimeter. Then

$$|E| = \int_{r=0}^{\infty} f(r) A_E(r) dr,$$

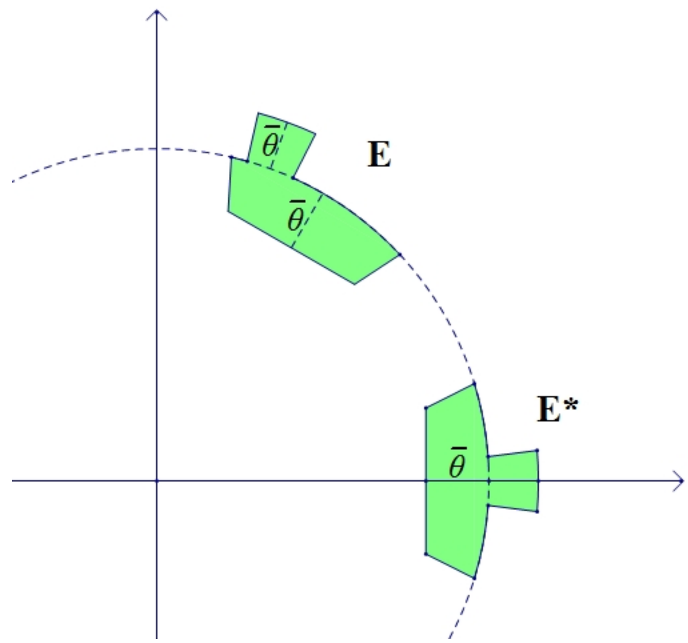

Figure 5: If the boundary of E is partly tangential, then the symmetrization E^* of a region E need not be congruent to E, even though they have the same perimeter.

Therefore, $|E| = |E^*|$ since by definition $A_{E^*} \equiv A_E$.

Regarding the inequality for the perimeter, it is known that spherical caps are the (unique) isoperimetric sets on spheres. Therefore, the inequality $P(E^*) \leq P(E)$ follows by integrating the radial variable and by using Jensen's inequality, as done in [FMP, Lemma 3.3].

If $P(E^*) = P(E)$, then each slice $E \cap S(r)$ must be empty, the whole sphere, or a spherical cap centered at some $\overline{\theta}(r)$ in $\mathbb{S}^{n-1}$. In particular, the symmetrization of each component of E is just a component of E^*. Since the boundary of each component is connected, if the slice is empty, it remains empty for all larger r; if it is the whole sphere, it remains the whole sphere for all smaller r. Since $P(E^*) = P(E)$ and the boundary is nontangential, for the spherical caps, $\overline{\theta}$ must be constant, so each component of E^* is just a rotation of a component of E. □

Remark. The necessity of requiring nontangential boundary is illustrated by Figure 5.

Corollary 3.9. *In the plane with smooth (except perhaps at a discrete set of points) radially nondecreasing density e^ψ with $|\nabla\psi|$ bounded, each component of an isoperimetric region is congruent to its spherical symmetrization.*

Proof. To apply Proposition 3.8, first note that the boundary is smooth (Proposition 3.1). Second, notice that each component of the region is a topological disc by Proposition 3.6, so its boundary is connected. Third suppose that the boundary is tangential at infinitely many points. By Proposition 2.6, the boundary is a circle about the origin, and the result holds trivially. Therefore the boundary of each component is smooth, connected, and nontangential except for finitely many points, and by Proposition 3.8, each component is congruent to its spherical symmetrization.

□

The density e^r turns out to be an especially nice density, as $|\nabla\psi| \leq 1$ (Lemma 3.10). It follows that the generalized curvature κ_ψ is greater than 1 (Lemma 3.13).

Lemma 3.10. *In $\mathbb{R}^2$ with density e^ψ where $\psi = r$, $|\nabla\psi| \leq 1$.*

Proof. Note that $|\nabla\psi| = |\hat{\mathbf{r}}| \leq 1$. □

Lemma 3.11. *For $\psi = r$, at a local maximum point of r on the boundary of a component of an isoperimetric region, $\frac{\partial\psi}{\partial\boldsymbol{n}} = -1$, where $\boldsymbol{n}$ points towards the interior of the component.*

Proof. At a local maximum point of r, $\hat{\mathbf{r}} = -\mathbf{n}$, and therefore $\partial\psi/\partial\mathbf{n} = \hat{\mathbf{r}} \cdot \mathbf{n} = -1$. □

