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GRAPHS WITH MINIMAL
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DIMENSION

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Abstract. We generalize a theorem by Brown and Nowakowski on the well-covered dimension of chordal graphs. Furthermore, we prove that the well-covered dimension of any Sierpinski gasket graph of order at least 2 is equal to 3, the simplicial clique number.

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1 Introduction

We start by giving a few graph theoretical definitions, leading up to the concepts of well-covered weighting, space, and dimension, which are the main object of study in this paper. We refer the reader to the text by West [11] for any notions we use but do not define, and for the more detailed treatment of related ideas.

Throughout, we assume that graphs are simple, connected, and undirected. A *graph* is a set of *vertices* $V(G)$ and *edges* $E(G)$ with specified connectivity relations. We usually write $G = (V(G), E(G))$ for a graph and vw for the edge connecting the vertices v and w . Also, we say that the *order* of G is $|V(G)|$ and that two vertices v and w are *adjacent* if there is an edge between them.

Let $I \subseteq V(G)$. The *neighborhood* of I , denoted by $N(I)$, is the set of all vertices that are adjacent to any vertex in I . The *closed neighborhood* of I is $N[I] = N(I) \cup I$. When $I = \{v\}$, we write $N(v)$ and $N[v]$. The *degree* of a vertex v is $|N(v)|$. A *complete graph* of order n , denoted by K_n , is a graph where each of the n vertices has degree $n - 1$; that is, each vertex is adjacent to every other vertex.

A *subgraph* of a graph $G = (V_1, E_1)$ is a graph $H = (V_2, E_2)$ with $V_2 \subseteq V_1$ and $E_2 \subseteq E_1$. The set V_2 *induces* the subgraph H if E_2 is the set of all edges of G that connect any two vertices in V_2 . We say that $X \subseteq V(G)$ is a *clique* of G if X induces the subgraph $K_{|X|}$. A clique of G is a *maximal clique* if it is not properly contained in any other clique of G . A graph is *chordal* if it has no induced cycles of length at least 4. A more thorough introduction to these ideas can be found in a 1985 work by Tarjan [9].

For conciseness, we write $\mathbb{N}_{\leq a}$ for $[1, a] \cap \mathbb{N}$, and $\mathbb{N}_{\geq a}$ for $[a, \infty) \cap \mathbb{N}$, where $\mathbb{N} = \{1, 2, 3, \dots\}$.

Our results rely heavily on the definitions that follow.

Definition 1. *Let G be a graph and $L \subset V(G)$. The set L is independent if no two vertices in L are adjacent. If L is not properly contained in any independent set of G , then we say that L is a maximal independent set (MIS) of G .*

Definition 2. *Let \mathbb{F} be a field and G be a graph.*

1. *A weighting of G is a map $f : V(G) \rightarrow \mathbb{F}$. If a weighting f is such that*

$$\sum_{v \in \mathcal{M}} f(v)$$

is constant for every MIS \mathcal{M} of G , then f is said to be a well-covered weighting of G .

2. *The vector space (over \mathbb{F}) of all well-covered weightings of G , which we designate with \mathcal{V} , is called the well-covered space of G .*
3. *The well-covered dimension of G over \mathbb{F} is $wcdim(G, \mathbb{F}) = \dim_{\mathbb{F}}(\mathcal{V})$.*

The well-covered dimension of any graph is, clearly, at most its order, as remarked by Caro and Yuster [4].

Remark 1. *There are graphs whose well-covered dimension depends on the characteristic of the field \mathbb{F} . Examples of these graphs can be found in the works of Brown and Nowakowski [2], and Birnbaum, McDonald, Kuneli, Urabe, and Vega [1]. The graphs we study in this article have well-covered dimension independent of field characteristic. Hence, we omit explicit reference to \mathbb{F} and write $wcdim(G)$ instead of $wcdim(G, \mathbb{F})$.*

Definition 3. *Let G be a graph.*

1. $v \in V(G)$ is a simplicial vertex of G if $N[v]$ is a maximal clique.
2. A clique of G that contains at least one simplicial vertex is a simplicial clique.
3. $\mathcal{C}(G)$ is the set of all simplicial cliques of G and $sc(G) := |\mathcal{C}(G)|$. We say that $sc(G)$ is the simplicial clique number of G and we denote the i -th member of $\mathcal{C}(G)$ by C_i .
4. A clique covering of G is a family of cliques whose union is $V(G)$.

The notions of well-covered weighting and well-covered space of a graph originate from the concept of well-coveredness of a graph. Well-covered graphs were first studied by Plummer [7], [8]. These graphs have the property that all of their *MIS*s have equal cardinality. Caro, Elingham and Ramey [3] noticed that the concept of well-coveredness of graphs may be stated in terms of weights of vertices. Let $f : V(G) \rightarrow \mathbb{Z}$ be such that $f(v) = 1$, for all $v \in V(G)$. Then, G is well-covered if for every *MIS* \mathcal{M} of G , $\sum_{v \in \mathcal{M}} f(v) = |\mathcal{M}|$ is constant.

It is now possible to ask the following central question: given a graph G , what properties need a weighting of G , f , have in order for $\sum_{v \in \mathcal{M}} f(v)$ to be constant, for any *MIS* \mathcal{M} of G ? In other words, what do the well-covered weightings of G look like? It is known that the well-covered space of any graph is non-empty [3], for the zero-function is a trivial well-covered weighting of any graph. But only some graphs have well-covered weightings other than the zero-function. Examples of graphs with unique well-covered weighting the zero-function are cycles of length at least 8 [1]. Thus, a graph that is not well-covered *can be made well-covered in terms of its vertex weights* via a well-covered weighting.

Caro and Yuster [4] studied the notions of well-covered space and well-covered dimension in the more general setting of hypergraphs H , and weightings whose domain need not be $V(H)$.

