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The strong symmetric genus spectrum of abelian groups

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Abstract. The strong symmetric genus of a group G, $\sigma^0(G)$, is the minimum genus of any compact surface on which G acts faithfully while preserving orientation. We investigate the set of positive integers which occur as the strong symmetric genus of a finite abelian group. This is called the strong symmetric genus spectrum. We prove that there are an infinite number of gaps in the strong symmetric genus spectrum of finite abelian groups. We also determine an upper bound for the size of a finite abelian group that can act faithfully on a surface of a particular genus and then find the genus of abelian groups in particular families. These formulas produce a lower bound for the density of the strong symmetric genus spectrum.

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1 Introduction

The strong symmetric genus of a group G, denoted $\sigma^0(G)$, is the minimum genus of a compact surface on which G acts faithfully while preserving orientation. This paper focuses on the strong symmetric genus of finite abelian groups. As of 2008, all groups of strong symmetric genus up to $\sigma^0 = 25$ had been classified [5]. It has also been shown by May and Zimmerman [10] that there is a group of every strong symmetric genus value. In a similar study, Conder and Tucker [4] investigated this same question for the symmetric genus parameter. The symmetric genus of a group G, $\sigma(G)$, allows the group to have both orientation preserving and orientation reversing elements. It has been conjectured that there is a group for every symmetric genus (see [4], [10]).

In contrast to what May and Zimmerman proved for all groups, we will show that there are integers that are not the strong symmetric genus of any finite abelian group. In fact, if $g = pq + 1$, where p is an odd prime and q equals 1 or a prime, with p and q not both 3, then there is no abelian group G for which $\sigma^0(G) = g$. Consequently, there are infinitely many gaps in the genus spectrum for abelian groups. We will also determine genus formulas for several abelian group families. These formulas yield a lower bound for the density of the integers $\sigma^0(G)$ for abelian groups G. In addition, we will show that $2(g + 3)$ is the upper bound for the size of a finite abelian group of rank three or more which acts faithfully on a surface of genus g. Section 2 of this paper will provide background information on the genus of compact surfaces and how groups act upon those surfaces. Then Section 3 will discuss past research in this field that we utilize as well as our own methodology. Finally, Section 4 and subsequent sections will encompass all of our research and findings.

2 Background

The genus of a surface describes the number of handles that are connected to that surface. This study will specifically deal with the genera of closed, orientable surfaces, which are compact surfaces with no boundaries. Figure 1 provides several examples of closed surfaces with small genera. The sphere has genus zero, while the torus, having one handle, has genus one. As additional handles are added to the surface, its genus value increases accordingly.

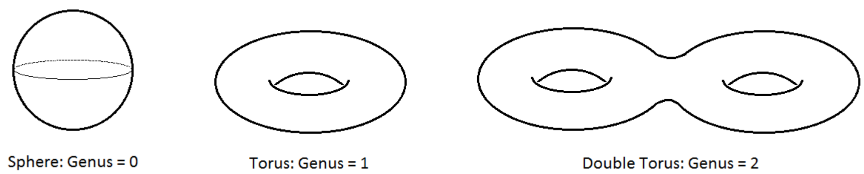

Figure 1: Surfaces of Small Genus

In order to better understand how groups act on surfaces it helps to visualize a few examples. The elements in any group can be thought of as symmetries or automorphisms of a surface. For example, a cyclic group $\mathbb{Z}_n$, of size n, acts on the sphere. Each automorphism in the group would be a rotation of the sphere about its central axis by $360°/n$. The variable n can be any positive integer since the sphere can have infinitely many degrees of rotation thus allowing any cyclic group to act on the sphere. Therefore, $\sigma^0(\mathbb{Z}_n) = 0$. Another example of this visualization of a group acting faithfully on a surface would be the way in which the group $\mathbb{Z}_n \times \mathbb{Z}_m$ acts on a torus. The first set of symmetries, $\mathbb{Z}_n$, would again be a rotation of $360°/n$ about the central axis. For the second set of symmetries, imagine the torus as a cylinder that has been bent into a ring. That cylinder has a central axis through its length and the torus can rotate about this central axis by $360°/m$, similar to a hair tie being rolled inward. The positive integers n and m can take on any value, thus all groups of the form $\mathbb{Z}_n \times \mathbb{Z}_m$ can act faithfully on the torus and $\sigma^0(\mathbb{Z}_n \times \mathbb{Z}_m) = 1$, for all n, $m > 1$.

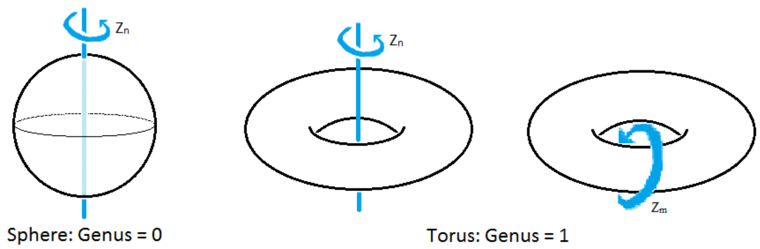

Figure 2: Automorphisms of Surfaces

3 Preliminaries

The relationship between groups and the surfaces they act on is a classical research topic that has been studied for over a century. One of the most influential people to work in this field was Hurwitz. He proved [6, p.424] that the order of any group of automorphisms on a surface of genus g was less than or equal to $84(g-1)$. This result and much of Hurwitz's other work has dramatically influenced the research of subsequent mathematicians in this field.