Proposition 3.12. *In $\mathbb{R}^2$ with density e^r each component of an isoperimetric region is convex.*

Proof. Given a component of an isoperimetric region, let z be the point on the boundary of the component farthest from the origin. Since $\partial\psi/\partial\mathbf{n}(z) = -1$ (Lemma 3.11), $\kappa_\psi = \kappa_0 - \partial\psi/\partial\mathbf{n}$ is constant on the curve, $|\nabla\psi| \leq 1$ (Lemma 3.10), and the curve is smooth (Corollary 3.3), κ_0 attains its minimum at z. Notice that at z, $\kappa_0(z) \geq 1/|z| > 0$, so it must be positive everywhere. This proves that the curve must be convex. Since the component is a topological disc by Proposition 3.6, the component is convex.

□

Lemma 3.13. *In $\mathbb{R}^2$ with density e^r, $\kappa_\psi > 1$ along the boundary of an isoperimetric region.*

Proof. By Proposition 3.5, the region is bounded and has a point p with maximum distance R from the origin. Since the region is contained in the disc about the origin of radius R, the classical curvature of the boundary at p satisfies

$$\kappa_0 \geq \frac{1}{R} > 0.$$

Therefore by Lemma 3.11,

$$\kappa_\psi = \kappa_0 - \frac{\partial\psi}{\partial\mathbf{n}} = \kappa_0 + 1 > 1.$$

Since κ_ψ is constant along the boundary of an isoperimetric region by Proposition 2.4, $\kappa_\psi > 1$. □

In the proof of Lemma 3.14, we derive a differential equation of the angle φ of the curve from the tangential direction. Using its solution, we find a relation between $\partial\psi/\partial\mathbf{n}$ and r. Corollary 3.15 states an equation for the classical curvature of an isoperimetric curve.

Lemma 3.14. *In $\mathbb{R}^2$ with density e^r, on curves with constant generalized curvature κ_ψ, for $r > 0$,*

$$\frac{\partial \psi}{\partial \boldsymbol{n}} = -\frac{\kappa_\psi}{r}(r - 1 - e^{-r}C),$$

where C is a real constant.

Proof. Let α be the angle with the horizontal, φ be the angle of the curve from the tangential direction (Figure 6) and κ_0 be the classical curvature. Then

$$\frac{d\alpha}{ds} = \kappa_0 = \kappa_\psi + \frac{\partial \psi}{\partial \mathbf{n}} = \kappa_\psi - \cos\varphi.$$

Since $\varphi = \alpha - \theta - \pi/2$,

$$\frac{d\varphi}{ds} = \frac{d\alpha}{ds} - \frac{d\theta}{ds} = \kappa_\psi - \cos\varphi - \frac{\cos\varphi}{r}.$$

Since $dr/ds = -\sin\varphi$,

$$\frac{d\varphi}{dr} = -\kappa_\psi \csc\varphi + (1 + \frac{1}{r})\cot\varphi.$$

Substitute $h = \cos\varphi$:

$$\frac{dh}{dr} = \kappa_\psi - \left(1 + \frac{1}{r}\right) h.$$

The solution to this differential equation is:

$$\cos\varphi = h = \frac{\kappa_\psi}{r}(r - 1 - e^{-r}C),$$

where C is a constant that depends on initial conditions. Thus

$$\frac{\partial \psi}{\partial \mathbf{n}} = -\cos\varphi = -\frac{\kappa_\psi}{r}(r - 1 - e^{-r}C).$$

□

Corollary 3.15. *On curves with constant generalized curvature κ_ψ, for $r > 0$,*

$$\kappa_0 = \kappa_\psi \left(\frac{1 + e^{-r}C}{r}\right).$$

Proof. By Lemma 3.14,

$$\kappa_0 = \kappa_\psi + \frac{\partial \psi}{\partial \mathbf{n}} = \kappa_\psi \left(\frac{1 + e^{-r}C}{r}\right).$$

□

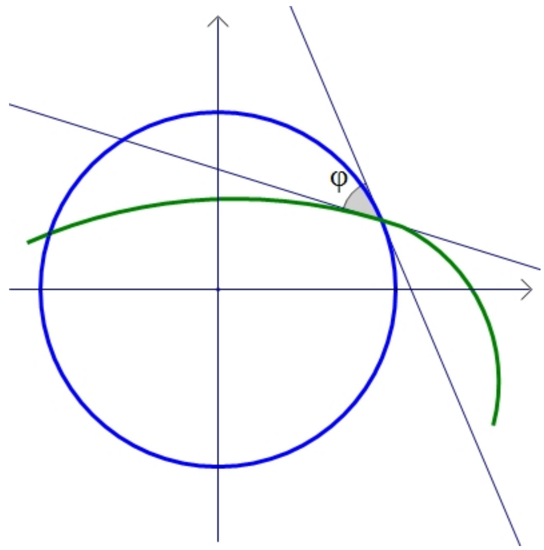

Figure 6: φ is the angle of the curve from the tangential direction.