Brown and Nowakowski [2] proved that for any graph G , $wcdim(G) \geq sc(G)$, and that equality holds if G is chordal. Thus, chordal graphs have *minimal well-covered dimension*. In the following sections, we show that there is a large family of graphs with minimal well-covered dimension (just like chordal graphs). This family contains the class of chordal graphs properly, and thus allows us to generalize Brown and Nowakowski's result. There are several open questions in the well-covered dimension theory of graphs. Some of these questions are the following:

1. What exactly does the well-covered dimension of a graph tell us about a graph?

2. How exactly are clique partitions and coverings, and the well-covered dimension of a graph related?
3. Find a non-trivial upper bound for the well-covered dimension of any graph; that is, an upper bound other than the order of a graph.
4. Classify all graphs according to their well-covered dimension. Suggestion: start by finding the largest possible class of graphs to which our results apply.

The remainder of this paper is organized as follows. In section (2), we define the class of *simplicial clique covered graphs*, the *SCCG* class, and prove that any graph in this class has well-covered dimension equal to the number of simplicial cliques. In section (3), we introduce the class of *simplicial clique sums* of a chordal graph with a *SCCG*, which results in a generalization of our main theorem in section (1) and Brown and Nowakowski's theorem on the well-covered dimension of chordal graphs. In section (4), we prove that all Sierpinski gasket graphs of order at least 2, have well-covered dimension equal to 3, which happens to be the simplicial clique number. This suggests that the main theorem in section (3) could be generalized further. A full generalization of this theorem would be a big step in answering open question 4., and would advance our overall understanding of the well-covered dimension theory of graphs.

2 The well-covered dimension of simplicial clique covered graphs

In this section we investigate a class of graphs that overlaps, but is not identical to, the class of chordal graphs. Our goal is to prove that the well-covered dimension of a graph in this class is equal to its simplicial clique number. From now on, all well-covered weightings are assumed to be non-trivial.

Definition 4. A graph G is a simplicial clique covered graph (SCCG) if $\mathcal{C}(G) \neq \emptyset$ and $\mathcal{C}(G)$ is a clique covering of G .

We now we present some technical definitions and notation.

Definition 5. Let G be a SCCG.

1. The connection set, \mathcal{W} , of G is the set of vertices that belong to at least two simplicial cliques of G .
2. For an independent set $I_m \subset \mathcal{W}$, $S(I_m)$ is the set of all simplicial cliques that are not contained in the closed neighborhood of I_m and $s_m := |S(I_m)|$.
3. $\overline{S}(I_m)$ is the complement of $S(I_m)$ in $\mathcal{C}(G)$.
4. For each $C_i \in \mathcal{C}(G)$, $W_i := C_i \cap \mathcal{W}$.

Remark 2. From the previous definition, it is evident that

1. I_m is a MIS of G if and only if $S(I_m) = \emptyset$.

2.

$$\mathcal{W} = \bigcup_{i=1}^{sc(G)} W_i.$$

Our first result is a classification of the MISs of any SCCG. We shall soon see that MISs reveal much about well-covered spaces and their dimension.

Theorem 1. Let G be a SCCG and \mathcal{M} be a MIS of G . Then, either

1. $\mathcal{M} = \{v_1, \dots, v_{sc(G)}\}$, where each $v_i \in \mathcal{M}$ is a simplicial vertex in a distinct $C_i \in \mathcal{C}(G)$,
or

2. $\mathcal{M} = I_m \cup \mathcal{M}'$, where I_m is an independent set of \mathcal{W} and \mathcal{M}' consists of one simplicial vertex per $C_i \in S(I_m)$.

Proof. Assume that \mathcal{M} is not as in (1). Then, $\mathcal{M} \cap \mathcal{W} \neq \emptyset$, and thus $\mathcal{M} \cap \mathcal{W} = I_m$, for some independent set I_m of \mathcal{W} .

If $\mathcal{M} - I_m = \emptyset$, then I_m is a MIS of G , and thus (see Remark 2) \mathcal{M} is as in (2) with $\mathcal{M}' = \emptyset$.

If $\mathcal{M} - I_m = \mathcal{M}' \neq \emptyset$, then it is clear that $\mathcal{W} \cap \mathcal{M}' = \emptyset$. It follows that each $v_i \in \mathcal{M}'$ must be simplicial and non-adjacent to vertices in I_m . Thus, \mathcal{M} is as in (2). \square

With all notation in place and with the help of Theorem 1, it is possible to count the MISs of any SCCG.

Theorem 2. Let G be a SCCG. Then, G has exactly

$$|\mathcal{I}| + \prod_{i=1}^{sc(G)} |C_i - W_i| + \sum_{m=1}^M \prod_{C_i \in S(I_m)} |C_i - W_i|$$

maximal independent sets, where \mathcal{I} is the family of all independent sets of \mathcal{W} that are MISs of G .

Proof. By Theorem 1, each MIS of G takes one of two forms. Let \mathcal{M} be a MIS of G .

Suppose that \mathcal{M} is of form (1) in Theorem 1. Then, each $v_i \in \mathcal{M}$ is exactly one out of the $|C_i - W_i|$ simplicial vertices of $C_i \in \mathcal{C}(G)$. Hence, there are

$$\prod_{i=1}^{sc(G)} |C_i - W_i|$$

MISs of this form.

Suppose that \mathcal{M} is of form (2) in Theorem 1 and that G has exactly M independent sets I_m . Since each $v_i \in \mathcal{M} - I_m$ can be exactly one out of the $|C_i - W_i|$ simplicial vertices of $C_i \in S(I_m)$, there are

$$\sum_{m=1}^M \prod_{C_i \in S(I_m)} |C_i - W_i|$$

MIS s of this form.

