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Abstract. This paper shows a simple tabular procedure derived from the method of undetermined coefficients for finding a particular solution to differential equations of the form:

$$\sum_{j=0}^{m} a_j \frac{d^j y}{dx^j} = P(x)e^{\alpha x} \tag{1}$$

This procedure reduces the derivatives of the product of an arbitrary polynomial and an exponential to rows of constants representing the coefficients of the terms. The rows are each multiplied by a_j and summed to produce a m^{th} order differential equation such that it's solution is the polynomial part of the particular solution of equation 1. Solving this corresponding differential equation determines the coefficients of the polynomial. The underlying algebra of this conversion and its formulaic implication are then discussed. Using the formula derived, the particular solution for equation 1 is found. This procedure is based on but different in application than the method of undetermined coefficients because while the method of undetermined coefficients requires substitution of a product of a polynomial, Q, and an exponential into the differential equation immediately, this procedure is derived from the examination of the substitution of the product of any function and an exponential. This allows for a richer understanding of the relationship between the differential equation for y and the differential equation for Q. Ultimately this method is better than the method of undetermined coefficients because it is more straightforward. In any case, both methods solve the same problem but KMO is faster.

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1 Introduction

Differential equations are a very old field of research. They began with Newton and his formulation of natural laws using calculus. In particular, they were used to model physical phenomena. Many famous mathematicians such as Leibniz, Euler, Laplace, and Lagrange also made contributions to the field[1]. Current researchers focus more on nonlinear equations due to the increase in computational abilities and geometric techniques. Linear differential equations are well understood especially those with constant coefficients. These equations were used to simplistically model physical systems. For example, non-homogenous second order linear equations may be used to model forced springs where the forcing is the nonhomogeneous part. So for such a system we have equations of the form $m\frac{d^2y}{dt^2}+\gamma\frac{dy}{dt}+ky=F(t)$ where m is the mass, γ is the dampening constant, k is the spring constant, and $F(t)$ is the forcing function[1]. Of non-homogenous equations the ones treated in this paper are the ones in the form of equation 1. The established method for solving these equations is the method of undetermined coefficients. This procedure involves the substitution of particular solution $y_p = e^{\alpha x}x^s\sum_{k'=1}^{n} c_{k'}x^{k'}$ or $y_p = e^{\alpha x}Q(x)$ where s is the number of times α is a root of the characteristic polynomial. This leads to a to a system of equations which yield $c_{k'}$. We instead consider the corresponding differential equation for Q first by means of a tabular procedure, then by means of a formula derived from the underlying algebra at play in the tabular procedure. Coupling this formula with triangular matrix inversion and a more general procedure we derive a particular solution of equation 1.

First, let us examine the method of undetermined coefficients by using it to solve the following equation:

$$2\frac{d^2y}{dx^2}+3\frac{dy}{dx}+y=(5x+3)e^{-2x} \tag{2}$$

We substitute $y=(ax+b)e^{-2x}$ into the equation to obtain with cumbersome applications of the product rule:

$$(3a)xe^{-2x}+(3b-5a)e^{-2x}=(5x+3)e^{-2x}$$

Now we can divide by e^{-2x} and equate $3a$ with 5 and $3b-5a$ with 3. We can then solve these equations to obtain a and b.

In this paper, we develop a technique to solve equation 1 which we call the KMO method. We begin explaining KMO by showing the tabular method from which it is derived. We do this by using it on the example equation just given. We then discuss the algebra at play in the tabular method and derive an equation that does the same thing as the table. Finally, we solve non-homogenous equations where the function on the right is a polynomial which together with the aforementioned equation completely solves the problem with our method. Along the way, we test the example equation with this method and compare the solution with that of the method of undetermined coefficients.

2 Example Application of the Tabular Procedure

We consider equation 2. The general solution to this nonhomogeneous equation is found by adding the particular solution to the solution of the corresponding homogenous equation. The homogenous solution is of the form: (with c_1 and c_2 constants)

$$y_h = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$

Where c_1 and c_2 are constants. By the quadratic equation, the solutions of the characteristic equation may be found:

$$\lambda_{1,2} = \frac{-3 \pm \sqrt{3^2 - 4(2)(1)}}{2(2)}$$

The KMO method is concerned exclusively with finding a particular solution. Thus henceforth let y mean y_p. Usually y is found with the method of undetermined coefficients as detailed in [1]. The method of undetermined coefficients is somewhat time consuming as it often involves arduous algebra. The tabular procedure begins with a solution of the form $y = Q(x)e^{\alpha x}$ where Q is a polynomial. In the case of equation 2 this is $y = Q(x)e^{-2x}$. Using an arbitrary polynomial eliminates the trial and error that sometimes occurs in the method of undetermined coefficients. We take derivatives of y:

$$y = Q(x)e^{-2x}$$

$$\frac{dy}{dx} = \frac{dQ}{dx}e^{-2x} - 2Q(x)e^{-2x}$$

$$\frac{d^2y}{dx^2} = \frac{d^2Q}{dx^2}e^{-2x} - 4\frac{dQ}{dx}e^{-2x} + 4Q(x)e^{-2x}$$

