Rose-
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Undergraduate
Mathematics
Journal

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Brennon Bauer[a] Amy Gifford[b]

Volume 15, No. 2, Fall 2014

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Rose-Hulman Institute of Technology
Department of Mathematics
Terre Haute, IN 47803
Email: mathjournal@rose-hulman.edu
http://www.rose-hulman.edu/mathjournal

[a]Department of Mathematics, Southern Utah University
[b]Department of Mathematics, Southern Utah University

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Abstract. We study a Lotka-Volterra competition model. By using the nondimensionalization method, we analyze the stability of the steady state solutions for this system. Also, a stable numerical scheme is proposed to verify the theoretical results of the system. Using the Principle of Mathematical Induction, we prove the unconditional stability and convergence of the numerical scheme.

Acknowledgements: We would like to thank our advisor Dr. Jianlong Han for his encouragement and helpful discussions. Also, we appreciate our referee's helpful suggestions.

1 Introduction

In biology, there are many different environments in which multiple predators must fight for the same prey. The following Lotka-Volterra competition model serves as a reasonable model for an environment where two predators compete for the same prey:

$$\begin{aligned} dx/dt &= \frac{r_1}{K_1}x(K_1 - x - \alpha_{12}y) \\ dy/dt &= \frac{r_2}{K_2}y(K_2 - y - \alpha_{21}x). \end{aligned} \tag{1}$$

In this system, x and y are the populations of two different species of predators, r_1 and r_2 are growth rates of each species, K_1 and K_2 are carrying capacities which designate the maximum populations that the environment can support over a long period of time, and α_{12} and α_{21} are interaction terms for the two predators. Note that all of these parameters are positive constants. Without the interaction terms, each population grows logistically. The signs on the interaction terms in both equations are negative because these two predators are competing over resources in the same environment, decreasing the overall rate of change for each species.

System (1) is similar to the Lotka-Volterra equations

$$\begin{aligned} dx/dt &= x(a - by) \\ dy/dt &= y(-b + dx) \end{aligned} \tag{2}$$

that were used to describe the dynamics of biological systems in which two species interact, one as a predator and the other as prey. System (2) was initially proposed by A.J. Lotka and V. Volterra in the 1920s, see [5] and [9]. It is still the basis of many models used in the analysis of population dynamics in ecology.

System (1) is also called a Lotka-Volterra competition model in honor of those mathematicians who made the first breakthrough in math biology. For more detail about the background of this system, we refer to [1], [4], [8], and references therein.

In order to reduce the number of parameters in this system, we use the following substitutions and apply the nondimensionalization method from [8] to derive a new equation. Setting $u = \frac{x}{K_1}, v = \frac{y}{K_2}, \tau = r_1 t, \rho = \frac{r_2}{r_1}, b_{12} = \alpha_{12}\frac{K_2}{K_1}, b_{21} = \alpha_{21}\frac{K_1}{K_2}$, we have

$$\begin{aligned} u_\tau &= u - u^2 - b_{12}uv \\ v_\tau &= \rho(v - v^2 - b_{21}uv). \end{aligned} \tag{3}$$

In the first part of this paper, we study the stability of the steady state solutions of system (3). In the second part, we develop a numerical scheme which inherits the properties of the true solution. This numerical scheme overcomes some of the weaknesses of the standard Euler's Method. We prove this numerical scheme is uniquely solvable and stable unconditionally. We also prove that the numerical approximation converges to the true solution of (3). In the last section, we present some results of computational experiments that verify the stability and convergence of the proposed difference scheme.

2 Stability of the Steady State Solutions

Setting the right-hand side of equation (3) equal to zero allows us to find all of the steady-state solutions:

$$\begin{aligned} u - u^2 - b_{12}uv &= 0 \\ \rho(v - v^2 - b_{21}uv) &= 0. \end{aligned} \tag{4}$$

Solving, we have four solutions:

$$\begin{cases} u = 0 \\ v = 0, \end{cases} \quad \begin{cases} u = 0 \\ v = 1, \end{cases} \quad \begin{cases} u = 1 \\ v = 0, \end{cases} \quad and \begin{cases} u = \frac{1-b_{12}}{1-b_{12}b_{21}} \\ v = \frac{1-b_{21}}{1-b_{12}b_{21}}. \end{cases} \tag{5}$$

To study the stability of these steady state solutions, we use the following theorem, which can also be found in [8].