The modern study of groups acting on surfaces involves Fuchsian groups, which are discrete groups of automorphisms of the hyperbolic plane. A Fuchsian group Γ has the following well-known presentation[8, p. 699]:

Generators: $x_1, x_2, \cdots, x_r, a_1, b_1, a_2, b_2, \cdots, a_{g_0}, b_{g_0}$
Relations: $x_1^{m_1} = x_2^{m_2} = \cdots = x_r^{m_r} = 1$, $\prod_{i=1}^{r} x_i \prod_{j=1}^{g_0} [a_j, b_j] = 1$,
where the m_i are integers greater than or equal to 2.

For any Fuchsian group Γ, the parameter g_0 is called the orbit genus of Γ and the m_i's are called the periods of Γ. Additionally, the group Γ can be represented by the signature

$(g_0; m_1, m_2, \cdots, m_r)$ [2, p. 8]. This group will have $2g_0 + r$ generators; however one of them is always redundant because the final relation allows for one generator to be solved in terms of the others. Thus a group of this presentation has $2g_0 + r - 1$ generators. Let G be a finite group that is the homomorphic image of the Fuchsian group Γ. The Riemann-Hurwitz formula [2, p. 8] relates the size and structure of a Fuchsian group to the genus of the surface on which its images act:

$$g - 1 = |G|(g_0 - 1) + \frac{|G|}{2}\sum_{i=1}^{r}(1 - \tfrac{1}{m_i}) \tag{1}$$

where $|G|$ is the order of the group G and g is the genus of the surface upon which G faithfully acts.

Another mathematician whose work is of particular importance to this paper is Maclachlan. He used the classical Riemann-Hurwitz formula to determine the strong symmetric genus of any finite abelian group. Maclachlan [8] showed that the minimum genus of any surface that a finite abelian group can faithfully act on results from its canonical form. Every finite abelian group has a unique representation $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_s}$, called the canonical form, in which $m_i | m_{i+1}$ for all i [1, p. 345]. The m_i's of the canonical form of an abelian group are called the invariants of that group, and the quantity of those invariants determines the group's rank.

Theorem 1. *[8, Thm. 4] Let A be an Abelian group with invariants $m_1, m_2, \cdots, m_s$ for $s > 2$, where $m_i | m_{i+1}$ and $|A| > 9$. Then the minimum genus σ^0 of a surface for which A is a group of automorphisms is given as follows:*

- *For s even:*
  $\frac{2(\sigma^0 - 1)}{|A|} = \min\limits_{0 \le 2\gamma \le s} \left\{ 2(\gamma - 1) + \sum\limits_{i=1}^{s-2\gamma}(1 - \frac{1}{m_i}) + (1 - \frac{1}{m_{s-2\gamma}}) \right\}$
  If $s = 2\gamma$, then m_0 is to be interpreted as 1.

- *For s odd:*
  $\frac{2(\sigma^0 - 1)}{|A|} = \min\limits_{0 \le 2\gamma < s} \left\{ 2(\gamma - 1) + \sum\limits_{i=1}^{s-2\gamma}(1 - \frac{1}{m_i}) + (1 - \frac{1}{m_{s-2\gamma}}) \right\}$

We use Maclachlan's formula for all of the σ^0 calculations in this paper.

Our research consisted of two main phases. First, we collected and analyzed data on genus values of abelian groups and then we proved several conjectures developed through that analysis. A main part in the data collection phase involved utilizing the Small Groups Library in the computational algebra software MAGMA [3]. We wrote a brief MAGMA program that runs through every abelian group of order one to order two thousand and computes its strong symmetric genus using Maclachlan's formula. It is important to note that since abelian groups of rank 1 have a strong symmetric genus equal to zero and abelian groups of rank 2 have a strong symmetric genus equal to one, the data we collected and analyzed only included groups of rank 3 or higher. An analysis of this data revealed patterns among the genus values, including arithmetic sequences, as well as patterns of missing genus values. Utilizing the work of Hurwitz and Maclachlan we were able to prove several of our conjectures about the genus spectrum of finite abelian groups.

4 Upper Bound of the Order of a Group on a Surface

Hurwitz's $84(g-1)$ bound [6, p. 424] applies to any group, not just abelian groups. More recently, Breuer [2, p. 32] used Maclachlan's formula to determine an upper bound for the order of an abelian group of automorphisms of a surface of genus g to be $4(g+1)$. This bound is attained for a family of rank 2 abelian groups, which have genus 1. For our study, we added the additional hypothesis that the strong symmetric genus of the finite abelian group must be greater than 1, which excludes all rank 1 and rank 2 groups. Expanding on Breuer's method, we were able to calculate the bound to be $2(g+3)$.

In order to facilitate the proof of this bound, we will use the following lemma.

Lemma 1. *Suppose Γ is a Fuchsian group with 4 periods and $g_0 = 0$, that is, Γ is a quadrilateral group. If Γ has two periods which are relatively prime, then any abelian image of Γ has rank 2 or less.*

Proof. Suppose $\Gamma(g_0; m_1, m_2, m_3, m_4)$ is a quadrilateral group with m_1 and m_2 relatively prime and A is an abelian image of Γ. Let x, y, and z be the generators so that $\Gamma = \langle x, y, z\rangle$. Therefore $A = \langle \bar{x}, \bar{y}, \bar{z}\rangle$, where $\bar{x}$ is the image of x, etc. Since the $o(\bar{x})$ and $o(\bar{y})$ are relatively prime, $A = \langle \bar{x}, \bar{y}, \bar{z}\rangle = \langle \bar{x}\cdot\bar{y}, \bar{z}\rangle$. Therefore, A has rank 2 or less.