Now we place the coefficients into a table:

	$Q(x)$	$\frac{dQ}{dx}$	$\frac{d^2Q}{dx^2}$
y	1	0	0
$\frac{dy}{dx}$	-2	1	0
$\frac{d^2y}{dx^2}$	4	-4	1

The tabular format is useful for examining the coefficients because when the derivatives of y are summed along with their coefficients, the coefficients of terms with like order derivatives of Q are summed together. For example, when the y term is added to the first derivative terms, the coefficient 1 of $Q(x)e^{-2x}$ in y is added to the coefficient -2 of $Q(x)e^{-2x}$ in $\frac{dy}{dx}$. However, before the summation the coefficients of the differential equation for y must be accounted for. Now we multiply each of the rows by the corresponding coefficients of the derivatives of y:

	$Q(x)$	$\frac{dQ}{dx}$	$\frac{d^2Q}{dx^2}$
$1 \cdot y$	1	0	0
$3 \cdot \frac{dy}{dx}$	-2	1	0
$2 \cdot \frac{d^2y}{dx^2}$	4	-4	1
	3	-5	2

The sum of the components in each column produces the coefficients for the derivatives of Q in the corresponding differential equation for Q. Since all terms contain the same exponential term, which is never zero, it may be divided out yielding:

$$2\frac{d^2Q}{dx^2} - 5\frac{dQ}{dx} + 3Q(x) = 5x + 3 \tag{3}$$

In equation 3, the $Q(x)$ term is the polynomial of highest degree and by equivalence must be of first degree. This means with $Q(x) = ax + b$, $\frac{dQ}{dx} = a$, and $\frac{d^2Q}{dx^2} = 0$. Substitution into equation 3 yields $-5a + 3ax + 3b = 5x + 3$. Algebraic manipulation of the last equation yields $a = \frac{5}{3}$ and $b = \frac{34}{9}$. Thus $y = (\frac{5}{3}x + \frac{34}{9})e^{-2x}$.

3 Removal of the Exponential

We will now show that the numbers in the table follow a certain pattern and thus we may derive a formula for the coefficients b_i of the corresponding differential equation for Q that will be useful for the larger problem at hand of solving equation 1.

Lemma 3.1. *The coefficients of the differential equation for Q, $\sum_{i=0}^{m} b_i \frac{d^iQ}{dx^i} = P(x)$, resulting from substitution of $Q(x)e^{\alpha x}$ for y into equation 1 and division by $e^{\alpha x}$ are:*

$$b_i = \sum_{j=0}^{m} \binom{j}{i} a_j \alpha^{j-i}$$

Proof. The goal is to find the value of $\mathbf{b}$ (where $\mathbf{b}$ is a vector whose i'th component is b_i) in the differential equation for Q. Since taking successive derivatives of a product of functions derives this equation, a rule for the n^{th} derivatives of this product is useful. The Leibniz rule states that $\frac{d^n}{dx^n} f(x)g(x) = \sum_{i=0}^{n} \binom{n}{i} \frac{d^if}{dx^i} \frac{d^{n-i}g}{dx^{n-i}}$ where $\binom{n}{i}$ is the binomial coefficient. If f is let to be Q and g is let to be $e^{\alpha x}$, then by the Leibniz rule, $\frac{d^jy}{dx^j} = \sum_{i=0}^{j} \binom{j}{i} \frac{d^iQ}{dx^i} \alpha^{j-i} e^{\alpha x}$. Substitution into the differential equation of Q and division by $e^{\alpha x}$ (a common nonzero factor) yields the following equation:

$$\sum_{j=0}^{m} a_j \sum_{i=0}^{j} \binom{j}{i} \frac{d^iQ}{dx^i} \alpha^{j-i} = P(x)$$