So far, we have produced an equation that governs the classical curvature of an isoperimetric curve. Theorem 3.19, our main result, shows that an isoperimetric region contains the origin in its interior and is convex. The proof follows from Propositions 3.16 and 3.18. Proposition 3.16 shows that the distance of a component of an isoperimetric curve from the origin has just two local extrema unless it is a circle about the origin. Moreover, by Proposition 3.18, if a component of an isoperimetric region does not contain the origin, then the classical curvature is decreasing from the closest point of the curve to the origin to the farthest point, contradicting the Four-Vertex Theorem.

Proposition 3.16. *In $\mathbb{R}^2$ with density e^r, the distance r from the origin, restricted to the boundary of a component of an isoperimetric region, has a unique local maximum and a unique local minimum unless that boundary of the component is a circle about the origin.*

Proof. Without loss of generality, assume that the given isoperimetric region is spherically symmetrized with respect to the positive x-axis (Corollary 3.9).

Let x_{min} and x_{max} be the minimum and maximum of x on the boundary of a component of the region (Figure 7). Define $r : \mathbb{R}^2 \to \mathbb{R}$ as the distance of a point in $\mathbb{R}^2$ from the origin and $R : [x_{min}, x_{max}] \to \mathbb{R}$ as the distance from the origin to the point on the component's boundary with x-coordinate x_0 for $x_0 \in [x_{min}, x_{max}]$. Note that $R(x_0) = |x_0|$ for $x_0 = x_{min}, x_{max}$. Due to convexity (Proposition 3.12), for $x_0 \in (x_{min}, x_{max})$, there are at most two points with x-coordinate x_0. They are symmetric with respect to the x-axis, and thus have the same distance from the origin. Therefore R is a well-defined function. Since the boundary is smooth (Corollary 3.3), R is a continuous function except possibly at the end points. Furthermore, x increases along the boundary of the isoperimetric region from x_{min} to x_{max}. Therefore, it suffices to prove that R is an increasing function of x unless the component is a disc about the origin.

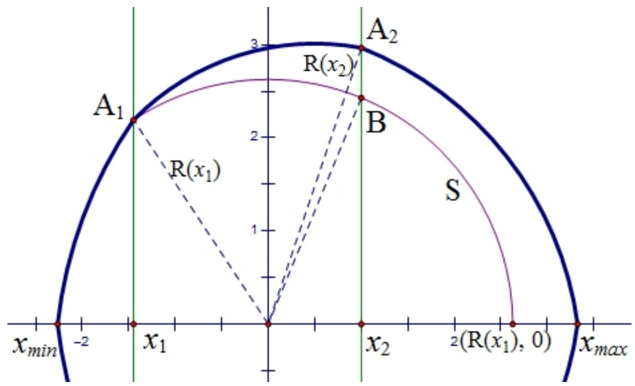

Figure 7: The boundary of a component of an isoperimetric region has a unique local minimum and a unique local maximum of the distance from the origin, unless it is a circle about the origin.

Choose two arbitrary values x_1, x_2 such that $x_{min} \leq x_1 < x_2 \leq x_{max}$, it suffices to prove that $R(x_1) \leq R(x_2)$. Construct lines $x = x_1$ and $x = x_2$. Let A_1 be the point of intersection between the line $x = x_1$ and the boundary of the region. Similarly define the point A_2. Then $r(A_1) = R(x_1)$, $r(A_2) = R(x_2)$. Consider the arc S centered at the origin from the point A_1 to point $(R(x_1), 0)$. By the definition of spherical symmetrization, the arc S is part of the region.