□

Theorem 2. *Let A be a finite abelian group with $\sigma^0(A) \geq 2$ and $A \neq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. If A acts faithfully on a compact surface of genus g, then $|A| \leq 2(g+3)$. Furthermore, this upper bound is attained for infinitely many values of g.*

Proof. Let $A \neq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ be a finite abelian group with $\sigma^0(A) \geq 2$ acting faithfully on a compact surface of genus g. Let $M = |A|$. Since $\sigma^0(A) \geq 2$, A has rank $s \geq 3$. There is a Fuchsian group Γ satisfying $s \leq 2g_0 + r - 1$ which maps onto A. This gives an action of the abelian group A on the surface of genus g by equation (1).

If $g_0 \geq 2$, then $g - 1 \geq M + \frac{M}{2}\sum_{i=1}^{r}(1 - \frac{1}{m_i}) \geq M$. Thus for $g_0 \geq 2$, $M \leq g - 1 < 2(g+3)$, regardless of the values of r and m_i.

If $g_0 = 1$, then $r \geq 2$ because $s \geq 3$ and $s \leq 2g_0 + r - 1 = r + 1$. This means that $\sum_{i=1}^{r}(1 - \frac{1}{m_i}) \geq 1$ and therefore $g - 1 \geq \frac{M}{2}$ by equation (1). Thus for $g_0 = 1$, $M \leq 2(g-1) < 2(g+3)$, regardless of the values of r and m_i. Therefore we may assume that $g_0 = 0$ and

$$g - 1 = -M + \frac{M}{2}\sum_{i=1}^{r}(1 - \frac{1}{m_i}). \tag{2}$$

Since $s \geq 3$ and $s \geq 2g_0 + r - 1 = r - 1$, it follows that $r \geq 4$. Without loss of generality, the periods can be ordered such that $m_1 \leq m_2 \leq \cdots \leq m_r$. Recall, $m_i \geq 2$, for all i. Since any $r-1$ of the r generators generate the abelian group A, $M = |A| \leq (m_1 \cdot m_2 \cdot \ldots \cdot m_r)/m_i$, for any i. We must now consider cases for the value of r.

Case 1: Let $r = 4$.

If $m_1 = 2$ and $m_2 = 2$, then $M \leq 4m_3$ and $M \leq 4m_4$. By equation (2), $g - 1 \geq -M + \frac{M}{2}\left(2 \cdot \frac{1}{2} + 2(1 - \frac{4}{M})\right) = -M + \frac{M}{2}\left(3 - \frac{8}{M}\right) = \frac{(M-8)}{2}$. Thus $M \leq 2(g+3)$. Note that in this case the Fuchsian group $\Gamma(2, 2, 2n, 2n)$ satisfies $m_3 = m_4 = \frac{M}{4}$ and so the equality is attained for any image of this Fuchsian group. This gives infinitely many values of g for which $g - 1 = \frac{(M-8)}{2}$.

If $m_1 = 2$ and $m_2 = 3$, then by Lemma 1, A would have rank 2 or less which is not possible because $\sigma^0(A) \geq 2$.

If $m_1 = 2$ and $m_2 = 4$, then $M \leq 8m_3$ and $M \leq 8m_4$. By equation (2), $g - 1 \geq -M + \frac{M}{2}\left(\frac{1}{2} + \frac{3}{4} + 2(1 - \frac{8}{M})\right) = -M + \frac{M}{2}\left(\frac{13}{4} - \frac{16}{M}\right) = \frac{(5M-64)}{8}$.

Comparing these two bounds yields $\frac{(M-8)}{2} \leq \frac{(5M-64)}{8}$, which simplifies to $4M - 32 \leq 5M - 64$, so $32 \leq M$. This is the point at which the first bound, $\frac{(M-8)}{2}$, drops below and thus includes all possible values greater than $\frac{(5M-64)}{8}$. Thus if $M \geq 32$, then $g - 1 \geq \frac{(M-8)}{2}$ and $M \leq 2(g+3)$.

Any abelian groups with $M < 32$ and $r = 4$ must be checked to see if they satisfy the bound $g - 1 \geq \frac{(M-8)}{2}$. There are only four such groups and using Maclachlan's formula it can be determined that each of them also satisfy this bound except the group $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Since $A \neq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ the bound $g - 1 \geq \frac{(M-8)}{2}$, or $M \leq 2(g+3)$ holds in this case.

If $m_1 = 2$ and $m_2 = 5$, then by Lemma 1, A would be of rank 2 or less which is not possible because $\sigma^0(A) \geq 2$.

If $m_1 = 2$ and $m_2 \geq 6$, then $m_3 \geq 6$ and $m_4 \geq 6$. By equation (2), $g - 1 \geq -M + \frac{M}{2}\left(\frac{1}{2} + 3 \cdot \frac{5}{6}\right) = -M + \frac{M}{2}(3) = \frac{M}{2}$. Therefore $M \leq 2(g-1) < 2(g+3)$.

If $m_1 = 3$ and $m_2 = 3$, then $m_3 \geq 3$ and $m_4 \geq 3$. However, when $m_1 = m_2 = m_3 = m_4 = 3$, the only abelian group of rank three that is the image of Γ is $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \neq A$, so this case can be ignored. Thus $m_1 = 3$, $m_2 = 3$, $m_3 > 3$ and $m_4 > 3$ and by Lemma 1, $m_3 \geq m_4 \geq 6$. By equation (2), $g - 1 \geq -M + \frac{M}{2}\left(2 \cdot \frac{2}{3} + 2 \cdot \frac{5}{6}\right) = -M + \frac{M}{2}(3) = \frac{M}{2}$. Therefore $M \leq 2(g-1) < 2(g+3)$.