Theorem 2.1 *Let (u_0, v_0) be a steady state solution of the system*

$$\begin{aligned} u' &= f(u,v) \\ v' &= g(u,v). \end{aligned} \tag{6}$$

Let $A(u,v) = \begin{pmatrix} f_u(u,v) & f_v(u,v) \\ g_u(u,v) & g_v(u,v) \end{pmatrix}.$

1. *If the matrix $A(u_0, v_0)$ has two negative real eigenvalues, then (u_0, v_0) is asymptotically stable.*

2. *If the matrix $A(u_0, v_0)$ has at least one positive real eigenvalue, then (u_0, v_0) is unstable.*

3. *If the matrix $A(u_0, v_0)$ has two complex eigenvalues with negative real parts, then (u_0, v_0) is asymptotically stable.*

4. *If the matrix $A(u_0, v_0)$ has two complex eigenvalues with positive real parts, then (u_0, v_0) is unstable.*

In order to use Theorem 2.1, we first compute $A(u,v)$ for the system (3):

$$A(u,v) = \begin{pmatrix} 1 - 2u - b_{12}v & -b_{12}u \\ -b_{21}\rho v & \rho - 2\rho v - b_{21}\rho u \end{pmatrix}. \tag{7}$$

2.1 Stability of $(u, v) = (0, 0)$

Substituting $u = 0, v = 0$ into (7), we have

$$A(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}.$$

Solving

$$|A(0,0) - \lambda I| = \begin{vmatrix} 1-\lambda & 0 \\ 0 & \rho - \lambda \end{vmatrix} = 0,$$

we obtain $\lambda = 1$ and $\lambda = \rho$. Because 1 and ρ are both positive constants, we can conclude that $(u, v) = (0, 0)$ is unstable.

2.2 Stability of $(u, v) = (0, 1)$

Here we have

$$A(0,1) = \begin{pmatrix} 1 - b_{12} & 0 \\ -b_{21}\rho & -\rho \end{pmatrix}.$$

In this case, we find $\lambda = -\rho$ and $\lambda = 1 - b_{12}$. Thus, we have two cases:

1. If $b_{12} < 1$, then we have at least one positive eigenvalue. $(u, v) = (0, 1)$ is unstable.
2. If $b_{12} > 1$, then we have two negative real eigenvalues. $(u, v) = (0, 1)$ is stable.

2.3 Stability of $(u, v) = (1, 0)$

In this instance,

$$A(1,0) = \begin{pmatrix} -1 & -b_{12} \\ 0 & \rho - b_{21}\rho \end{pmatrix},$$

and we find similar eigenvalues and cases: $\lambda = -1$ and $\lambda = \rho(1 - b_{21})$.

1. If $b_{21} < 1$, then there is at least one positive eigenvalue, $(u, v) = (1, 0)$ is unstable.
2. If $b_{21} > 1$, then there are two negative real eigenvalues, $(u, v) = (1, 0)$ is stable.

2.4 Stability of $(u, v) = (\frac{1-b_{12}}{1-b_{12}b_{21}}, \frac{1-b_{21}}{1-b_{12}b_{21}})$

Set $d = 1 - b_{12}b_{21}$. Then we have

$$A(\frac{1 - b_{12}}{1 - b_{12}b_{21}}, \frac{1 - b_{21}}{1 - b_{12}b_{21}}) = \begin{pmatrix} \frac{-1+b_{12}}{d} & \frac{b_{12}(b_{12}-1)}{d} \\ \frac{\rho b_{21}(b_{21}-1)}{d} & \frac{\rho(b_{21}-1)}{d} \end{pmatrix}.$$

Setting

$$\begin{vmatrix} \frac{-1+b_{12}}{d} - \lambda & \frac{b_{12}(b_{12}-1)}{d} \\ \frac{\rho b_{21}(b_{21}-1)}{d} & \frac{\rho(b_{21}-1)}{d} - \lambda \end{vmatrix} = 0, \tag{8}$$

we obtain

$$\begin{aligned} \lambda = & \frac{\rho(b_{21} - 1) + (b_{12} - 1)}{2d} \\ & \pm \frac{\sqrt{[\rho(b_{21} - 1) + (b_{12} - 1)]^2 - 4\rho d(b_{12} - 1)(b_{21} - 1)}}{2d}. \end{aligned} \tag{9}$$

Now we have the following four cases to consider:

1. $b_{12} < 1$ and $b_{21} < 1$

2. $b_{12} > 1$ and $b_{21} > 1$
3. $b_{12} < 1$ and $b_{21} > 1$
4. $b_{12} > 1$ and $b_{21} < 1$

In order to interpret each case, we must look at the signs of equation (9). Set $a = \rho(b_{21} - 1) + (b_{12} - 1)$ and $k = -4\rho d(b_{12} - 1)(b_{21} - 1)$. Then we have the following equation:

$$\lambda = \frac{a \pm \sqrt{a^2 + k}}{2d}. \tag{10}$$

Case 1: $b_{12} < 1$ and $b_{21} < 1$

If $b_{12} < 1$ and $b_{21} < 1$, then a and k are both negative, and d is positive. Now we can consider the sign of (10). Because $a < \pm\sqrt{a^2 + k}$ if $a^2 + k \geq 0$, we have two negative eigenvalues. If $a^2 + k < 0$, then the real part (10) is negative. Therefore, if $b_{12} < 1$ and $b_{21} < 1$, then $(\frac{1-b_{12}}{1-b_{12}b_{21}}, \frac{1-b_{21}}{1-b_{12}b_{21}})$ is stable.