Case 1: S intersects the line $x = x_2$.
Let B be the intersection of S and the line $x = x_2$ (with positive y-coordinate), then $r(B) = R(x_1)$. Note that since A_2 is on the boundary of the component while B is in the interior, by convexity, the y-coordinate of B is not greater than that of A_2. Since they are on the same vertical line, the distance from origin to B is not greater than that to A_2, i.e.

$$R(x_1) = r(B) \leq r(A_2) = R(x_2).$$

Case 2: S does not intersect the line $x = x_2$.
Then

$$R(x_1) \leq r(A_2) = R(x_2).$$

In both cases we have $R(x_1) \leq R(x_2)$ given $x_1 < x_2$. Therefore R is a non-decreasing function of x.

If R is not an increasing function of x, then there exists x_1, x_2 such that $x_1 < x_2$ and $R(x_1) = R(x_2)$. Since R is non-decreasing and continuous in the open interval (x_{min}, x_{max}), $R(x) = R(x_1) = R(x_2)$ for all $x_1 \leq x \leq x_2$. Thus all the boundary points with x-coordinate between x_1 and x_2 are critical points. By Proposition 2.6, the given component of the isoperimetric region is a disc about the origin.

$\square$

Lemma 3.17. *In $\mathbb{R}^2$ with density e^r, if a component of an isoperimetric region does not contain the origin in its interior, then*

$$C \geq -1,$$

where C is the constant in Lemma 3.14.

Proof. Let a, b denote the minimum and maximum of r of the given component of an isoperimetric region. By Lemma 3.14,

$$\frac{\partial \psi}{\partial \mathbf{n}} = -\frac{\kappa_\psi}{r}(r - 1 - e^{-r}C)$$

holds at every point on the boundary of the component for $r > 0$.

Case 1: The component contains the origin in its boundary.
Assume that $C < -1$. Since the function r is continuous on the boundary and $r = 0$ at the origin, we can choose a point on the boundary such that its distance from the origin r_ϵ satisfies $0 < r_\epsilon < ln(-C)$, i.e. $-1 - e^{r_\epsilon}C \geq 0$. Then at this point,

$$\frac{\partial \psi}{\partial \mathbf{n}} = -\frac{\kappa_\psi}{r}(r - 1 - e^{-r}C) \leq -\frac{\kappa_\psi}{r}(r) < -1,$$

which contradicts Lemma 3.10. Therefore $C \geq -1$.

Case 2: The component does not contain the origin.
By Lemma 3.11,

$$\left.\frac{\partial \psi}{\partial \mathbf{n}}\right|_{r=b} = -1.$$

Since the component does not contain the origin, at the minimum point of r on its boundary, the distance from the origin is strictly greater than 0, and the radial vector is in the same direction as the normal vector (towards the interior of the component), thus

$$\left.\frac{\partial \psi}{\partial \mathbf{n}}\right|_{r=a} = 1.$$

Adding the last two equations yields that

$$-\frac{\kappa_\psi}{a}(a - 1 - e^{-a}C) - \frac{\kappa_\psi}{b}(b - 1 - e^{-b}C) = 0.$$

After simplification,

$$-(a + b) + 2ab - C(be^{-a} + ae^{-b}) = 0. \qquad (1)$$

Define $f(C) = -(a + b) + 2ab - C(be^{-a} + ae^{-b})$. Notice that

$$f(-1) = -(a + b) + 2ab + (ae^{-b} + be^{-a}) = a(e^{-b} + b - 1) + b(e^{-a} + a - 1) > 0,$$

$$f'(C) = -(ae^{-b} + be^{-a}) < 0.$$

Thus for $C \leq -1$, $f(C) > 0$, a contradiction of (1). □

Proposition 3.18. *In $\mathbb{R}^2$ with density e^r, if a component of an isoperimetric region does not contain the origin in its interior, then the classical curvature κ_0 is decreasing along its boundary from the minimum point of r to the maximum point of r.*

Proof. Without loss of generality, assume that the given component of an isoperimetric region is symmetrized with respect to the positive x-axis (Corollary 3.9). By Proposition 3.16, it suffices to show that the classical curvature κ_0 is a decreasing function of r.

Since $\kappa_0 = \kappa_\psi + \partial\psi/\partial\mathbf{n}$, where κ_ψ is a constant throughout the whole boundary, it suffices to prove that $\partial\psi/\partial\mathbf{n}$ is a decreasing function of r.