Finally, if $m_1 \geq 4$ then $m_2 \geq m_3 \geq m_4 \geq 4$. By equation (2), $g - 1 \geq -M + \frac{M}{2}(4 \cdot \frac{3}{4}) = -M + \frac{M}{2}(3) = \frac{M}{2}$. Therefore $M \leq 2(g-1) < 2(g+3)$. Thus if $g_0 = 0$ and $r = 4$, $M \leq 2(g+3)$ is the upper bound for the size of the abelian group A with $\sigma^0(A) \geq 2$ acting faithfully on a compact surface of genus g, unless $A = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

Case 2: Let $r = 5$.

If $m_1 = 2$, $m_2 = 2$, and $m_3 = 2$, then $M \leq 8m_4$ and $M \leq 8m_5$. By equation (2), $g - 1 \geq -M + \frac{M}{2}\left(3 \cdot \frac{1}{2} + 2(1 - \frac{8}{M})\right) = -M + \frac{M}{2}\left(\frac{7}{2} - \frac{16}{M}\right) = \frac{(3M-32)}{4}$. Comparing the two bounds yields $\frac{(M-8)}{2} \leq \frac{(3M-32)}{4}$, which simplifies to $2M - 16 \leq 3M - 32$, so $16 \leq M$. This is the point at which the first bound, $\frac{(M-8)}{2}$, drops below and thus includes all possible values greater than $\frac{(3M-32)}{4}$. There are no abelian groups with $M < 16$ and $r = 5$ so $g - 1 \geq \frac{(M-8)}{2}$, or $M \leq 2(g+3)$, is the upper bound for M in this case.

If $m_1 = 2$, $m_2 = 2$, and $m_3 \geq 3$, then $m_4 \geq m_5 \geq 3$. By equation (2), $g - 1 \geq -M + \frac{M}{2}\left(2 \cdot \frac{1}{2} + 3 \cdot \frac{2}{3}\right) = -M + \frac{M}{2}(3) = \frac{M}{2}$. Therefore $M \leq 2(g-1) < 2(g+3)$.

If $m_1 \geq 2$ and $m_2 \geq 3$, then $m_3 \geq 3$, $m_4 \geq 3$ and $m_5 \geq 3$. By equation (2), $g - 1 \geq -M + \frac{M}{2}\left(\frac{1}{2} + 4 \cdot \frac{2}{3}\right) = -M + \frac{M}{2}(\frac{19}{6}) = \frac{7M}{12}$. Therefore $M \leq \frac{12}{7}(g-1) < 2(g+3)$. Thus if $g_0 = 0$ and $r = 5$, $M \leq 2(g+3)$ is the upper bound for the size of the abelian group A with $\sigma^0(A) \geq 2$ acting faithfully on a compact surface of genus g.

Case 3: Let $r \geq 6$.

Since every m_i is a positive integer greater than 1, $1 - \frac{1}{m_i} \geq \frac{1}{2}$, for all i. So $g - 1 \geq -M + \frac{M}{2}\sum_{i=1}^{6}(\frac{1}{2}) \geq -M + \frac{3M}{2} \geq \frac{M}{2}$, by equation (2). Thus for $g_0 = 0$ and $r \geq 6$, $M \leq 2(g-1) < 2(g+3)$, regardless of the m_i values.

Consequently, given any possible values of g_0, r, and the periods, the finite abelian group $A \neq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ acting faithfully on a compact surface of genus g will have order $M \leq 2(g+3)$.