Case 2: $b_{12} > 1$ and $b_{21} > 1$

If $b_{12} > 1$ and $b_{21} > 1$, then a and d are negative. So k is positive. Then one eigenvalue of (10) is positive. Thus, if $b_{12} > 1$ and $b_{21} > 1$, then $(\frac{1-b_{12}}{1-b_{12}b_{21}}, \frac{1-b_{21}}{1-b_{12}b_{21}})$ is unstable.

Case 3: $b_{12} < 1$ and $b_{21} > 1$

If $b_{12} < 1$ and $b_{21} > 1$, then we must consider two subcases: $b_{12}b_{21} < 1$ and $b_{12}b_{21} > 1$.

1. If $b_{12} < 1$, $b_{21} > 1$ and $b_{12}b_{21} < 1$, then a can be positive or negative. However, since $d = 1 - b_{12}b_{21} > 0$, k is positive. Once again, since $|a| < \sqrt{a^2 + k}$, at least one eigenvalue of (10) is positive. So $(\frac{1-b_{12}}{1-b_{12}b_{21}}, \frac{1-b_{21}}{1-b_{12}b_{21}})$ is unstable.

2. If $b_{12} < 1$, $b_{21} > 1$ and $b_{12}b_{21} > 1$, then a can be positive or negative, but since $d = 1 - b_{12}b_{21} < 0$, k is negative. In this case, we will have two subcases:

   When $\rho|b_{21} - 1| > |b_{12} - 1|$, in this case $a > 0$, since $a > \pm\sqrt{a^2 + k}$ and $d < 0$, both eigenvalues of (10) are negative. So $(\frac{1-b_{12}}{1-b_{12}b_{21}}, \frac{1-b_{21}}{1-b_{12}b_{21}})$ is stable.

   When $\rho|b_{21} - 1| < |b_{12} - 1|$, in this case $a < 0$, since $d < 0$, at least one eigenvalue of (10) is positive. So $(\frac{1-b_{12}}{1-b_{12}b_{21}}, \frac{1-b_{21}}{1-b_{12}b_{21}})$ is unstable.

Case 4: $b_{12} > 1$ and $b_{21} < 1$

We can find the following results for Case 4 using a proof similar to Case 3:

1. If $b_{12} > 1$, $b_{21} < 1$, $b_{12}b_{21} > 1$, and $\rho|b_{21} - 1| < |b_{12} - 1|$, then $(\frac{1-b_{12}}{1-b_{12}b_{21}}, \frac{1-b_{21}}{1-b_{12}b_{21}})$ is stable.

2. If $b_{12} > 1$, $b_{21} < 1$, $b_{12}b_{21} > 1$, and $\rho|b_{21} - 1| > |b_{12} - 1|$, then $(\frac{1-b_{12}}{1-b_{12}b_{21}}, \frac{1-b_{21}}{1-b_{12}b_{21}})$ is unstable.

3. If $b_{12} > 1$, $b_{21} < 1$, and $b_{12}b_{21} < 1$, then $(\frac{1-b_{12}}{1-b_{12}b_{21}}, \frac{1-b_{21}}{1-b_{12}b_{21}})$ is unstable.

Remark 2.2 *The stability of the steady state solution* $(\frac{1-b_{12}}{1-b_{12}b_{21}}, \frac{1-b_{21}}{1-b_{12}b_{21}})$ *in case 1 and case 2 is also discussed in [8]. In case 3 and case 4, mathematically we give sufficient conditions of the stability of the steady state solution* $(\frac{1-b_{12}}{1-b_{12}b_{21}}, \frac{1-b_{21}}{1-b_{12}b_{21}})$ *which are not included in [8].*

3 A Semi-Implicit Numerical Scheme

Since we cannot find the solution to system (3) explicitly, we would like to solve system (3) numerically using Euler's Method. First, we discretize the t interval by letting $t_k = t_0 + k\Delta t$, where $k = 0, 1, \cdots$ and Δt is the step size. We can approximate the population at $t = t_k$ by $u^k \approx u(t_k)$. We have

$$\begin{aligned} u^{k+1} &= u^k + \Delta t(u^k - (u^k)^2 - b_{12}u^k v^k) \\ v^{k+1} &= v^k + \Delta t(\rho v^k - \rho(v^k)^2 - \rho b_{21}u^k v^k). \end{aligned} \tag{11}$$

However, when using this explicit numerical scheme, a large initial population value can imply that the population at time t_k will be negative, which does not make sense with our model.

To correct this potential problem, following the ideas in [1], [2], [3], [6], and [7], we propose a semi-implicit numerical scheme for this system that guarantees $u^k > 0, v^k > 0$ if $u^0 > 0, v^0 > 0$. The semi-implicit numerical scheme can be written as follows:

$$\begin{aligned} u^{k+1} &= u^k + \Delta t(u^k - u^{k+1}u^k - b_{12}u^{k+1}v^k) \\ v^{k+1} &= v^k + \Delta t(\rho v^k - \rho v^{k+1}v^k - \rho b_{21}v^{k+1}u^k). \end{aligned} \tag{12}$$