By Lemma 3.14,

$$\frac{\partial\psi}{\partial\mathbf{n}} = -\frac{\kappa_\psi}{r}(r - 1 - e^{-r}C).$$

Now

$$\frac{d}{dr}\frac{\partial\psi}{\partial\mathbf{n}} = \frac{d}{dr}\left(-\frac{\kappa_\psi}{r}(r-1-e^{-r}C)\right) = -\frac{\kappa_\psi}{r^2}(1+(r+1)e^{-r}C) \leq -\frac{\kappa_\psi}{r^2}(1-(r+1)e^{-r}).$$

In the last inequality, we have used the fact that $C \geq -1$ by Lemma 3.17. Notice that the last expression is negative since by a Taylor expansion $e^r \geq 1 + r$ and $\kappa_\psi > 1$ by Lemma 3.13. As a result, $\partial\psi/\partial\mathbf{n}$ is a decreasing function of r, as desired. □

Now we present our main result:

Theorem 3.19. *An isoperimetric region in $\mathbb{R}^2$ with density e^r contains the origin in its interior and is convex.*

Proof. Suppose that some component of an isoperimetric region does not contain the origin in its interior. Recall that each component of an isoperimetric region is a topological disc by Proposition 3.6. By Proposition 3.18, the classical curvature along the boundary increases from the unique minimum point of r to the unique maximum point of r. Thus there are only two extrema of κ_0 on the boundary. However, since an isoperimetric curve is smooth by Proposition 3.2, this contradicts the Four-Vertex Theorem [O], which states that there are at least four extrema of curvature on a smooth Jordan curve. Thus the components of an isoperimetric region must contain the origin in their interior. Notice that since each component is a topological disc, at most one component can contain the origin, and thus the region is connected. Therefore since there is only one component the region is convex by Proposition 3.12.

□

Having established that an isoperimetric region contains the origin in its interior and is convex, we still find it difficult to prove that it is a disc about the origin. The rest of this section develops some relevant tools and provides numerical evidence.

Proposition 3.20. *A circle in the plane with density e^r has constant generalized curvature if and only if its center is at the origin.*

Proof. Consider a circle on the plane with radius R. By rotational symmetry, we can suitably choose a Cartesian coordinate system such that the center is on the nonnegative x-axis. Let d be the distance between the center and the origin; then the coordinate of the center is $(d, 0)$.

Recall that

$$\kappa_\psi = \kappa_0 - \frac{\partial \psi}{\partial \mathbf{n}}.$$

Since for a circle κ_0 remains constant, we need only to show that $\partial\psi/\partial\mathbf{n}$ is not constant. Evaluate $\partial\psi/\partial\mathbf{n}$ at two points on the circle $(d+R, 0)$ and (d, R):

$$\left.\frac{\partial \psi}{\partial \mathbf{n}}\right|_{(d+R,0)} = \frac{\partial \psi}{\partial x} = 1,$$
$$\left.\frac{\partial \psi}{\partial \mathbf{n}}\right|_{(d,R)} = \frac{\partial \psi}{\partial y} = \frac{R}{\sqrt{d^2+R^2}}.$$

These two values are equal if and only if $d^2 = 0$, i.e. $d = 0$. Thus a circle has constant generalized curvature if and only if its center is at the origin. $\square$

Corollary 3.22 derives a differential equation for $r(\theta)$, the polar equation describing an isoperimetric curve. The differential equation is not solvable by analytic methods: in fact, it is not Lipschitz continuous, as shown in Proposition 3.25. Lemma 3.21 sets up a lower bound on the constant of integration C; if equality holds for this bound, then the only solution to the differential equation is a circle about the origin (Proposition 3.26).

Lemma 3.21. *In $\mathbb{R}^2$ with density e^r, an isoperimetric curve with constant generalized curvature κ_ψ satisfies*

$$C \geq \frac{1-\kappa_\psi}{\kappa_\psi} e^{1/(\kappa_\psi - 1)},$$

where C is the constant in Lemma 3.14.

Proof. If $C \geq -1$,

$$\frac{1-\kappa_\psi}{\kappa_\psi} e^{1/(\kappa_\psi - 1)} = \frac{-e^{1/(\kappa_\psi-1)}}{1 + 1/(\kappa_\psi - 1)} < \frac{-e^0}{1+0} = -1 \leq C,$$

because $1/(\kappa_\psi - 1) > 0$ by Lemma 3.13 and $-e^r/(1+r)$ is a decreasing function of r.