□

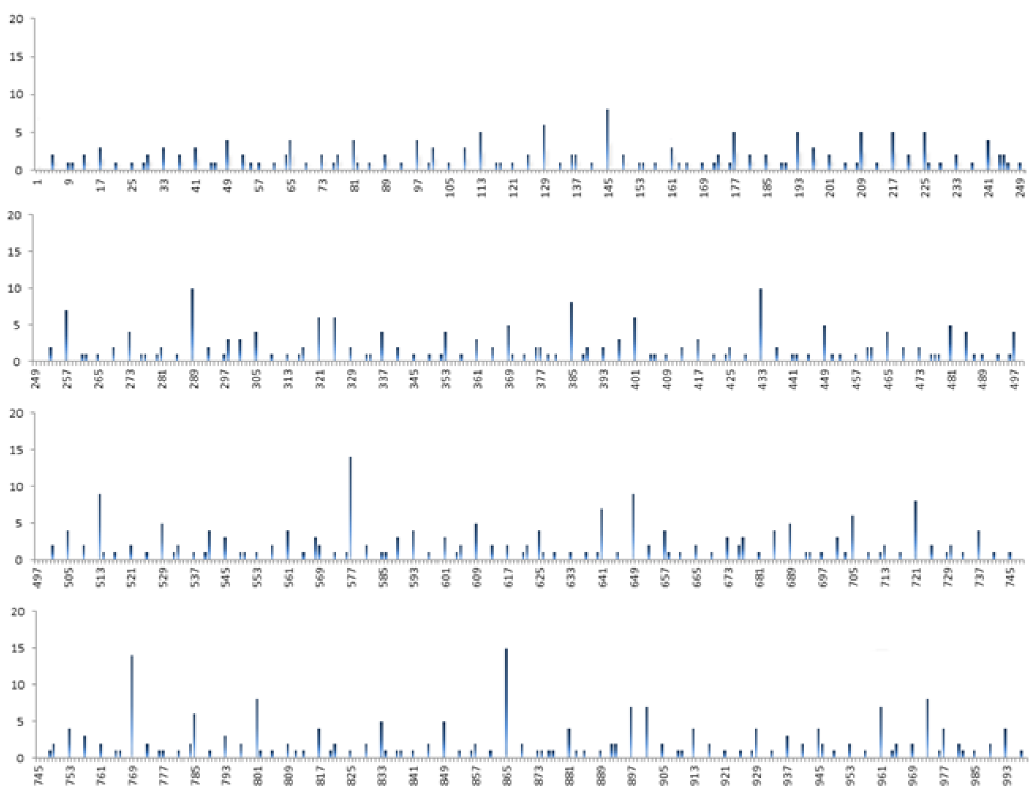

Figure 3: Frequency of σ^0 for Finite Abelian Groups

The main application of this bound for our research was to determine the completeness of the genus data. Although we calculated the strong symmetric genus value of every abelian group with order 2000 or less using MAGMA, it was possible that a finite abelian group of larger order could have a genus value that fell within our data range. We wanted to know the highest integer, n, for which we could positively determine whether or not n was the genus of an abelian group. Additionally, we would be able to identify every finite abelian group with the strong symmetric genus of n. Using our bound $2(g+3) = 2000$, when $g = 997$. This means that for every integer n up to and including 997, if n is the genus of an abelian group, then that group will have order 2000 or less. Thus, our calculations yield a classification of all finite abelian groups G with strong symmetric genus $\sigma^0(G) \leq 997$. Figure 3 shows the frequency of every strong symmetric genus value up to 997 for finite abelian groups. This added completeness of the genus data allowed us to better observe patterns and gaps within the strong symmetric genus spectrum.

5 Group Families with Arithmetic Genus Formulas

Using Maclachlan's formula for the genus of a group, stated in Theorem 1, we obtain a general formula for the strong symmetric genus of a family of groups.

Theorem 3. *Let $A = \mathbb{Z}_a \times \mathbb{Z}_a \times \mathbb{Z}_{an}$, where $a \,\epsilon\, \mathbb{Z}$ and $a \geq 2$. Then the strong symmetric genus of A is given by $\sigma^0(A) = (a^3 - a^2)n - (a^2 - 1)$.*

Proof. Let $A = \mathbb{Z}_a \times \mathbb{Z}_a \times \mathbb{Z}_{an}$, where $a \,\epsilon\, \mathbb{Z}$ and $a \geq 2$. So A is a group of rank 3 with invariants a, a, and an and $s = 3$. First, it is necessary to determine which value of γ yields the minimum value of the right hand side of Maclachlan's formula. Since $s = 3$, $0 \leq 2\gamma < s$ so $\gamma = 0$ or $\gamma = 1$.

Case 1: $\gamma = 0$

$$-2 + \sum_{i=1}^{3}(1 - \tfrac{1}{m_i}) + (1 - \tfrac{1}{m_3}) = -2 + 2(1 - \tfrac{1}{a}) + 2(1 - \tfrac{1}{an}) = 2 - \tfrac{2}{a} - \tfrac{2}{an}$$

Case 2: $\gamma = 1$

$$0 + \sum_{i=1}^{1}(1 - \tfrac{1}{m_i}) + (1 - \tfrac{1}{m_1}) = 2(1 - \tfrac{1}{a}) = 2 - \tfrac{2}{a}$$

For groups of the form $\mathbb{Z}_a \times \mathbb{Z}_a \times \mathbb{Z}_{an}$, the case $\gamma = 0$ will always yield the minimum value because $2 - \frac{2}{a} - \frac{2}{an} < 2 - \frac{2}{a}$. This allows us to calculate a general genus formula for these groups by setting this minimum value equal to the left hand side of the equation. Since

$|A| = a \cdot a \cdot an = a^3n$, we have:

$$\frac{2(\sigma^0 - 1)}{(a^3n)} = 2 - \tfrac{2}{a} - \tfrac{2}{an}$$
$$2(\sigma^0 - 1) = 2a^3n - 2a^2n - 2a^2$$
$$\sigma^0 = (a^3 - a^2)n - (a^2 - 1)$$

□

The formulas for the first four groups in this family are shown below:

- $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2n} : \sigma^0 = (2^3 - 2^2)n - (2^2 - 1) = 4n - 3$
- $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3n} : \sigma^0 = (3^3 - 3^2)n - (3^2 - 1) = 18n - 8$
- $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_{4n} : \sigma^0 = (4^3 - 4^2)n - (4^2 - 1) = 48n - 15$
- $\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_{5n} : \sigma^0 = (5^3 - 5^2)n - (5^2 - 1) = 100n - 24$

By a similar process, the strong symmetric genus formula for many other families of abelian groups can be determined.

Theorem 4. *Let $A = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3n}$. Then the strong symmetric genus of A is given by $\sigma^0(A) = 81n - 26$.*

Proof. Let $A = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3n}$. The invariants of A are 3, 3, 3, and $3n$ and $s = 4$ because A is a group of rank 4. First, it is necessary to determine which value of γ yields the minimum value of the right hand side of Maclachlan's formula. Since $s = 4$, $0 \leq 2\gamma \leq s$ so $\gamma = 0$, $\gamma = 1$, or $\gamma = 2$.

Case 1: $\gamma = 0$

$$-2 + \sum_{i=1}^{4}(1 - \tfrac{1}{m_i}) + (1 - \tfrac{1}{m_4}) = -2 + 3(1 - \tfrac{1}{3}) + 2(1 - \tfrac{1}{3n}) = 2 - \tfrac{2}{3n}$$

Case 2: $\gamma = 1$

$$0 + \sum_{i=1}^{2}(1 - \tfrac{1}{m_i}) + (1 - \tfrac{1}{m_1}) = 3(1 - \tfrac{1}{3}) = 3 - \tfrac{3}{3} = 2$$

Case 3: $\gamma = 2$

$$2 + \sum_{i=1}^{0}(1 - \tfrac{1}{m_i}) + (1 - \tfrac{1}{m_0}) = 2 + 0 + 1 - 1 = 2$$

For groups of the form $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3n}$, the case $\gamma = 0$ will always yield the minimum value because $2 - \frac{2}{3n} < 2$. This allows us to calculate a general genus formula for these groups by setting this minimum value equal to the left hand side of the equation. Since $|A| = 3 \cdot 3 \cdot 3 \cdot 3n = 81n$, we have:

$$\frac{2(\sigma^0 - 1)}{81n} = 2 - \tfrac{2}{3n}$$
$$2(\sigma^0 - 1) = 162n - 54$$
$$\sigma^0 = 81n - 26$$

□

The proofs of the following two theorems are similar to those of Theorems 3 and 4. We omit the details.