Rearranging (12) to isolate u^{k+1} and v^{k+1}, we obtain

$$\begin{aligned} (1 + u^k\Delta t + b_{12}v^k\Delta t)u^{k+1} &= (1 + \Delta t)u^k \\ (1 + \rho v^k\Delta t + \rho b_{21}u^k\Delta t)v^{k+1} &= (1 + \rho\Delta t)v^k. \end{aligned} \tag{13}$$

This gives us the following set of equations:

$$\begin{aligned} u^{k+1} &= \frac{(1 + \Delta t)u^k}{1 + u^k\Delta t + b_{12}v^k\Delta t} \\ v^{k+1} &= \frac{(1 + \rho\Delta t)v^k}{1 + \rho v^k\Delta t + \rho b_{21}u^k\Delta t}. \end{aligned} \tag{14}$$

Remark 3.1 *In (12), we use $u^{k+1}u^k$ and $u^{k+1}v^k$ instead of $(u^k)^2$ and $u^k v^k$ in the first equation, and we use $v^{k+1}v^k$ and $v^{k+1}u^k$ instead of $(v^k)^2$ and $u^k v^k$ in the second equation. When u^{k+1} and v^{k+1} are isolated on the left-hand side, all negative terms on the right-hand side will be moved to the left-hand side. This will guarantee $u^{k+1} > 0, v^{k+1} > 0$ if $u^0 > 0, v^0 > 0$ no matter how large $\triangle t$ is. However, the standard Euler's Method can not guarantee the positivity of the numerical solutions if u^0 and v^0 are too large. Our numerical scheme will overcome this weakness of (11), since it guarantees positivity of u^{k+1} and v^{k+1}. More examples about this semi-implicit numerical scheme can be found in [1] and references therein.*

4 The Stability of the Semi-Implicit Numerical Scheme

Now we will prove that after any time k, our numerical scheme will be nonnegative.

Theorem 4.1 *If $u^0, v^0 \geq 0$, then $u^k, v^k \geq 0$.*

Proof 4.2 *Proof by Induction:*

If $k = 0$, by assumption, $u^0, v^0 \geq 0$, so our statement is true for $k = 0$.

Assume our statement is true for $k > 0$, that is, if $u^0, v^0 \geq 0$, then $u^k, v^k \geq 0$. We want to prove that it is also true for $k+1$. In fact, since $u^k \geq 0$, $v^k \geq 0$, and $\Delta t \geq 0$, we have $(1 + \Delta t)u^k \geq 0$ and $1 + u^k\Delta t + b_{12}v^k\Delta t \geq 0$. Thus, $\frac{(1+\Delta t)u^k}{1+u^k\Delta t+b_{12}v^k\Delta t} = u^{k+1} \geq 0$. Similarly, $v^{k+1} \geq 0$. Therefore, by the Principle of Mathematical Induction, we have proved $u^k, v^k \geq 0$ for all $k \geq 0$ if $u^0, v^0 > 0$.

Now we will prove that given any initial conditions, our semi-implicit numerical scheme for equation (3) will be unconditionally stable. Here, we use the term "unconditionally stable" to mean u^k and v^k will be bounded by some constants which are independent of $\triangle t$ and k. There are 4 possible cases for the initial data.

Theorem 4.3 *Stability of the Numerical Scheme*

1. *If $u^0 \leq 1$ and $v^0 \leq 1$, then $u^k \leq 1$ and $v^k \leq 1$.*
2. *If $u^0 \geq 1$ and $v^0 \leq 1$, then $u^k \leq u^0$ and $v^k \leq 1$.*
3. *If $u^0 \leq 1$ and $v^0 \geq 1$, then $u^k \leq 1$ and $v^k \leq v^0$.*
4. *If $u^0 \geq 1$ and $v^0 \geq 1$, then $u^k \leq u^0$ and $v^k \leq v^0$.*

So the semi-implicit numerical scheme is unconditionally stable.

We begin by proving Case 1 by induction:

Proof 4.4 *If $k = 0$, by assumption, $u^0 \leq 1$ and $v^0 \leq 1$. So our statement is true for $k = 0$.*

Assume that the statement is true for k. That is, if $u^0 \leq 1$ and $v^0 \leq 1$, then $u^k \leq 1$ and $v^k \leq 1$. We want to show that the statement is true for $k+1$, that is, if $u^0 \leq 1$ and $v^0 \leq 1$, then $u^{k+1} \leq 1$ and $v^{k+1} \leq 1$. We know $u^k \leq 1$, so

$$u^k \leq 1 + \Delta t b_{12} v^k. \tag{15}$$

Adding $\Delta t u^k$ to each side,

$$(1 + \Delta t)u^k \leq 1 + u^k\Delta t + b_{12}v^k\Delta t. \tag{16}$$

This implies

$$\frac{(1+\Delta t)u^k}{1+u^k\Delta t+b_{12}v^k\Delta t} \leq 1. \tag{17}$$

Thus $u^{k+1} \leq 1$. *Similarly,* $v^{k+1} \leq 1$ *and we have our desired result. Therefore, by the Principle of Mathematical Induction, we have proved the statement in Case 1 is true for all* $k \geq 0$.