If $C < -1$, consider a critical point of

$$\frac{\partial \psi}{\partial \mathbf{n}} = -\frac{\kappa_\psi}{r}(r - 1 - e^{-r}C)$$

as a function of r:

$$0 = \frac{d}{dr}\frac{\partial \psi}{\partial \mathbf{n}} = \frac{d}{dr}\left(-\frac{\kappa_\psi}{r}(r-1-e^{-r}C)\right) = -\frac{\kappa_\psi}{r^2}(1 + (r+1)e^{-r}C).$$

Thus the only critical point is at $r = r_0$, where r_0 is the only root of

$$C = \frac{-e^r}{1+r}$$

for $C < -1$. Note that at $r = r_0$,

$$\left.\frac{d^2}{dr^2}\frac{\partial\psi}{\partial\mathbf{n}}\right|_{r=r_0} = \frac{\kappa_\psi}{r_0^3}(2 + C(r_0^2 + 2r_0 + 2)e^{-r_0}) = -\frac{\kappa_\psi}{r_0(1+r_0)} < 0.$$

Thus the critical point is always a maximum point.

Recall from Lemma 3.11 that a local maximum point of r on an isoperimetric curve satisfies

$$\frac{\partial\psi}{\partial\mathbf{n}} = -1.$$

Therefore

$$\left.\frac{\partial\psi}{\partial\mathbf{n}}\right|_{r=r_0} = -\frac{\kappa_\psi}{r_0}(r_0 - 1 - e^{-r_0}C) \geq -1,$$

which yields

$$r_0 \leq \frac{1}{\kappa_\psi - 1}.$$

Since the function $-e^r/(1+r)$ is strictly decreasing,

$$C = \frac{-e^{r_0}}{1+r_0} \geq \frac{-e^{1/(\kappa_\psi-1)}}{1+1/(\kappa_\psi-1)} = \frac{1-\kappa_\psi}{\kappa_\psi}e^{1/(\kappa_\psi-1)}.$$

□

Proposition 3.22. *In polar coordinates, a point (r,θ) on a smooth curve satisfies*

$$\frac{dr}{d\theta} = -r\tan\varphi$$

where φ is the angle of the curve from the tangential direction.

Proof. At the point (r,θ), consider the infinitesimal right-angled triangle with two sides dr, $rd\theta$ and hypotenuse ds. The angle between ds and $rd\theta$ is $-\varphi$, thus we have

$$-\tan\varphi = \frac{1}{r}\frac{dr}{d\theta}.$$

The result follows. □

Corollary 3.23. *In $\mathbb{R}^2$ with density e^r, smooth curves with constant generalized curvature κ_ψ satisfy*

$$\frac{dr}{d\theta} = -(\text{sgn}\varphi)r\sqrt{\left(\frac{r}{\kappa_\psi(r-1-e^{-r}C)}\right)^2 - 1},$$

where C is the constant in Lemma 3.14 and φ is the angle of the curve from the tangential direction, with $\text{sgn}\varphi = 1$ for $\varphi \geq 0$ and $\text{sgn}\varphi = -1$ for $\varphi < 0$.

Proof. By Proposition 3.22,

$$\frac{dr}{d\theta} = -r\tan\varphi.$$

On smooth curves with constant generalized curvature κ_ψ, by Lemma 3.14,

$$\frac{dr}{d\theta} = -r\tan\varphi = -(\mathrm{sgn}\varphi)r\sqrt{\frac{1}{\cos^2\varphi} - 1} = -(\mathrm{sgn}\varphi)r\sqrt{\left(\frac{r}{\kappa_\psi(r-1-e^{-r}C)}\right)^2 - 1}$$

since

$$\tan\varphi = (\mathrm{sgn}\varphi)\sqrt{\frac{1}{\cos^2\varphi} - 1}.$$

□

To better understand the solution to the isoperimetric problem, we analyze numerically what the isoperimetric curve looks like based on the differential equation from Corollary 3.23:

$$\frac{dr}{d\theta} = -(\mathrm{sgn}\varphi)r\sqrt{\left(\frac{r}{\kappa_\psi(r-1-e^{-r}C)}\right)^2 - 1} \qquad r(0) = r_0, \tag{2}$$

where r_0 is a place where $dr/d\theta = 0$, i.e. r_0 is a root of the equation

$$\cos\varphi = \frac{\kappa_\psi}{r}(r-1-e^{-r}C) = 1.$$

Lemma 3.24. *Given a real constant $\kappa_\psi > 1$, the equation*

$$\frac{\kappa_\psi}{r}(r-1-e^{-r}C) = 1$$

has one positive root if $C \geq -1$ and two positive roots if $-1 > C \geq (1-\kappa_\psi)e^{1/(\kappa_\psi-1)}/\kappa_\psi$. In particular, for $C = (1-\kappa_\psi)e^{1/(\kappa_\psi-1)}/\kappa_\psi$, the roots are degenerate.