Theorem 5. *Let $A = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2n}$. Then the strong symmetric genus of A is given by $\sigma^0(A) = 12n - 7$.*

Theorem 6. *Let $A = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2n}$. Then the strong symmetric genus of A is given by $\sigma^0(A) = 32n - 15$.*

A summary of the genus formulas for families of abelian groups that were calculated in this section is given in Table 1.

Group	Genus	
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2n}$	$\sigma^0 = 4n - 3$	$\equiv 1 \pmod 4$
$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3n}$	$\sigma^0 = 18n - 8$	$\equiv 10 \pmod{18}$
$\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_{4n}$	$\sigma^0 = 48n - 15$	$\equiv 33 \pmod{48}$
$\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_{5n}$	$\sigma^0 = 100n - 24$	$\equiv 76 \pmod{100}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2n}$	$\sigma^0 = 12n - 7$	$\equiv 5 \pmod{12}$
$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{3n}$	$\sigma^0 = 81n - 26$	$\equiv 55 \pmod{81}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2n}$	$\sigma^0 = 32n - 15$	$\equiv 17 \pmod{32}$

Table 1: Genus Formulas for Selected Families of Groups

6 Density of the Strong Symmetric Genus Spectrum

The above formulas can be used to find a lower bound for the density of the spectrum of strong symmetric genus values for all finite abelian groups. The density δ of a set is calculated by taking the limit of the number of integers in the set that are less than or equal to n divided by n, as n tends towards infinity. So if the set $J = \{g \mid g$ is the strong symmetric genus of a finite abelian group} and the counting function $F(n)$ is defined as $F(n) = \{$the number of values in J that are $\leq n\}$, then:

$$\delta(J) = \lim_{n\to\infty} \frac{F(n)}{n}$$

The genus formulas in Table 1 can be combined to produce a lower bound for the density of all integers that are genus values for abelian groups. The calculation of the densities of arithmetic sequences is very straight forward.

The number of positive integers congruent to 1 (mod 4) that are less than or equal to n is $\frac{n}{4}$ because every fourth number will have a remainder of 1 when divided by 4. Thus, $\delta(x \equiv 1 \pmod 4) = \lim_{n\to\infty} (\frac{n/4}{n}) = \frac{n}{4n} = \frac{1}{4}$. The density of each of the arithmetic sequences can be calculated the same way. It is possible for there to be overlap between two or more of these arithmetic sequences which must be calculated and subtracted from the combined density value. Numbers congruent to $1 \pmod 4$ are all odd while numbers congruent to 10 (mod 18) are all even, so there is no overlap to consider worry about between those two sets. However, every number congruent to $33 \pmod{48}$, $5 \pmod{12}$ or $17 \pmod{32}$ is congruent to $1 \pmod 4$ and must be left out of the density calculation.

The remaining two genera families will have some overlap, but not the complete overlap that was seen in previous cases. The numbers congruent to $76 \pmod{100}$ are all even so they overlap with $10 \pmod{18}$. By reducing these into a system of congruencies with relatively prime moduli and then applying the Chinese Remainder Theorem this overlap can be calculated to be once every 900 numbers, or $1/900$. Finally, the numbers congruent to 55 (mod 81) can be either even or odd so they must be compared to all of the remaining sets in our calculation. The overlap between $55 \pmod{81}$ and $1 \pmod 4$ is $1/324$. The overlap between $55 \pmod{81}$ and $10 \pmod{18}$ is $1/162$. The overlap between $55 \pmod{81}$ and 76 (mod 100) is $1/8100$. Since $55 \pmod{81}$, $10 \pmod{18}$, and $76 \pmod{100}$ all give even values, it is possible for there to be values congruent to all three moduli which were counted twice in the overlap calculations. This double elimination occurs once in every 8100 numbers, or $1/8100$.

Genus Formula	Density
$\sigma^0 = 4n - 3$	$\delta(x \equiv 1 \pmod 4) = 1/4$
$\sigma^0 = 18n - 8$	$\delta(x \equiv 10 \pmod{18}) = 1/18$
$\sigma^0 = 100n - 24$	$\delta(x \equiv 76 \pmod{100}) = 1/100$
$\sigma^0 = 81n - 26$	$\delta(x \equiv 55 \pmod{81}) = 1/81$

Table 2: Densities of Selected Genus Sequences

With the density values and their overlaps calculated, it is now possible to calculate a lower bound to the density of the spectrum of strong symmetric genus values of all finite abelian groups. Eliminating the genera families that have complete overlap gives a set of densities that will be needed for this calculation, shown in Table 2.