Case 2: Let $k = 0$. *By assumption,* $u^0 = u_0$ *and* $v^0 \leq 1$. *So our statement is true for* $k = 0$.

Assume that the statement is true for k. *That is, if* $u^0 \geq 1$ *and* $v^0 \leq 1$, *then* $u^k \leq u^0$ *and* $v^k \leq 1$. *We want to show that it also true for* $k+1$. *We know* $u^0 \geq 1$, *so*

$$u^k \leq u^0 u^k. \tag{18}$$

Multiplying both sides by Δt,

$$u^k\Delta t \leq u^0 u^k \Delta t. \tag{19}$$

Since $u^k \leq u^0 + \Delta t u^0 b_{12} v^k$, *we can add these terms into (19) to obtain*

$$u^k + u^k\Delta t \leq u^0 + u^0u^k\Delta t + u^0 b_{12}v^k\Delta t. \tag{20}$$

This implies

$$\frac{u^k(1+\Delta t)}{1+u^k\Delta t+b_{12}v^k\Delta t} \leq u^0. \tag{21}$$

Thus $u^{k+1} \leq u^0$. *Using the same methods as Case 1, we can see* $v^{k+1} \leq 1$. *Therefore, by the Principle of Mathematical Induction, we have our desired result for Case 2. Similarly, we can prove Case 3 and Case 4.*

5 The Convergence of the Semi-Implicit Numerical Scheme

Consider the definition of the derivative and Taylor expansion. By using the Taylor Expansion, there exists $\bar{t} \in (t, t+\Delta t)$ such that

$$u(t+\Delta t) = u(t) + u'(t)\Delta t + \frac{1}{2!}u''(\bar{t})(\Delta t)^2.$$

Therefore

$$\frac{u(t+\Delta t)-u(t)}{\Delta t} = u'(t) + \frac{1}{2}u''(\bar{t})(\Delta t).$$

If $|u''(t)|$ is bounded, we have

$$\frac{u(t+\Delta t)-u(t)}{\Delta t} = u'(t) + O(\Delta t).$$

Using this notation in system (3), we will have

$$\frac{u(t+\Delta t)-u(t)}{\Delta t} = u - u^2 - b_{12}uv + O(\Delta t) \tag{22}$$

$$\frac{v(t+\Delta t)-v(t)}{\Delta t} = \rho(v - v^2 - b_{21}uv) + O(\Delta t). \tag{23}$$

Discretizing the interval $[0, T]$ into N subintervals, the length of each subinterval is $\triangle t = \frac{T}{N}$, and boundaries of each interval are $[t_i, t_{i+1}]$, where $t_i = i\triangle t$ for $i = 0, 1, \cdots, N-1$. At each t_i, we have:

$$\frac{u(t_i+\Delta t)-u(t_i)}{\Delta t} = u(t_i) - [u(t_i)]^2 - b_{12}u(t_i)v(t_i) + O(\Delta t) \tag{24}$$

$$\frac{v(t_i+\Delta t)-v(t_i)}{\Delta t} = \rho(v(t_i) - [v(t_i)]^2 - b_{21}u(t_i)v(t_i)) + O(\Delta t). \tag{25}$$

Let $U^i = u(t_i)$ and $V^i = v(t_i)$. Then,

$$\begin{aligned} \tfrac{U^{i+1}-U^i}{\Delta t} &= U^i - (U^i)^2 - b_{12}U^iV^i + O(\Delta t) \\ \tfrac{V^{i+1}-V^i}{\Delta t} &= \rho(V^i - (V^i)^2 - b_{21}U^iV^i) + O(\Delta t) \end{aligned} \tag{26}$$

for $i = 0, 1, 2, ..., N-1$.
Multiplying through by Δt, we have

$$\begin{aligned} U^{i+1} - U^i &= U^i\Delta t - (U^i)^2\Delta t - b_{12}U^iV^i\triangle t + O(\Delta t)^2 \\ V^{i+1} - V^i &= \rho(V^i\triangle t - (V^i)^2\triangle t - b_{21}U^iV^i\triangle t) + O(\Delta t)^2. \end{aligned} \tag{27}$$

Similarly, our numerical scheme gives us

$$\begin{aligned} u^{i+1} - u^i &= u^i\Delta t - u^{i+1}u^i\Delta t - b_{12}u^{i+1}v^i\Delta t \\ v^{k+1} - v^k &= \rho(v^k\Delta t - v^{k+1}v^k\Delta t - b_{21}v^{k+1}u^k\Delta t). \end{aligned} \tag{28}$$

We want to see how close the true solution (U^i, V^i) is to the numerical solution (u^i, v^i) from equation (14). Our main result is:

Theorem 5.1 *If $u_0(0) > 0, v_0(0) > 0$, and $T > 0$, then the numerical solution to (28) converges to the true solution of (27) uniformly as $\triangle t \to 0$ on $[0, T]$ and the convergence rate is $O(\triangle t)$.*

Lemma 5.2 *$f(x) = (1 + \frac{a}{x})^x$ is an increasing function on $(0, \infty)$ if $a > 0$.*