Observe that

$$\prod_{C_i \in S(I_m)} |C_i - W_i|$$

vanishes for those I_m that are MIS s of G because $S(I_m) = \emptyset$. Thus, we must add all independent sets of \mathcal{W} that are MIS s of G . Letting \mathcal{I} be the family of all such sets, we add $|\mathcal{I}|$ MIS s to complete our count. \square

Next, we look at the defining properties of well-covered weightings of $SCCG$ s.

Lemma 1. *Let G be a $SCCG$. Let \mathcal{W} be a connection set and f be a well-covered weighting of G . Then, f is constant on $C_i - W_i$ for each $C_i \in \mathcal{C}(G)$.*

Proof. Let \mathbb{F} be a field and let $f : V(G) \rightarrow \mathbb{F}$ be a well-covered weighting of G .

Pick an arbitrary $C_i \in \mathcal{C}(G)$. By (1) of Theorem 1 together with Theorem 2, we can find $|C_i - W_i|$ MIS s of G of cardinality $sc(G)$ that have $sc(G) - 1$ vertices in common and as the $sc(G)$ -th vertex, a distinct $v_i \in C_i - W_i$. Then, all of the vertices in $C_i - W_i$ have the same weight under f . Since C_i was chosen arbitrarily, the result follows. \square

Lemma 2. *Let G be a $SCCG$. Let \mathcal{W} be a connection set and f be a well-covered weighting of G . For any $w \in \mathcal{W}$,*

$$f(w) = \sum f(v),$$

where the sum is taken over a set of simplicial vertices, each of which belongs to a distinct $C_i \in \overline{S}(\{w\})$.

Proof. Let $w \in \mathcal{W}$ and let $I_m = \{w\}$. For any $i \in \mathbb{N}_{\leq sc(G)}$, let u_i and v_i be simplicial vertices of G such that each u_i belongs to a distinct $C_i \in S(I_m)$ and each v_i belongs to a distinct $C_i \in \overline{S}(I_m)$. Consider a set $\mathcal{M} = I_m \cup \{u_1, \dots, u_{s_m}\}$ and a set $\{v_1, \dots, v_{sc(G)-s_m}\}$. Note that \mathcal{M} is of form (2) in Theorem 1 and $\mathcal{M}' = (\mathcal{M} - I_m) \cup \{v_1, \dots, v_{sc(G)-s_m}\}$ is of form (1) in Theorem 1. It follows that

$$f(w) = \sum f(v),$$

where each v in the sum is simplicial and belongs to a distinct $C_i \in \overline{S}(\{w\})$. \square

Remark 3. *When needed, we use the following notation for vectors in \mathbb{F}^n :*

$$(a_1^{n_1} \mid a_2^{n_2} \mid \dots \mid a_k^{n_k}) := (\underbrace{a_1, \dots, a_1}_{n_1\text{-times}}, \underbrace{a_2, \dots, a_2}_{n_2\text{-times}}, \dots, \underbrace{a_k, \dots, a_k}_{n_k\text{-times}}),$$

where $n = \sum_{i=1}^k n_i$.

Let G be a *SCCG*. We identify each well-covered weighting of G with an n -tuple $\mathbf{x} = (f(v_1), \dots, f(v_n)) \in \mathbb{F}^n$. We call the vector space of all such n -tuples \mathbb{V} . It is clear that $wcdim(G) = \dim(\mathbb{V})$.

Let \mathcal{W} be a connection set of G . For any $i \in \mathbb{N}_{\leq sc(G)}$, let $t_i = |C_i - W_i|$ and let $l = |\mathcal{W}|$. Suppose that G has order $n = k + l$, where $k = \sum_{i=1}^{sc(G)} t_i$. Then, using Lemma 1, any vector $\mathbf{x} \in \mathbb{V}$ may be expressed as

$$\mathbf{x} = (f(w_1), \dots, f(w_l) \mid a_1^{t_1} \mid a_2^{t_2} \mid \dots \mid a_{sc(G)}^{t_{sc(G)}}),$$

where we have placed the weights of the connection vertices of G first. Now we can use Lemma 2 to get that the first l components of \mathbf{x} are linear combinations of the a_i 's. It follows that every $\mathbf{x} \in \mathbb{V}$ can be written as a linear combination of at most $sc(G)$ linearly independent vectors. This means that $wcdim(G) \leq sc(G)$. Since we already knew that $wcdim(G) \geq sc(G)$, for any graph G , we obtain the main theorem of this section, which is stated below.

Theorem 3. *Let G be a *SCCG*. Then, $wcdim(G) = sc(G)$.*

We now give some examples of applications of Theorem 3.

Example 1. *Consider the graph G given by*

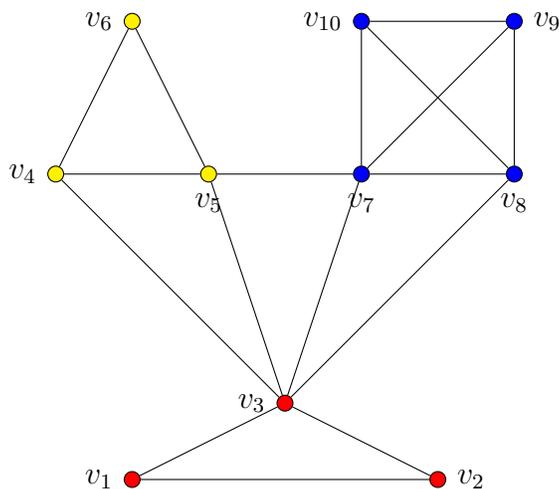


Figure 1: A *SCCG* with $sc(G) = 3$ and empty connection set.

G is a *SCCG* with $\mathcal{C}(G) = \{C_1, C_2, C_3\}$, where $C_1 = \{v_1, v_2, v_3\}$, $C_2 = \{v_4, v_5, v_6\}$ and $C_3 = \{v_7, v_8, v_9, v_{10}\}$. For this particular G , $\mathcal{W} = \emptyset$. Note that $\mathcal{C}(G)$ is a minimum clique partition of G .