The a_j can be moved inside the inner summation since either way all terms with j are summed by the first summation. The inner summation may be summed to m instead since terms indexed by $i > j$ vanish due to the binomial coefficient. This is useful since as a result the orders of summation may now be switched due to the commutative property of summation. With this new ordering, $\frac{d^iQ}{dx^i}$ may be moved outside of the inner summation to yield the following equation:

$$\sum_{i=0}^{m} \frac{d^iQ}{dx^i} \sum_{j=0}^{m} \binom{j}{i} a_j \alpha^{j-i} = P(x)$$

The equation may be recognized as the differential equation for Q where b_i is the inner summation. □

3.1 Application to Equation 2

The vector **a** may be quickly seen to be $[1, 3, 2]$, $\alpha = -2$, and $m = 2$. Thus the expression for b_i is as follows:

$$\sum_{j=0}^{2} \binom{j}{i} a_j (-2)^{j-i}$$

The calculations of the coefficients are as follows:

$$b_0 = (1 \cdot 1 \cdot 1) + (1 \cdot 3 \cdot (-2)) + (1 \cdot 2 \cdot 4) = 3$$

$$b_1 = 0 + (1 \cdot 3 \cdot 1) + (2 \cdot 2 \cdot (-2)) = -5$$

$$b_2 = 0 + 0 + (1 \cdot 2 \cdot 1) = 2$$

Thus the result is the same as equation 3.

4 Particular Solution of Equation 4

Before we state the particular solution we find, we establish some convenient notation. Let n be the degree of P and let p be the index of the first non-vanishing coefficient of the differential equation in Q. Let T and R be two sets such that $T = \{i \mid i \in \mathbb{Z}, n - m + p < i \leq n\}$ and $R = \{i \mid i \in \mathbb{Z}, 0 \leq i \leq n-m+p\}$. Also, let k_l be the coefficients such that $P(x) = \sum_{l=0}^{n} k_l x^l$. Finally, let $a_i^j = \frac{(i+j)!}{i!} b_j$ and $n_T = m - p$.

Theorem 4.1. *A particular solution is $Q(x)e^{\alpha x}$ where $Q(x) = \sum_{k'=p}^{n+p} c_{k'} x^{k'}$. The leading coefficient of Q is the leading coefficient of P divided by a_n^p where n is the degree of P; the remaining coefficients are determined recursively [3] in the following way. For t such that $n - t \in T$, $c_{p+n-t} = \frac{1}{a_{n-t}^p}(k_{n-t} - \sum_{i=1}^{t} a_{n-t}^{p+i} c_{p+n-t+i})$ and for t such that $n - t \in R$, $c_{p+n-t} = \frac{1}{a_{n-t}^p}(k_{n-t} - \sum_{i=1}^{n_T} a_{n-t}^{p+i} c_{p+n-t+i})$.*

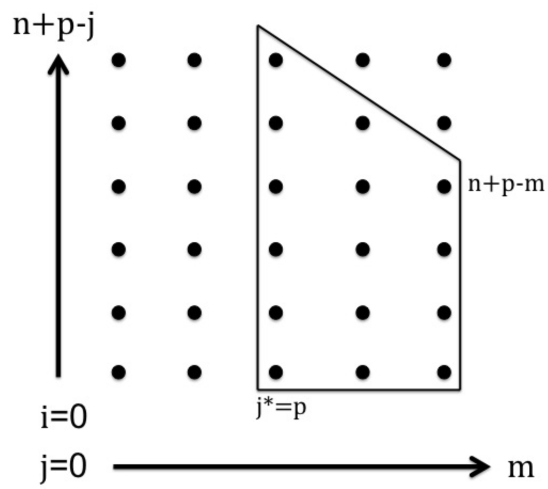

Figure 1: Region over which the summation takes place

Proof. Let $P_n(x)$ specify the degree of P as n. The converted differential equation of m^{th} order may now be explicitly written as the following:

$$\sum_{j=0}^{m} b_j \frac{d^j Q}{dx^j} = P_n(x) \tag{4}$$

The degree of Q depends on the value of p such that b_p is the first non-vanishing coefficient. This may be attributed to the fact that if p=3 then the lowest order derivative is $\frac{d^3 Q}{dx^3}$ meaning that if $n = 4$, $\frac{d^3 Q}{dx^3}$ must be of degree 4 since there are no higher degree polynomials being summed. The maximum degree of Q is $n + p$ since the p^{th} derivative of Q reduces to a polynomial of degree n, which is the same degree as P. This also implies $c_{k'}$ for $k' < p$ can be chosen to be zero since they do not appear in the equation and therefore are not determined by it. Consequently, $Q(x) = \sum_{i=0}^{n+p} c_i x^i$ where the constant coefficients, c_i, are to be found. Since $\frac{d^j Q}{dx^j} = \sum_{i=0}^{n+p-j} \frac{(i+j)!}{i!} c_{i+j} x^i$, substitution into equation 4 yields $\sum_{j=p}^{m} b_j \sum_{i=0}^{n+p-j} \frac{(i+j)!}{i!} c_{i+j} x^i = P_n(x)$ or more simply:

$$\sum_{j=p}^{m} \sum_{i=0}^{n+p-j} a_i^j c_{i+j} x^i = P_n(x) \tag{5}$$

As previously noted, vectors indexed by the dummy variable of the outer sum may be moved into the inner sum. The product of $\frac{(i+j)!}{i!}$ and b_j produces a_i^j as previously defined. The sum on the left hand side of equation 5 may be imagined as summation over the parallelogram of integers pictured (see figure 1).

The i^{th} row represents the i^{th} degree term of the resulting polynomial on the left hand side of the equation. The unknown variable in question is **c**. As Q is differentiated, the $c_{k'}$

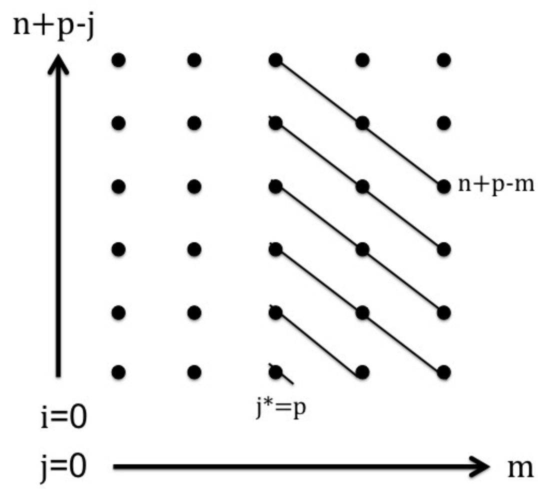

Figure 2: One-dimensional subregions over which $k^{'}$ is constant

coefficients are pushed back to terms of lower degrees, this means that if a $c_{k'}$ in column h is on row t, in column $h+1$, that $c_{k'}$ is on row $t-1$. Thus the components of $\mathbf{c}$ are on the diagonals pictured (see figure 2).

Regardless of m, p, or n, $a_n^p c_{n+p}$ is not summed with any other coefficient and thus solving for c_{n+p} is simple. It is the solution to $a_n^p c_{n+p} = k_n$ which is $c_{n+p} = \frac{k_n}{a_n^p}$. There are three possibilities for the rest of the summation. These possibilities depend on shape of the region over which the summations take place. If $p = 0$ and $m \geq n$, the region is triangular. If $p = m$, the region is the rectangular strip of a_s^m with $s = 0 \dots n$. If otherwise, the region is trapezoidal. If $p = m$ it is easy to see that the $c_{k'}$ are solved in the same way as c_{n+p}. For triangles and trapezoids, even though there is a summation taking place, the solution for the third from last coefficient depends on the last and second from last coefficients. This is due to the fact that the diagonals depend on the same coefficient. The triangular region T may be seen to occur from $i = n+p-m+1$ to n since for i less than that all terms of degree p or more are being summed. The rectangular region R is thus $i \leq n+p-m$ if it exists. The sum of a row in T that is t units from the top is the sum of the p^{th} term to the t^{th} term from it and the sum of these coefficients is the t^{th} term from the leading coefficient of P. So we have:

$$k_{n-t} = a_{n-t}^p c_{n+p-t} + \sum_{i=1}^{t} a_{n-t}^{p+i} c_{n+p-t+i}$$

As previously discussed, the $c_{k'}$ in the summation are already known; therefore with algebra we may solve for c_{n+p-t}. That is, subtraction by the summation and division by a_{n-t}^p yields:

$$c_{p+n-t} = \frac{1}{a^p_{n-t}}(k_{n-t} - \sum_{i=1}^{t} a^{p+i}_{n-t} c_{p+n-t+i})$$

The expression for $c_{k'}$ whos indices are in R are similar except that the upper index of the sum has no dependence on t rather is fixed by the number of js from p such that p plus this number is m. This is n_T. Thus the $c_{k'}$ may be determined recursively by $c_{p+n-t} = \frac{1}{a^p_{n-t}}(k_{n-t} - \sum_{i=1}^{t} a^{p+i}_{n-t} c_{p+n-t+i})$ and $c_{p+n-t} = \frac{1}{a^p_{n-t}}(k_{n-t} - \sum_{i=1}^{n_T} a^{p+i}_{n-t} c_{p+n-t+i})$.