Proof. Let $f(r) = \kappa_\psi(r-1-e^{-r}C)/r$, $r > 0$.

If $C \geq -1$, then

$$\frac{df}{dr} = \frac{\kappa_\psi}{r^2}(1+(r+1)e^{-r}C) > 0.$$

Thus $f(r)$ is a strictly increasing differentiable function in the domain $(0,\infty)$, and $f(r) = 0$ has at most one root. Since $f(r) \to -\infty$ as $r \to 0$, and $f(r) \to \kappa_\psi > 1$ as $r \to \infty$, $f(r) = 1$ for some $r > 0$ by the intermediate value theorem.

If $-1 > C \geq (1-\kappa_\psi)e^{1/(\kappa_\psi-1)}/\kappa_\psi$, then critical points of $f(r)$ satisfy

$$\frac{df}{dr} = \frac{\kappa_\psi}{r^2}(1+(r+1)e^{-r}C) = 0,$$

i.e. $C = -e^r/(r+1)$. The function $-e^r/(r+1)$ is decreasing for $r \geq 0$, and equals -1 when $r = 0$. Therefore, for each $C < -1$, there is a unique positive number r_* satisfying

$$C = \frac{-e^{r_*}}{r_* + 1}.$$

Thus f has exactly one critical point r_*. Since $f''(r)$ is always positive, r_* is a minimum point of f.

The value at the minimum point is $\kappa_\psi r_*/(r_* + 1)$. Notice

$$\frac{-e^{r_*}}{r_* + 1} = C \geq \frac{1 - \kappa_\psi}{\kappa_\psi} e^{1/(\kappa_\psi - 1)} = \frac{-e^{1/(\kappa_\psi - 1)}}{1/(\kappa_\psi - 1) + 1}.$$

Since $-e^r/(r+1)$ is a decreasing function for $r \geq 0$, we get

$$r_* \leq \frac{1}{\kappa_\psi - 1}.$$

Hence

$$\frac{r_*}{r_* + 1}\kappa_\psi \leq 1.$$

Thus we have

$$f(r) \to \infty \text{ as } r \to 0, \qquad f(r) \to \kappa_\psi > 1 \text{ as } r \to \infty, \qquad \text{and} \qquad f(r_*) = \frac{r_*}{r_* + 1}\kappa_\psi \leq 1,$$

and also

$$f'(r) < 0 \text{ for } r < r_*, \qquad\qquad f'(r) > 0 \text{ for } r > r_*.$$

It follows by the intermediate value theorem that the equation $f(r) = 1$ has two roots r_1, r_2 such that $0 < r_1 \leq r_* \leq r_2$. The two roots r_1 and r_2 coincide if and only if

$$C = \frac{1 - \kappa_\psi}{\kappa_\psi} e^{1/(\kappa_\psi - 1)}.$$

□

Lemma 3.25. *The right hand side*

$$r\sqrt{\left(\frac{r}{\kappa_\psi(r - 1 - e^{-r}C)}\right)^2 - 1}$$

of the differential equation (2) is not Lipschitz continuous at $r = r_0$ for $C < -1$, where r_0 is a root of the equation

$$\cos\varphi = \frac{\kappa_\psi}{r}(r - 1 - e^{-r}C) = 1.$$

Proof. This function is differentiable at $r = r_0$, therefore it suffices to show that its first derivative is unbounded.

$$\frac{d}{dr}\left(r\sqrt{\left(\frac{r}{\kappa_\psi(r-1-e^{-r}C)}\right)^2-1}\right)$$
$$=\sqrt{\left(\frac{r}{\kappa_\psi(r-1-e^{-r}C)}\right)^2-1}-\frac{r^2(1+Ce^{-r}(1+r))}{\kappa_\psi^2(r-1-e^{-r}C)^3\sqrt{(r/(\kappa_\psi(r-1-e^{-r}C))^2)-1}}.$$

Since $\kappa_\psi(r_0-1-e^{-r_0}C)/r_0 = 1$, the first term is 0 and the denominator of the second term is also 0, while the numerator is negative for $C < -1$. Thus this derivative is unbounded at $r = r_0$. The result follows.