A basis for the well-covered space of G is

$$B_G = \{(1^3 \mid 0^7), (0^3 \mid 1^3 \mid 0^4), (0^6 \mid 1^4)\},$$

using Lemma 1, and hence $wcdim(G) = 3$. This is consistent with the result we would have obtained had we used Theorem 3.

Problems may arise if a *SGGC*, G , were presented in some unrecognizable form. In that case, the problem of finding the well-covered dimension of G is comparable to the problem of finding a minimum clique cover of G .

Let G be a *SCCG* with $sc(G) = k$, for some $k \in \mathbb{N}$, and let $C_i \in \mathcal{C}(G)$. Observe that the well-covered dimension of G does not depend on $|C_i|$. Informally speaking, if we let $|C_i| \rightarrow \infty$, the well-covered dimension G is still k .

Example 2. Each graph in Figure 2 is a *SCCG* with simplicial clique number equal to 2. So the well-covered dimension of each of these graphs is 2, although the set of simplicial cliques is distinct in each case.

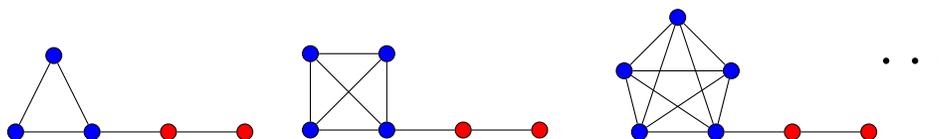
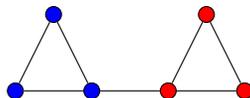


Figure 2: An infinite family of *SCCGs* with $wcdim = 2$.

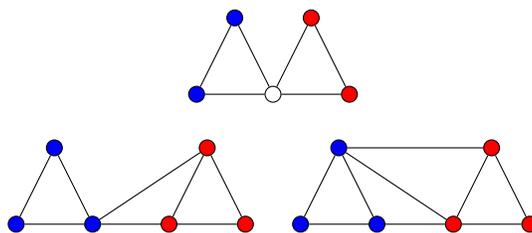
Thus, there is an infinite number of *SCCGs* with well-covered dimension any positive integer.

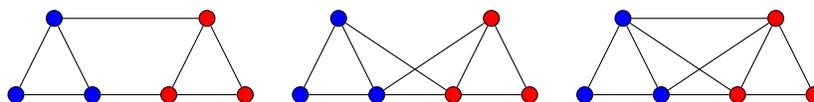
There are other ways of obtaining a family of *SCCGs* with some desired well-covered dimension. In particular, if we add or remove edges between vertices of a given *SCCG*, making sure that we do not create or delete a simplicial clique, we obtain a family of *SCCGs* with the same well-covered dimension. However, this resulting family might not be infinite.

Example 3. Consider the following *SCCG*.



For this particular graph, we may obtain only finitely many *SCCGs* by removing and adding edges. Examples of these graphs are illustrated below.





All of these graphs have well-covered dimension equal to 2 and the same set of simplicial cliques.

3 The well-covered dimension of simplicial clique sums

In this section, we obtain a class of graphs with well-covered dimension equal to the simplicial clique number that contains both chordal graphs and *SCCGs*.

Definition 6. Let G_1 and G_2 be subgraphs of a graph \mathcal{G} such that $\mathcal{C}(G_1)$ and $\mathcal{C}(G_2)$ are non-empty. We say that \mathcal{G} is the simplicial clique sum (SCS) of G_1 and G_2 if

1. $V(G_1) \cup V(G_2) = V(\mathcal{G})$,
2. $E(G_1) \cup E(G_2) = E(\mathcal{G})$, and
3. $V(G_1) \cap V(G_2)$ is a simplicial clique of G_1 , G_2 and \mathcal{G} .

Remark 4. Let \mathcal{G} be the SCS of G_1 and G_2 . Then, for any $u \in V(G_1) - (V(G_1) \cap V(G_2))$ and any $v \in V(G_2) - (V(G_1) \cap V(G_2))$, $uv \notin E(\mathcal{G})$.

In 1961, G.A. Dirac proved that every chordal graph has a simplicial vertex [5]. Note that this means that every chordal graph has at least one simplicial clique. Let G be a chordal graph and G' be a *SCCG*. We know that $\mathcal{C}(G)$ and $\mathcal{C}(G')$ are non-empty. Then, G may be understood to be the *SCS* of G_1 , the complete subgraph induced by some $C \in \mathcal{C}(G)$, with $G_2 = G$. Likewise, G' may be understood to be the *SCS* of the complete subgraph induced by some $C' \in \mathcal{C}(G')$ with G' itself. This is a remarkable fact because it allows us to view the *SCS* class of graphs, as a class that contains all chordal graphs and all *SCCGs*. Figure 3 is an example of the *SCS* of a *SCCG* that is not chordal (red) and a chordal graph that is not a *SCCG* (black). The yellow simplicial clique is their intersection.

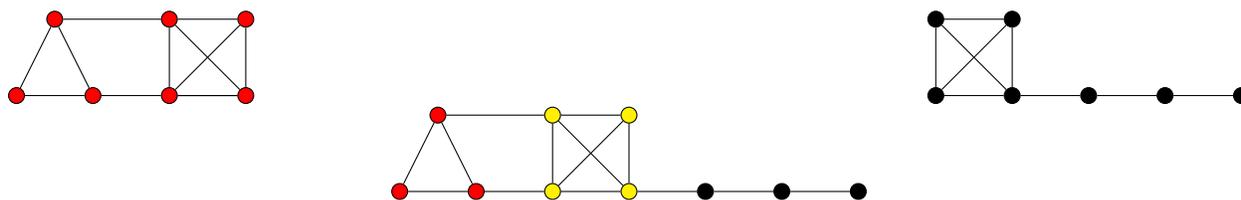


Figure 3: *SCS* of a *SCCG* and a chordal graph.