□

Combining the results from section 3 and this section we have the three equations determining the coefficients of the solution:

$$a^j_i = \frac{(i+j)!}{i!} \sum_{k=0}^{m} \binom{k}{i} a_k \alpha^{k-i} \tag{6}$$

$$c_{p+n-t} = \frac{1}{a^p_{n-t}}(k_{n-t} - \sum_{i=1}^{t} a^{p+i}_{n-t} c_{p+n-t+i}) \quad \text{if } p+n-t \in T \tag{7}$$

$$c_{p+n-t} = \frac{1}{a^p_{n-t}}(k_{n-t} - \sum_{i=1}^{n_T} a^{p+i}_{n-t} c_{p+n-t+i}) \quad \text{if } p+n-t \in R \tag{8}$$

An application of KMO consists of computing the a^j_j using equation 6 and then using either the equation 7 or both equations 7 and 8 to find the coefficients c_i, $0 \le i \le n+p$. The solution is then $P(x)e^{\alpha x}$.

4.1 Application to Equation 2

We are solving

$$2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = (5x+3)e^{-2x}$$

First we note the corresponding differential equation for Q is

$$2\frac{d^2Q}{dx^2} - 5\frac{dQ}{dx} + 3Q(x) = 5x + 3$$

A quick examination of order and vanishing coefficients yields that $m = 2$, $n = 1$, and $p = 0$. As a result, the leading coefficient of Q is c_1 and the region over which the summation takes place is the triangular one pictured (see figure 3).

Since $c_{p+n} = \frac{k_n}{a^p_n}$ and the leading coefficient of P is 5, $c_1 = \frac{5}{a^0_1}$. Since $a^j_i = \frac{(i+j)!}{i!} b_j$ and $b_0 = 3$, $a^0_1 = 1(3) = 3$. Also we see $c_1 = \frac{5}{3}$. Since $c_{p+n-t} = \frac{1}{a^p_{n-t}}(k_{n-t} - \sum_{i=1}^{t} a^{p+i}_{n-t} c_{p+n-t+i})$,

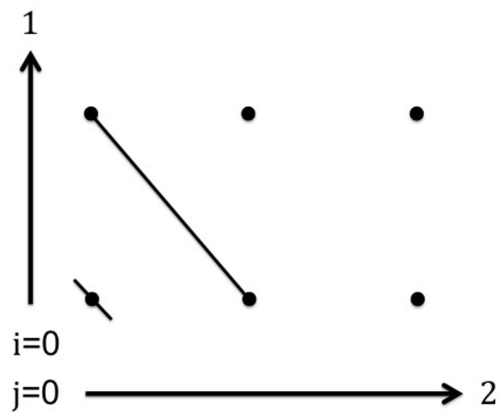

Figure 3: Region over which the summation takes place for equation 2

$c_0 = \frac{1}{a_0^0}(3 - a_0^1(\frac{5}{3}))$. $a_0^0 = 3$ and $a_0^1 = -5$ so therefore $c_0 = \frac{4}{3}$. Combining this information, $Q(x) = \frac{5}{3}x + \frac{34}{9}$. The solution is the same as previously shown.

5 Conclusion

The KMO method provides a quick alternative to solving differential equations by using the formulas derived. Substitution of parameters **a**, **k** and α into the formulae quickly yield a particular solution to equation 4 in the form of a vector **c**. Furthermore, these formulae may be programed into a computer with fractional arithmetic to obtain exact solutions. Included is the numerical MATLAB code for KMO with triangular summation regions equivalent to triangular matrix inversion.

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A MATLAB code for KMO

```
function [ c ] = KMO(a, k, alpha)
m = numel(a) - 1;
n = numel(k) - 1;

% calculate b
b = zeros(1, m + 1);
for i=0:m
    for j=i:m
        b(i+1) = b(i+1) + nchoosek(j,i)*a(j+1)*(alpha^(j-i));
    end
end

% get p, nT, and the order of Q
p = find(b, 1, 'first')-1;
nT = m - p;
c = zeros(1,n + p + 1);

% calculate c
c(p + n + 1) = (1/(b(p + 1)*factorial(p + n))/factorial(n))*(k(n + 1));
for t=1:nT - 1
    sum = 0;
    for i=1:t;
        sum = sum + (c(p + n - t + i + 1)*(b(p + 1 + i)*factorial(p + n ...
            - t + i))/factorial(n - t));
    end
    c(p + n - t + 1) = (1/(b(p + 1)*factorial(p + n - t))/factorial(n - ...
        t))*(k(n - t + 1)-sum);
end
end
```