Theorem 7. *Let $J = \{g \mid g$ is the strong symmetric genus for an abelian group$\}$. The density of J is $\delta(J) \geq \frac{643}{2025} > \frac{5}{16}$.*

Proof. The families of groups for which this paper has found genus formulas represent a subset of all abelian groups. Thus the density of strong symmetric genera for the families that have been analyzed will be less than or equal to the density of the full spectrum of strong symmetric genus values for all finite abelian groups. The density of all the arithmetic sequences is the sum of their individual densities, minus the density of each calculated overlap, with the density of the double cancellation added back in. Therefore,

$$\delta(J) \geq \frac{1}{4} + \frac{1}{18} + \frac{1}{81} + \frac{1}{100} - \frac{1}{162} - \frac{1}{324} - \frac{1}{900} - \frac{1}{8100} + \frac{1}{8100} = \frac{643}{2025}$$

□

7 Gaps in the Spectrum

With a lower bound set for the density of the spectrum of strong symmetric genus values for all finite abelian groups it is natural to then look for a corresponding upper bound. This involves analyzing those positive integers which cannot be the strong symmetric genus for any finite abelian group and determining the number of these gaps in the spectrum of strong symmetric genus values.

Theorem 8. *Suppose p is an odd prime and q equals 1 or a prime. Assume p and q are not both 3. Then there is no finite abelian group A such that $\sigma^0(A) = pq + 1$.*

Proof. Let the finite abelian group A have the canonical form $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_s}$, where the m_i values are the invariants. Assume that $\sigma^0(A) = pq+1$. Let $M = |A|$ and let $\hat{m}_i = \frac{M}{m_i}$. Since $\sigma^0(A) > 1$, we know $s = \text{rank}(A) \geq 3$. Maclachlan's formula applies to groups of order greater than 9. The only group of rank 3 that has an order 9 or less is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and its strong symmetric genus is 1. Thus, $|A| > 9$ and $\sigma^0(A)$ is given by Maclachlan's formula.

For some γ satisfying $\frac{s}{2} \geq \gamma \geq 0$, Maclachlan's formula gives:

$$\frac{2(\sigma^0(A) - 1)}{|A|} = 2(\gamma - 1) + \sum_{i=1}^{s-2\gamma}(1 - \tfrac{1}{m_i}) + (1 - \tfrac{1}{m_{s-2\gamma}}) \tag{3}$$

Let $n = s - 2\gamma$. Using M and $\hat{m}_i$ the genus equation (3) can be re-written as:

$$2(\sigma^0(A) - 1) = 2M(\gamma - 1) + \sum_{i=1}^{n}(M - \hat{m}_i) + (M - \hat{m}_n)$$

$$2(\sigma^0(A) - 1) = M(2\gamma + n - 1) - \sum_{i=1}^{n}(\hat{m}_i) - \hat{m}_n$$

Now assume that the genus $\sigma^0(A) = pq + 1$, where p is an odd prime and q equals 1 or a prime with $p \neq 3$ or $q \neq 3$. Thus $\sigma^0(A) - 1 = pq$. This makes the equation:

$$2pq = M(2\gamma + n - 1) - \sum_{i=1}^{n}(\hat{m}_i) - \hat{m}_n \tag{4}$$

Case 1: Let $s = 3$.

Then the invariants of A satisfy $m_2 = am_1$ and $m_3 = bm_2 = abm_1$, for some $a, b \,\epsilon\, \mathbb{Z}^+$ Therefore, $M = m_1 \cdot m_2 \cdot m_3 = a^2b(m_1)^3$, $\hat{m}_1 = \frac{M}{m_1} = a^2b(m_1)^2$, $\hat{m}_2 = \frac{M}{m_2} = ab(m_1)^2$, and $\hat{m}_3 = \frac{M}{m_3} = a(m_1)^2$. Since $s = 3$, obviously $\gamma = 0$ or $\gamma = 1$.

First assume $\gamma = 0$. Then $n = 3$ and equation (4) simplifies to:

$$\begin{aligned}
2pq &= 2M - \hat{m}_1 - \hat{m}_2 - \hat{m}_3 - \hat{m}_3 \\
&= 2a^2b(m_1)^3 - a^2b(m_1)^2 - ab(m_1)^2 - 2a(m_1)^2 \\
&= a(m_1)^2(2abm_1 - ab - b - 2) \\
&= a(m_1)^2 \cdot x, \text{ where } x = 2abm_1 - ab - b - 2
\end{aligned}$$

Since $2pq$ has at most three prime factors, so does $a(m_1)^2 \cdot x$. Since $m_1 \geq 2$, it follows that either $a = 1$ or $x = 1$ or both.

Suppose $a > 1$. Then $x = 1$, but $x = 2abm_1 - ab - b - 2 = ab(2m_1 - 1) - b - 2$. So $x \geq 3ab - b - 2 \geq b(3a - 1) - 2 \geq b(5) - 2 \geq 3$. This is a contradiction. Hence $a = 1$ and x is 1 or a prime. Now $x = 2bm_1 - b - b - 2 = 2b(m_1 - 1) - 2 = 2(b(m_1 - 1) - 1)$. Thus $2pq = 1 \cdot (m_1)^2 \cdot x = 2(m_1)^2(b(m_1 - 1) - 1)$. Let $y = b(m_1 - 1) - 1$, so $2pq = 2(m_1)^2 \cdot y$. If $m_1 = 2$, then $y = b - 1$, and therefore $2pq = 2(m_1)^2 \cdot 1 = 2^3$. This would force $p = q = m_1 = 2$ which is not possible because p is an odd prime. If $m_1 = 3$, then $y = 2b - 1$, so it is possible for $2pq = 2(m_1)^2 \cdot 1 = 2 \cdot 3^2$ when $b = 1$. However this forces $p = q = 3$, which is excluded. Finally, if $m_1 > 3$, then $y > 3b - 1 > 1$, for all values of b. This means that $2pq = 2(m_1)^2 \cdot y$ is the product of four integers not equal to 1, which is a contradiction.