Next, is our main result on maximal independent sets of *SCSs*.

Theorem 4. *Let \mathcal{G} be the SCS of G_1 and G_2 . Let M_1 be a MIS of G_1 , M_2 be a MIS of G_2 , and $M_1 \cap M_2 = \{v\}$, for some $v \in V(G_1) \cap V(G_2)$. Then, \mathcal{M} is a MIS of \mathcal{G} if and only if $\mathcal{M} = M_1 \cup M_2$.*

Proof. This proof consists of two parts: verifying that (I) $M_1 \cup M_2$, as described in the hypothesis of the theorem, is a MIS of \mathcal{G} and that (II) all MISs of \mathcal{G} are of this form.

(I) Suppose, for a contradiction, that $M_1 \cup M_2$ is not a MIS of \mathcal{G} . Then, either $M_1 \cup M_2$ is dependent or $M_1 \cup M_2$ is independent but not maximal. In the first case, there exist $u, v \in M_1 \cup M_2$ such that $uv \in E(\mathcal{G})$. There are two possibilities: (1) $u, v \in M_i$, where $i = 1, 2$ or (2) $u \in M_1 - M_2$ and $v \in M_2 - M_1$. But (1) implies that $uv \in E(G_i)$ so that M_i is a dependent set of G_i , and (2) contradicts Remark 4, since $M_1 - M_2 \subset V(G_1) - V(G_1) \cap V(G_2)$ and $M_2 - M_1 \subset V(G_2) - V(G_1) \cap V(G_2)$. If $M_1 \cup M_2$ is independent but not maximal, there is $w \in V(\mathcal{G})$ such that $M_1 \cup M_2 \cup \{w\}$ is independent. Say, $M_1 \cup \{w\}$ is an independent set of G_1 . But then, M_1 is not maximal in G_1 . Therefore, $M_1 \cup M_2$ is a MIS of \mathcal{G} .

(II) Any MIS of \mathcal{G} must contain vertices of G_1 and G_2 . This is because $\mathcal{C}(G_1)$ and $\mathcal{C}(G_2)$ are non-empty, and any MIS of any graph must contain a vertex per simplicial clique. Thus, any MIS of \mathcal{G} can be expressed as $I_1 \cup I_2$, where $I_1 \subset V(G_1)$ and $I_2 \subset V(G_2)$ are non-empty. Without loss of generality, suppose that I_1 is not a MIS of G_1 . Clearly, if I_1 is dependent, so is $I_1 \cup I_2$, so we discard that possibility. Suppose that I_1 is independent but not maximal in G_1 . Then, $I_1 \cup \{y\}$ is independent, for some $y \in V(G_1)$. But then, if $I_2 \cup \{y\}$ is independent, so is $I_1 \cup I_2 \cup \{y\}$. Otherwise, $I_2 \cup \{y\}$ is dependent and so is $I_1 \cup I_2 \cup \{y\}$. Thus, I_1 and I_2 must be MISs of G_1 and G_2 . By Remark 4, the only vertices at which I_1 and I_2 can intersect belong to $V(G_1) \cap V(G_2)$. Observe that $I_1 \cap I_2$ can contain at most one vertex from $V(G_1) \cap V(G_2)$, since any two vertices in this set are adjacent. Observe too that if $I_1 \cap I_2 = \emptyset$, then each vertex in $V(G_1) \cap V(G_2)$ must be adjacent to vertices outside of $V(G_1) \cap V(G_2)$. But this forces $V(G_1) \cap V(G_2)$ to be non-simplicial. Therefore, every MIS of \mathcal{G} is of the form $I_1 \cup I_2$, where I_1 and I_2 are MISs of G_1 and G_2 that intersect at a single vertex from $V(G_1) \cap V(G_2)$. \square

Corollary 1. *Let \mathcal{G} be the SCS of G_1 and G_2 , and let $c = |V(G_1) \cap V(G_2)|$. Let $v_i \in V(G_1) \cap V(G_2)$, let l_i be the number of MISs of G_1 that contain v_i , and let m_i be the number of MISs of G_2 that contain v_i . Then, \mathcal{G} has exactly*

$$\sum_{i=1}^c l_i m_i$$

maximal independent sets.

Proof. We know the structure of the MISs of \mathcal{G} from Theorem 4. Observe that per each of the l_i MISs of G_1 , we can form m_i MISs of \mathcal{G} that contain v_i . That is, we can form $l_i m_i$ MISs of \mathcal{G} that contain v_i . Since there are c such v_i , the result follows. \square

Theorem 5. *Let \mathcal{G} be the SCS of G_1 and G_2 . Then,*

$$wcdim(\mathcal{G}) = wcdim(G_1) + wcdim(G_2) - 1.$$

Proof. Let \mathcal{V} be the well-covered space of \mathcal{G} , $f \in \mathcal{V}$ and $u \in V(G_1) \cap V(G_2)$. Define W_{G_1} to be the vector space of all f that assign zero to vertices outside of $V(G_1)$ with the added property that for any $g \in W_{G_1}$, $g(u) = \frac{f(u)}{2}$. Similarly, define W_{G_2} to be the vector space of all f that assign zero to vertices outside of $V(G_2)$ and such that for any $h \in W_{G_2}$, $h(u) = \frac{f(u)}{2}$. By Theorem 4, we know what the MISs of \mathcal{G} look like. Namely $\mathcal{M} = M_1 \cup M_2$, where $M_1 \cap M_2 = \{s\}$ for some $s \in V(G_1) \cap V(G_2)$. Observe that all functions in W_{G_1} and W_{G_2} have domain $V(\mathcal{G})$, which is the domain of f , and thus addition of these functions is defined. Then, for any $g \in W_{G_1}$ and any $h \in W_{G_2}$,

$$\begin{aligned} \sum_{v \in \mathcal{M}} (g + h)(v) &= \sum_{v \in \mathcal{M}} g(v) + \sum_{v \in \mathcal{M}} h(v) \\ &= \left(\sum_{v \in \mathcal{M} - \{s\}} g(v) + g(s) \right) + \left(\sum_{v \in \mathcal{M} - \{s\}} h(v) + h(s) \right) \\ &= \sum_{v \in \mathcal{M} - \{s\}} g(v) + \sum_{v \in \mathcal{M} - \{s\}} h(v) + f(s) \\ &= \sum_{v \in \mathcal{M}} f(v). \end{aligned}$$