Proof 5.3 *Note that*

$$f'(x) = \left(1 + \frac{a}{x}\right)^x \left[\ln\left(1 + \frac{a}{x}\right) - \frac{a}{x+a}\right]. \tag{29}$$

Consider $g(x) = \ln x$ *on* $[1, 1 + \frac{a}{x}]$. *By the Mean Value Theorem, there exists* $c \in (1, 1 + \frac{a}{x})$ *such that*

$$g\left(1 + \frac{a}{x}\right) - g(1) = g'(c)[1 + \frac{a}{x} - 1] = \frac{1}{c} \cdot \frac{a}{x}. \tag{30}$$

Since $1 < c < \frac{x+a}{x}$, *we have*

$$1 > \frac{1}{c} > \frac{x}{x+a}.$$

This implies that

$$\frac{1}{c} \cdot \frac{a}{x} > \frac{a}{x+a}.$$

Substituting into (30), we have

$$g\left(1 + \frac{a}{x}\right) - g(1) = \frac{1}{c} \cdot \frac{a}{x} > \frac{a}{x+a}. \tag{31}$$

Since $g(1 + \frac{a}{x}) = \ln(1 + \frac{a}{x})$ *and* $g(1) = 0$, *inequality (31) implies that*

$$\ln\left(1 + \frac{a}{x}\right) > \frac{a}{x+a}. \tag{32}$$

Substituting (32) into (29), we have

$$f'(x) > 0.$$

Therefore $f(x)$ *is an increasing function.*

Proof of Theorem 5.1:

Proof 5.4 *Subtracting (28) from (27),*

$$\begin{aligned} U^{i+1} - u^{i+1} - (U^i - u^i) \\ &= (U^i - u^i)\Delta t - ((U^i)^2 - u^{i+1}u^i)\Delta t \\ &- b_{12}(U^iV^i - u^{i+1}v^i)\Delta t + O(\Delta t)^2 \end{aligned} \tag{33}$$

$$\begin{aligned} V^{i+1} - v^{i+1} - (V^i - v^i) \\ &= \rho[(V^i - v^i)\Delta t - ((V^i)^2 - v^{i+1}v^i)\Delta t \\ &- b_{21}(U^iV^i - u^{i+1}v^i)\Delta t] + O(\Delta t)^2. \end{aligned} \tag{34}$$

Now, let $E^i = U^i - u^i$ *and* $F^i = V^i - v^i$. *From equations (33) and (34), we have*

$$\begin{aligned} E^{i+1} - E^i = &E^i\Delta t - ((U^i)^2 - u^{i+1}u^i)\Delta t \\ &- b_{12}(U^iV^i - u^{i+1}v^i)\Delta t + O(\Delta t)^2 \end{aligned} \tag{35}$$

$$\begin{aligned} F^{i+1} - F^i =& \rho[F^i\Delta t - ((V^i)^2 - v^{i+1}v^i)\Delta t \\ & - b_{21}(U^iV^i - u^{i+1}v^i)\Delta t] + O(\Delta t)^2. \end{aligned} \tag{36}$$

Equations (35) and (36) imply that

$$\begin{gathered} |E^{i+1}| \leq |E^i| + |E^i|\Delta t + |((U^i)^2 - u^{i+1}u^i)|\Delta t \\ |b_{12}(U^iV^i - u^{i+1}v^i)|\Delta t + O(\Delta t)^2 \end{gathered} \tag{37}$$

$$\begin{gathered} |F^{i+1}| \leq |F^i| + \rho[|F^i|\Delta t + |((V^i)^2 - v^{i+1}v^i)|\Delta t \\ |b_{21}(U^iV^i - u^{i+1}v^i)|\Delta t] + O(\Delta t)^2. \end{gathered} \tag{38}$$

First, we need to estimate $|(U^i)^2 - u^{i+1}u^i|$. *Factoring and substituting, we have*

$$\begin{aligned} |(U^i)^2 - u^{i+1}u^i| &= |(U^i)^2 - U^iu^i + U^iu^i - u^{i+1}u^i| \\ &= |(U^i)(U^i - u^i) + U^iu^i - U^iu^{i+1} + U^iu^{i+1} - u^{i+1}u^i| \\ &\leq |U^iE^i| + |U^i(u^i - u^{i+1})| + |u^{i+1}E^i| \\ &\leq C_0|E^i| + C_1|E^i| + C_2|u^i - u^{i+1}|. \end{aligned} \tag{39}$$

But $|u^k - u^{k+1}| = \Delta t|u^k - u^{k+1}u^k - b_{12}u^{k+1}v^k| = O(\Delta t)$. *So* $|u^i - u^{i+1}| = O(\Delta t)$. *Thus,* $|(U^i)^2 - u^{i+1}u^i| \leq C_0|E^i| + C_1|E^i| + O(\Delta t)$.
Equivalently, $|(U^i)^2 - u^{i+1}u^i| \leq C_2|E^i| + O(\Delta t)$. *We also need to estimate* $|U^iV^i - u^{i+1}v^i|$.