□

As a result, uniqueness of the differential equation is not guaranteed. For $C < -1$, when r_0 is taken to be the greater root, the curve is either a circle about the origin or a curve that does not close up, as shown in Figure 1. When r_0 is taken to be the smaller root, the curve is again either a circle about the origin or a curve that does not close up, as shown in Figure 2.

The curves in Figure 1 and 2 correspond to a nondegenerate solution to the ODE. There is also a degenerate solution $r(\theta) = r_0$ for any given parameters κ_ψ and C, which describes a circle about the origin (Figure 3). This circle does *not* have the generalized curvature κ_ψ, as specified in the differential equation. Indeed, for a nondegenerate solution of the ODE, the curvature at the initial point is given by

$$\kappa_0 = \kappa_\psi + \frac{\partial\psi}{\partial\mathbf{n}} = \kappa_\psi - 1,$$

which is a constant given a fixed κ_ψ. As a result, the differential equation has a unique nondegenerate solution, which corresponds to a unique constant-generalized-curvature curve given κ_ψ and C. The numerical evidence shows that when C reaches its minimum, this nondegenerate solution coincides with the degenerate solution, and gives the unique constant-generalized-curvature curve, which is precisely a circle about the origin.

Furthermore, consider the angle α that the isoperimetric curve $r(\theta)$ makes with the horizontal at $r(\pi)$ (Figure 8). If $\alpha = \pi/2$, the curve will be a circle about the origin. If $\alpha \neq \pi/2$, then because the curve must be symmetric about the x-axis, the curve would not be smooth at the point of intersection with the $x-$axis, contradicting Corollary 3.3.

Considering the angle α as a function of r_0, the plots that come from the numerical solution indicate that $\alpha = \pi/2$ (i.e. the curve is a circle about the origin) for a particular value of r_0 coinciding with there being a unique root to the equation $\kappa_\psi(r-1-e^{-r}C)/r = 1$. That is, the isoperimetric curve is a circle when $C = (1-\kappa_\psi)e^{1/(\kappa_\psi-1)}/\kappa_\psi$ by Lemma 3.24. A rigorous proof of this behavior of the angle α would prove that discs about the origin are isoperimetric:

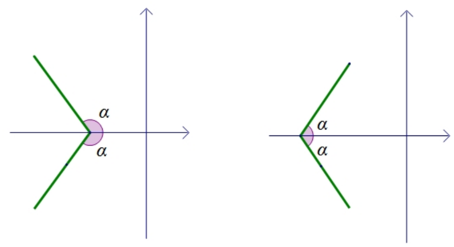

Figure 8: When $\alpha \neq \pi/2$, the boundary of the region is not smooth at $r(\pi)$.

Proposition 3.26. *In $\mathbb{R}^2$ with density e^r, discs about the origin are isoperimetric if the following statement holds:*

If an isoperimetric curve $r(\theta)$ is perpendicular to the horizontal at $r(\pi)$, then

$$C = \frac{1-\kappa_\psi}{\kappa_\psi} e^{1/(\kappa_\psi - 1)},$$

where C is the constant in Lemma 3.14.

Proof. Let α be the angle that an isoperimetric curve $r(\theta)$ makes with the horizontal at $r(\pi)$. Since $dr/d\theta = 0$ at $r(0) = r_0$, it is a critical point of r. By Proposition 2.5, an isoperimetric region is symmetric with respect to the line through the origin and $r(0)$, i.e. the x-axis. Therefore the region is also symmetric at $r(\pi)$. If $\alpha \neq \pi/2$, the boundary of the region is not smooth at $r(\pi)$, contradicting Proposition 3.2 (Figure 8). Thus $\alpha = \pi/2$ at $r(\pi)$. Assume the given statement holds, then

$$C = \frac{1-\kappa_\psi}{\kappa_\psi} e^{1/(\kappa_\psi - 1)}.$$

At this value of C,

$$r_0 = \frac{1}{\kappa_\psi - 1}$$

is the only solution of the equation

$$\frac{\kappa_\psi}{r}(r - 1 - e^{-r}C) = 1.$$

The function

$$r(\theta) = \frac{1}{\kappa_\psi - 1}$$

is the unique solution to the differential equation (2) that has classical curvature

$$\kappa_0 = \kappa_\psi + \frac{\partial \psi}{\partial \mathbf{n}} = \kappa_\psi - 1$$

at $r(0) = r_0$. Therefore, it corresponds to the unique isoperimetric curve with constant generalized curvature κ_ψ, which is a circle about the origin with radius $1/(\kappa_\psi - 1)$. □

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