This shows that $W_{G_1} + W_{G_2} = \mathcal{V}$. Since $W_{G_1} \cap W_{G_2}$ is the set of all f that are zero everywhere, except at $V(G_1) \cap V(G_2)$, where they are constant, we get that $\dim(\mathcal{V}) = \dim(W_{G_1}) + \dim(W_{G_2}) - 1$, and the result follows. \square

Now we are finally able to state and prove our result generalizing Brown and Nowakowski's theorem on the well-covered dimension of chordal graphs.

Theorem 6. *Let \mathcal{G} be the SCS of G_1 , a SCCG, and G_2 , a chordal graph. Then, $wc\dim(\mathcal{G}) = sc(\mathcal{G})$.*

Proof. By Theorem 5, $wc\dim(\mathcal{G}) = sc(G_1) + sc(G_2) - 1 = sc(\mathcal{G})$. \square

The question is now whether there are any graphs that have minimal well-covered dimension and are not SCS graphs. In order to answer this question, we turn to the study of the well-covered dimension of Sierpinski gasket graphs, of which there are infinitely many. The Sierpinski gasket graphs are not part of the family of graphs to which Theorem 6 applies. In spite of this, all Sierpinski gasket graphs have well-covered dimension the simplicial clique number.

4 The well-covered dimension of Sierpinski gasket graphs

In this section, we study the well-covered dimension of the Sierpinski gasket graph, which we denote by S_n , for any $n \in \mathbb{N}$. The Sierpinski gasket graph is constructed recursively, in

the same way the Sierpinski gasket is constructed.

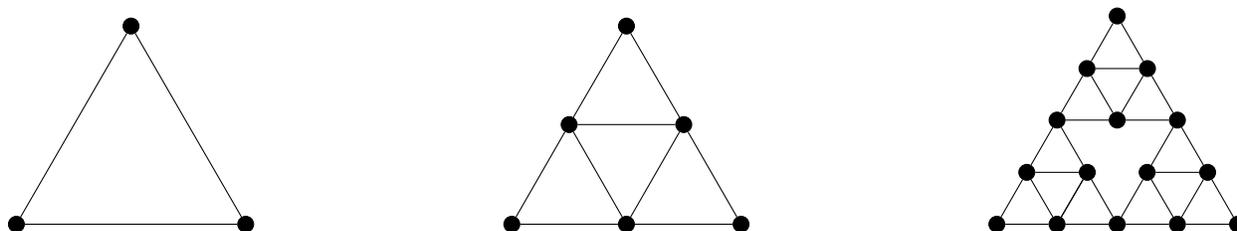


Figure 4: The first three Sierpinski graphs, S_1, S_2 and S_3 .

From Figure 4, it should be clear that S_1 is a K_3 and that S_2 is both a $SCCG$ and a chordal graph with $sc(S_2) = 3$. Thus, $wcdim(S_1) = 1$ and $wcdim(S_2) = 3$. However, for $n \in \mathbb{N}_{\geq 3}$, S_n is not a $SCCG$, not a chordal graph, and not the SCS of a chordal graph and a $SCCG$. All we know about the well-covered dimension of S_n , for $n \in \mathbb{N}_{\geq 3}$, is stated below and is a direct consequence of Lemma 10 in the work of Brown and Nowakowski [2].

Remark 5. For $n \in \mathbb{N}_{\geq 3}$, $wcdim(S_n) \geq 3$.

From the recursive construction of S_n , it follows that S_n has all of its predecessors as subgraphs. In particular, S_n has exactly three S_{n-1} subgraphs. Observe that for $n \in \mathbb{N}_{\geq 3}$, S_n has *sides* that are paths of length at least 5, and that an S_{n-1} subgraph of S_n has *corners* that are the *former simplicial cliques* of the S_{n-1} iterate. An S_{n-1} subgraph of S_n has three corners, only one of which is a simplicial clique of S_n . These ideas are exemplified in Figure 5. In the graph to the left, a *side* of S_4 is colored yellow and an S_3 subgraph of S_4 is colored blue. In the graph to the right, the simplicial clique corner of an S_3 subgraph of S_4 is colored red.

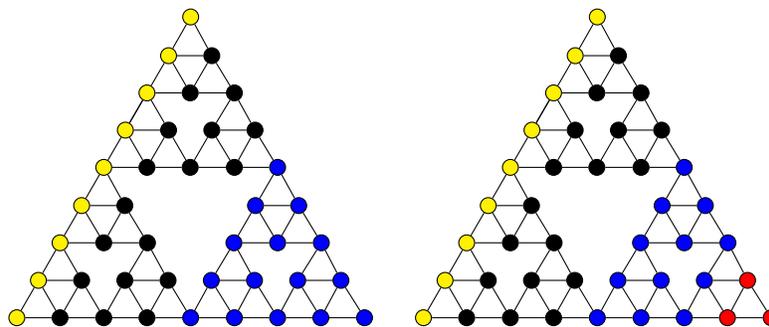


Figure 5: Subgraphs, sides and corners of S_4 .

For any $n \in \mathbb{N}$, $|V(S_n)| = \frac{3(3^{n-1}+1)}{2}$ [10]. For any $n \in \mathbb{N}_{\geq 3}$, $V(S_n) - \bigcup_{i=1}^3 C_i$ is the set of all vertices that do not belong to a simplicial clique of S_n . Note that $V(S_n) - \bigcup_{i=1}^3 C_i \neq \emptyset$ because when $n \in \mathbb{N}_{\geq 3}$, $|\bigcup_{i=1}^3 C_i| = 9$ and $|V(S_n)| \geq 15$. The figure that follows shows $V(S_3) - \bigcup_{i=1}^3 C_i$ in blue.