$$\begin{aligned} |U^iV^i - u^{i+1}v^i| &= |U^iV^i - U^iv^i + U^iv^i - u^{i+1}v^i| \\ &= |U^i(V^i - v^i) + U^iv^i - u^iv^i + u^iv^i - u^{i+1}v^i| \\ &\leq |U^i(V^i - v^i)| + |v^i(U^i - u^i)| + |v^i(u^i - u^{i+1})| \\ &\leq C_3|F^i| + C_4|E^i| + O(\Delta t). \end{aligned} \tag{40}$$

Substituting (39) and (40) into (37), we now have the following equation:

$$\begin{aligned} |E^{i+1}| \leq& |E^i| + |E^i|\Delta t + [C_2|E^i| + O(\Delta t)]\Delta t \\ & + b_{12}[C_3|F^i| + C_4|E^i| + O(\Delta t)]\Delta t + O(\Delta t)^2. \end{aligned} \tag{41}$$

Similarly, we can rewrite (38) as

$$\begin{aligned} |F^{i+1}| \leq& |F^i| + \rho[|F^i|\Delta t + [C_5|F^i| + O(\Delta t)]\Delta t \\ & + b_{21}[C_6|F^i| + C_7|E^i| + O(\Delta t)]\Delta t] + O(\Delta t)^2. \end{aligned} \tag{42}$$

Now, we can add (41) and (42) to derive a new equation:

$$|F^{i+1}| + |E^{i+1}| \leq |F^i| + |E^i| + C_8\Delta t|F^i| + C_9\Delta t|E^i| + O(\Delta t)^2. \tag{43}$$

And if we let $C_{10} = \max\{C_8, C_9\}$*, then*

$$|F^{i+1}| + |E^{i+1}| \leq |F^i| + |E^i| + C_{10}\Delta t|F^i| + C_{10}\Delta t|E^i| + O(\Delta t)^2. \tag{44}$$

Next, let $W^i = |F^i| + |E^i|$*. We have*

$$\begin{aligned} W^{i+1} \leq& (1 + C_{10}\Delta t)W^i + O(\Delta t)^2 \\ W^i \leq& (1 + C_{10}\Delta t)W^{i-1} + O(\Delta t)^2 \\ &\vdots \\ W^1 \leq& (1 + C_{10}\Delta t)W^0 + O(\Delta t)^2. \end{aligned} \tag{45}$$

Note that $W^0 = 0$*. Therefore*

$$\begin{aligned} W^{i+1} \leq& (1 + C_{10}\Delta t)^2 W^{i-1} + O(\Delta t)^2 + \\ & (1 + C_{10}\Delta t)O(\Delta t)^2 \\ \leq& (1 + C_{10}\Delta t)^3 W^{i-2} + O(\Delta t)^2 + \\ & (1 + C_{10}\Delta t)O(\Delta t)^2 + (1 + C_{10}\Delta t)^2 O(\Delta t)^2 \\ & \vdots \\ \leq& (1 + C_{10}\Delta t)^{i+1} W^0 + O(\Delta t)^2[(1 + C_{10}\Delta t) + \\ & (1 + C_{10}\Delta t)^2 + \ldots + (1 + C_{10}\Delta t)^i] \\ =& O(\Delta t)^2[\frac{(1 + C_{10}\Delta t)^{i+1} - 1}{C_{10}\Delta t}]. \end{aligned}$$

We claim that $[\frac{(1+C_{10}\Delta t)^{i+1}-1}{C_{10}}]$ *is bounded by a constant.*
In fact,

$$\begin{aligned} \frac{(1 + C_{10}\Delta t)^{i+1} - 1}{C_{10}} &\leq \frac{(1 + C_{10}\Delta t)^N - 1}{C_{10}} \\ &\leq \frac{[1 + C_{10}(\frac{T}{N})]^N - 1}{C_{10}} \\ &= \frac{[1 + \frac{C_{11}}{N}]^N - 1}{C_{10}}. \end{aligned} \tag{46}$$

Since

$$\lim_{N\to\infty} \frac{(1 + \frac{C_{11}}{N})^N - 1}{C_{10}} = \frac{e^{C_{11}} - 1}{C_{10}}, \tag{47}$$

and by Lemma 5.2, $[1 + \frac{C_{11}}{N}]^N$ *is an increasing function as* $N \to \infty$*, we have*

$$\frac{(1 + \frac{C_{11}}{N})^N - 1}{C_{10}} \leq \frac{e^{C_{11}} - 1}{C_{10}} = C_{12}. \tag{48}$$

Therefore,

$$W^{i+1} \leq O(\Delta t)C_{12} = O(\triangle t). \tag{49}$$

This completes the proof.

6 The Numerical Experiments

In this section, we present some results of computational experiments to show that the proposed difference scheme is stable and gives reasonable solutions.

Case 1: $b_{12} = 0.5$, $b_{21} = 0.5$, $\rho = 1$, $\alpha = 0.4$, $\beta = 0.5$.