Now assume $\gamma = 1$. Then $n = 1$ and equation (4) simplifies to:

$$\begin{aligned}
2pq &= 2M - 2\hat{m}_1 = 2a^2b(m_1)^3 - 2a^2b(m_1)^2 \\
&= 2a^2b(m_1)^2(m_1 - 1) = 2a^2b(m_1)^2 \cdot x, \text{ where } x = m_1 - 1
\end{aligned}$$

Since $2pq$ has at most three prime factors, so does $2a^2b(m_1)^2 \cdot x$. This can only be true if a, b, and x all equal 1. However, this would force $p = q = m_1 = 2$ which is not possible because p is an odd prime, so we have a contradiction in this case as well.

Case 2: Let $s > 3$.

Since $s > 3$, M is the product of at least 4 invariants and every $\hat{m}_i$ is the product of at least 3 invariants. Thus $(m_1)^3 | \hat{m}_i$ for all $\hat{m}_i$ and $(m_1)^4 | M$.

If $\frac{s}{2} > \gamma \geq 0$ it is possible to factor out $(m_1)^3$ from the right hand side of equation (4).

$$2pq = (m_1)^3 \left(\frac{M}{(m_1)^3}(2\gamma + n - 1) - \sum_{i=1}^{n} \left(\frac{\hat{m}_i}{(m_1)^3} \right) - \frac{\hat{m}_n}{(m_1)^3} \right)$$

Therefore $x = \frac{M}{(m_1)^3}(2\gamma + n - 1) - \sum_{i=1}^{n}(\frac{\hat{m}_i}{(m_1)^3}) - \frac{\hat{m}_n}{(m_1)^3}$ is an integer. This simplifies the genus equation to $2pq = (m_1)^3 \cdot x$. Since $m_1 \geq 2$, the only possibility is $x = 1$ and $p = q = m_1 = 2$. This is a contradiction, since p is an odd prime.

When s is even, it is possible that $\gamma = \frac{s}{2}$. In this case $n = 0$ and equation (4) simplifies to:

$$2pq = M(s-1) - \sum_{i=1}^{0}(\hat{m}_i) - \hat{m}_0$$

According to Maclachlan [8, pp. 711], when $s = 2\gamma$, the m_0 is to be interpreted as 1. Thus $\hat{m}_0 = M$ and $2pq = M(s-2)$. That means it is possible to factor out $(m_1)^4$ from the right hand side of the equation.

$$2pq = (m_1)^4 \cdot \frac{M}{(m_1)^4} \cdot (s-2)$$

Since $(m_1)^4 | M$, the number $x = \frac{M}{(m_1)^4} \cdot (s-2)$ is an integer. This makes the genus equation:

$$2pq = (m_1)^4 \cdot x, \text{ where } x \in \mathbb{Z}$$

Since $2pq$ has at most three prime factors, so does $(m_1)^4 \cdot x$. The number $(m_1)^4 \cdot x$ is the product of four or more integers not equal to 1. This is a contradiction.

In conclusion, when p is an odd prime and q equals 1 or a prime with $p \neq 3$ or $q \neq 3$, then there is no finite abelian group A such that $\sigma^0(A) = pq + 1$.

□

Theorem 9. *There are an infinite number of gaps in the spectrum of strong symmetric genera for finite abelian groups.*

Proof. It has long been known that there are an infinite number of prime numbers. It follows that there must therefore be an infinite number of integers of the form $g = pq + 1$, where p is an odd prime and q is 1 or a prime. Since no such value g can be the strong symmetric genus of a finite abelian group, there are an infinite number of gaps in the spectrum of strong symmetric genera for abelian groups.

□

8 Possible Further Research

During our study into the genus spectrum of abelian groups we encountered several topics that we feel merit further research. Two such points of interest are the density of the gaps in the genus spectrum and patterns in the occurrence and extreme frequency for some genus values.

We proved that there does not exist any finite abelian group with strong symmetric genus $g = pq + 1$, where p is an odd prime, q equals 1 or a prime, and p and q are not both 3. Using our data from MAGMA we found 671 integers less than or equal to 997 that are not the strong symmetric genus of any finite abelian group. Of these 671 gaps, 458 or 68.2% are of the form $g = pq + 1$. Although this set of numbers accounts for many of the gaps in the spectrum of σ^0, it does not contribute to an upper bound for the density of genera. This is because the density of numbers that are the product of two primes is zero [7]. If an arithmetic sequence of gaps in the genus spectrum were discovered an upper bound for the density of genus values could be revealed. For our full classification of all finite abelian groups with strong symmetric genus less than or equal to 997, there were only 326 integers that were genus values, resulting in a density of approximately .327. This data supports the following conjecture:

Conjecture 1. *Let $J = \{g \mid g$ is the strong symmetric genus for an abelian group$\}$. The density of J is $\delta(J) < \frac{1}{2}$.*

The genus data we calculated revealed several interesting patterns within the genus spectrum. For example, there were significantly fewer even genus values than odd. For $\sigma^0 \leq 997$, there were only 64 even values, compared to the 694 odd values. Equally as intriguing was the fact that there were only 8 strong symmetric genus values less than 997 that were congruent to $3 \pmod 4$. We also observed several occurrences of extreme frequency in the genus values, which can be easily seen in Figure 3. The mode of the genus values, 865, occurred with 15 groups and the next most frequent values, 577 and 865, occurred in 14 groups. Conversely, every even genus value occurred for only 3 or fewer groups. Further research into genus frequencies could determine if there is a maximum to the number of finite abelian groups that can share the same strong symmetric genus value and explain why some genus values are more frequent than others. If the genus values of finite abelian groups beyond those in the MAGMA Small Groups Library were computed, it is possible that other similar patterns in the genus spectrum could be discovered and investigated.

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