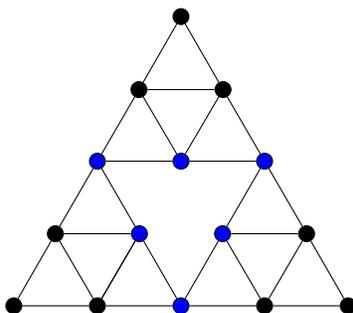


Figure 6: The set of all vertices not in a simplicial clique of S_3 .

Moreover, if $w \in V(S_n)$ is non-simplicial, then $|N(w)| = 4$. Otherwise, $w \in V(S_n)$ is simplicial and $|N(w)| = 2$. In the figure below, we show the simplicial vertices of S_1, S_2 and S_3 in red, the neighborhood of a simplicial vertex in blue, and that of a non-simplicial vertex in green.

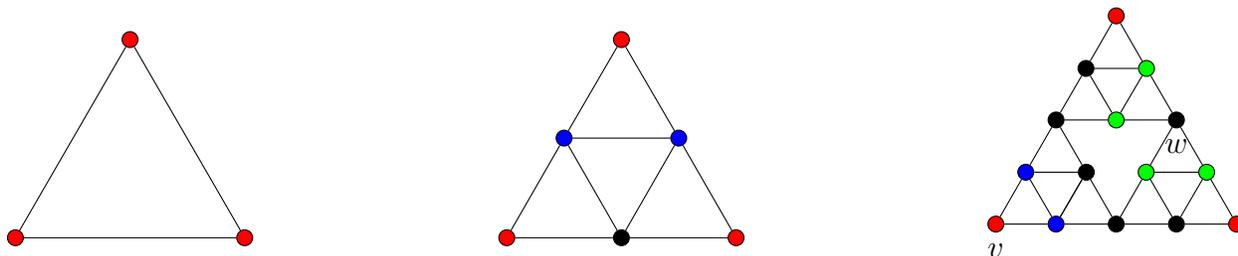


Figure 7: Simplicial vertices and neighborhoods.

Let $n \in \mathbb{N}_{\geq 3}$ and u, v be a pair of non-simplicial vertices that are adjacent. It is always the case that either (1) u and v are adjacent to a unique third vertex a or that (2) u and v are adjacent to exactly two vertices b and c . This means that in case (1), $|N[u] \cap N[v]| = |\{u, v, a\}|$, while in case of (2), $|N[u] \cap N[v]| = |\{u, v, b, c\}|$. As a result, $6 \leq |N[\{u, v\}]| \leq 7$, which shows that $V(S_n) - N[\{u, v\}] \neq \emptyset$, since $|V(S_n) - N[\{u, v\}]| \geq 8$. Additionally, $N[\{u, v\}] - \{u, v\} \neq \emptyset$, since $|N[\{u, v\}] - \{u, v\}| \geq 4$.

Lemma 3 (Brown and Nowakowski [2]). *Let u and v be vertices of a graph G . Suppose that I is an independent set, and that $I \cup \{u\}$ and $I \cup \{v\}$ are MISs of G . Then, for any well-covered weighting f of G , $f(u) = f(v)$.*

Theorem 7 (Birnbaum, McDonald, Kuneli, Urabe, and Vega [1]). *Let $n \in \mathbb{N}_{\geq 5}$, P_n be a path on n vertices, and f be a well-covered weighting of P_n . Then, f is constant on each of the two simplicial cliques of P_n , while f is zero at the remaining vertices of P_n .*

In the proofs of results that follow, we make repeated use of the algorithm that is outlined below. See page 2 in Plummer's survey [8].

Definition 7. *A greedy algorithm is a tool for constructing maximal independent sets of graphs. It is executed in the following manner. Let G be a graph.*

1. *Select $v_1 \in V(G)$ and set $I = \{v_1\}$.*
2. *Delete v_1 and its neighborhood in G . The remaining vertices of G induce a subgraph G_{v_1} .*
3. *Select any vertex v_2 of G_{v_1} and put it in the set I .*
4. *Repeat step (2) to obtain a subgraph $(G_{v_1})_{v_2}$ of G_{v_1} .*
5. *Continue this process until all vertices of G either have been added to I or have been deleted.*

The resulting set I is a MIS of G .

We note that a greedy algorithm need not start with I in the above definition being a singleton. In fact, we can use this algorithm to extend an independent set of any size to a maximal independent set. We can also carry out this algorithm on a subgraph of a given graph to extend some independent set into another independent set that is maximal with respect to that subgraph. We are now ready to prove the first result of this section.

Lemma 4. *Let $n \in \mathbb{N}_{\geq 3}$ and f be any well-covered weighting of S_n . Then, f is constant on $V(S_n) - \bigcup_{i=1}^3 C_i$.*

Proof. This proof is by induction on $n \in \mathbb{N}_{\geq 3}$. Choose a pair of adjacent vertices $v_1, v_2 \in V(S_3) - \bigcup_{i=1}^3 C_i$. Let T be a set that contains vertices adjacent to each vertex in the neighborhood of v_1 and of v_2 , excluding v_1 and v_2 .

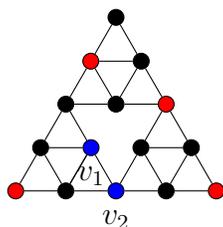


Figure 8: A MIS T in red.

Since $T \cup \{v_1\}$ and $T \cup \{v_2\}$ are *MISs* of S_3 , by Lemma 3, $f(v_1) = f(v_2)$, for any well-covered weighting f of S_3 . Repeating this process for each of the remaining five pairs of adjacent vertices in $V(S_3) - \bigcup_{i=1}^3 C_i$, we obtain that f is constant on this set.