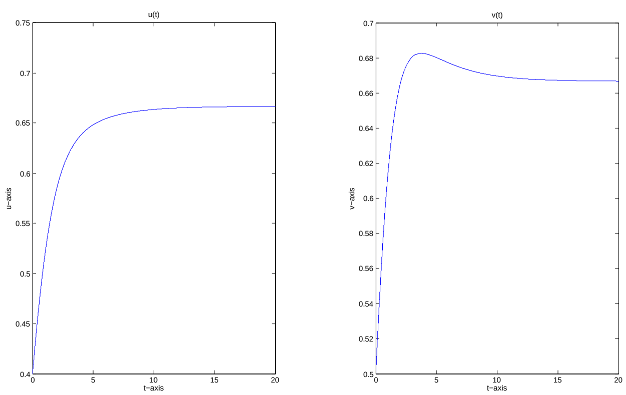

Figure 1: Solutions for $b_{12} = b_{21} = 0.5$, $u(0) = 0.4$, $v(0) = 0.5$.

Note that the interaction rates between the two predators are the same, so neither predator is significantly stronger than the other. The graph shows that the numerical solution (u, v) will converge to $(0.66, 0.66)$ which is a stable steady solution of equation (3). This is consistent with our theoretical results in Case 1 of Section (2.4).

Case 2: $b_{12} = 1.5$, $b_{21} = 0.5$, $\rho = 1$.

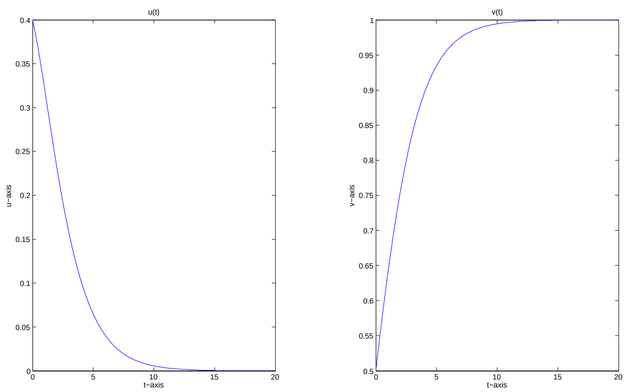

Figure 2: Solutions for $b_{12} = 1.5, b_{21} = 0.5$, $u(0) = 0.4$, $v(0) = 0.5$.

In this case, the interaction rate $b_{12} > b_{21}$, so predator v is a stronger animal than predator u, meaning species u should approach extinction. The graph shows that the numerical solution (u, v) will converge to $(0, 1)$ which means the first species will die out and the second species will approach its carrying capacity. This is in agreement with the theoretical result found in Section 2 where $(0, 1)$ is stable.

Case 3: $b_{12} = .5$, $b_{21} = 1.5$, $\rho = 1$.

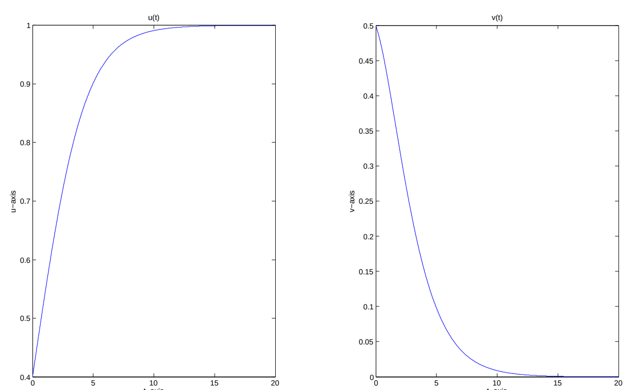

Figure 3: Solutions for $b_{12} = .5, b_{21} = 1.5$, $u(0) = 0.4$, $v(0) = 0.5$.

The graph shows that the numerical solution (u, v) will converge to $(1, 0)$ which means the first species will approach its carrying capacity and the second species will die out. This is in agreement with the theoretical result found in Section 2 where $(1, 0)$ is stable.

Case 4: $b_{12} = 4$, $b_{21} = 2$, $\rho = 1$.

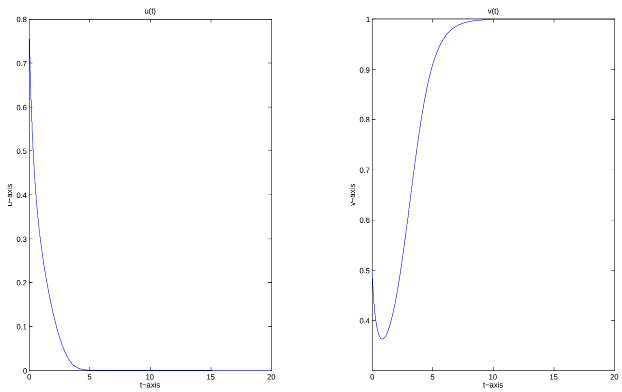

Figure 4: Solutions for $b_{12} = 4, b_{21} = 2$, $u(0) = 0.8$, $v(0) = 0.5$.

The graph shows that the numerical solution (u, v) will converge to $(0, 1)$ which means the first species will die out and the second species will approach its carrying capacity.

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