Assume that the result holds for all $m < n$. Pick any S_{n-1} subgraph of S_n . By induction, all vertices not in a corner of this S_{n-1} subgraph have the same weight. Choose one out of the two S_{n-1} corners that are not simplicial cliques of S_n . Let t be any vertex in this corner and t' be a vertex adjacent to t that lies outside of this corner. Note that the vertex t' could belong to the chosen S_{n-1} subgraph or to a corner of another S_{n-1} subgraph in the vicinity. This is illustrated in Figure 9. Let $\mathcal{T} \subset V(S_n) - N[\{t, t'\}]$ be an independent set that contains vertices adjacent to each vertex in $N[\{t, t'\}] - \{t, t'\}$. Extend \mathcal{T} with vertices from $V(S_n) - N[\{t, t'\}]$ via a greedy algorithm. Make this set as large as possible and call it \mathcal{I} .

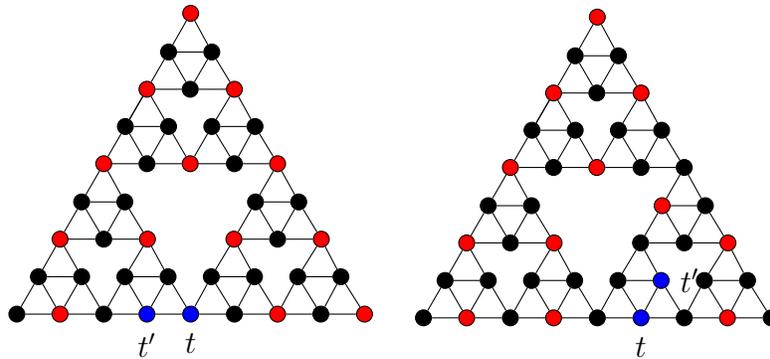


Figure 9: Examples of t , t' and \mathcal{I} .

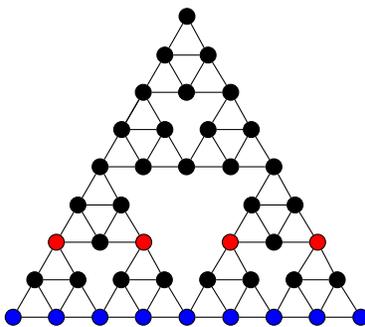
We have that $\mathcal{I} \cup \{t\}$ and $\mathcal{I} \cup \{t'\}$ are independent sets. Now, if $w \in V(S_n) - N[\{t, t'\}]$, w is adjacent to some vertex in \mathcal{I} . If $w \in N[\{t, t'\}] - \{t, t'\}$, w is adjacent to either a vertex in \mathcal{I} , to t or t' . Thus, for any $w \in V(S_n) - \{t, t'\}$, $(\mathcal{I} \cup \{t\}) \cup \{w\}$ and $(\mathcal{I} \cup \{t'\}) \cup \{w\}$ are dependent sets. Hence, $\mathcal{I} \cup \{t\}$ and $\mathcal{I} \cup \{t'\}$ are *MISs* of S_n so that by Lemma 3, $f(t) = f(t')$, for any well-covered weighting f of S_n .

Since the S_{n-1} subgraph of S_n was chosen without loss of generality, it follows that any well-covered weighting of S_n is constant on the set of all vertices that do not belong to a simplicial clique of S_n . \square

We now further examine the properties of weightings in the well-covered space of S_n , for $n \in \mathbb{N}_{\geq 3}$. This result and the corollary that follows it, give us a full description of the well-covered weightings of S_n .

Lemma 5. *Let $n \in \mathbb{N}_{\geq 3}$ and f be a well-covered weighting of S_n . Then, $f(v) = 0$, for all v on the sides of S_n , unless v belongs to a simplicial clique of S_n , where f is constant.*

Proof. Let P_n be a side of S_n and $\mathcal{K} \subset V(S_n) - N[P_n]$ be an independent set that contains vertices adjacent to each $v \in N[P_n] - P_n$.

Figure 10: An example of \mathcal{K} (red) for S_4 .

Via a greedy algorithm, we can extend \mathcal{K} with vertices from $\mathcal{K} \subset V(S_n) - N[P_n]$ into an independent set \mathcal{K}' that is as large as possible. Note that for any independent set $\mathcal{M} \subset P_n$ that is maximal with respect to P_n , $\mathcal{K}' \cup \mathcal{M}$ is a MIS of S_n . Thus, the vertices of P_n behave as if they were the vertices of an isolated path. By Theorem 7, since $|P_n| \geq 5$, the two vertices at each end of P_n have the same weight, while all other vertices of P_n have weight zero, under any well-covered weighting of S_n . \square

Corollary 2. For any well-covered weighting f of S_n with $n \in \mathbb{N}_{\geq 3}$, $f(v) = 0$, for all $v \in V(S_n) - \bigcup_{i=1}^3 C_i$.

Proof. Let P_n be a side of S_n and suppose that $v \in V(P_n)$ but that $v \notin C_i$, for all $i \in \mathbb{N}_{\leq 3}$. Let f be any well-covered weighting of S_n . By Lemma 5, $f(v) = 0$, and by Lemma 4, it follows that f is zero on $V(S_n) - \bigcup_{i=1}^3 C_i$. \square

In conclusion, if $n \in \mathbb{N}_{\geq 3}$, any well-covered weighting of S_n is a linear combination of at most three linearly independent functions that assign a distinct, non-zero scalar to each simplicial clique, and zero to all other vertices of S_n . Then, $wcdim(S_n) \leq 3$ and by Remark 5, we obtain the final theorem of this section.

Theorem 8. For any $n \in \mathbb{N}_{\geq 3}$, $wcdim(S_n) = 3$.

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