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Graceful Labelings of Pendant Graphs

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Graceful Labelings of Pendant Graphs

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Abstract. In 1967, Alexander Rosa introduced a new type of graph labeling called a “graceful labeling.” This paper will provide some background on graceful labelings and their relation to certain types of graphs called “pendant graphs.” We will also present new results concerning a specific type of graceful labeling of pendant graphs as well as further areas of research.

Acknowledgements: I would like to give a special thanks to my mentor Jeff Rushall for all of his help and guidance, and to Ian Douglas for sharing his programming expertise.

1. Introduction

From elementary school students to university seniors, most students have some exposure to graphs. The phrases "graph the function" and "based on the graph" are a common occurrence in most standardized math courses. They conjure up images of coordinate axes and intersecting curves, which often indicates that calculations involving slopes and area are on the horizon. As the only exposure students have to graphs occur in geometry and calculus-related courses, which only study graphs of this form, most students believe that all graphs involve axes and functions. However, that is simply not true.

In fact, most types of graphs do not have coordinate axes or involve a specific function. Outlines for essays, the game board for *The Settlers of Catan*®, and fire evacuation route plans are all examples of graphs without these characteristics. To understand why, consider the definition of a graph.

Definition 1.1 A *graph* is a collection of points together with a collection of lines connecting some subset of those points. The points are called *vertices* and the lines are called *edges*.

Notice that this definition does not incite an image students typically associate with graphs. Instead, it seems to be more akin to childhood games and activities, such as "connect-the-dots" puzzles and the Chinese board game *Go*. A closer look at the informal definition of a graph of a function, however, shows otherwise.

Definition 1.2 The *graph of a function f* is the set of ordered pairs showing the values taken by f over the domain of f.

Thus, a graph of a function can be thought of as a graph of vertices with no edges connecting them! The curve does not look like a set of distinct vertices because there are infinitely many vertices "very close" together. Therefore, to a graph theorist, all of the images in Figure 1 are considered to be graphs.

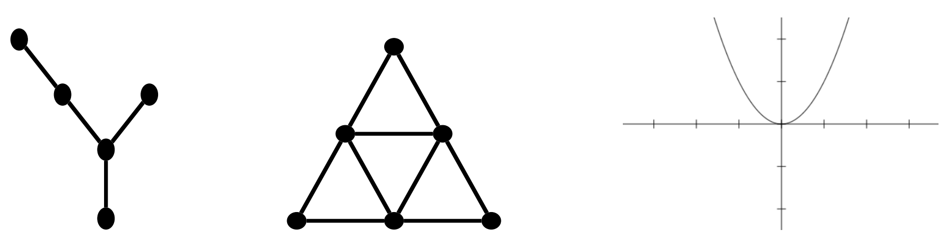


Figure 1

In the next few sections, we will discuss some of the basic properties of graphs, including "graph labelings," and how these can be assigned to different graphs. We will then define a "graceful labeling" and study a few examples to understand when such a labeling exists and how we can produce these labelings for certain types of graphs. The remainder of the paper will then focus on "pendant graphs" and their graceful labelings. To conclude this paper, we will see some new results as well as provide some open questions.

2. Graph Theory Basics

The study of the properties and structures of graphs is called *graph theory*. Ideas and results from graph theory have been used to help model a variety of systems, from engineering blueprints to outlining a college essay. Different properties may be more valuable for modeling certain systems than others, so it is important to categorize graphs in terms of their structure.

Example 2.1 Consider the graphs in Figure 2.

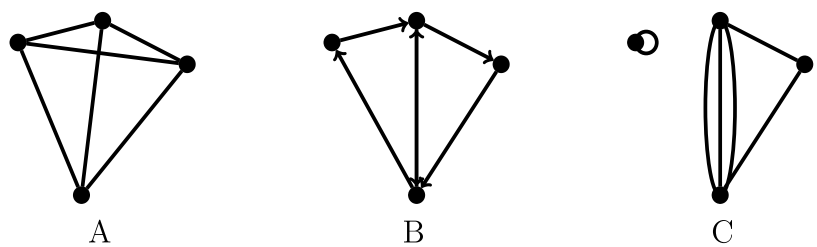


Figure 2

Although each graph contains 4 vertices and 6 edges (note that each arrow direction of graph B determines an edge, so the segment connecting the top and bottom vertices is really 2 edges), some differences between the graphs are apparent. For example, graph C has a vertex that is not connected by an edge to any other vertex in the graph. We say graph C is *not connected*. To differentiate these graphs further, note the following definitions.

Definition 2.1 A graph is *connected* if there exists a sequence of edges (called a *path*) that can be transversed from any vertex to any other vertex in the graph.

Definition 2.2 A graph is *directed* if the edges of the graph are assigned an initial and terminal vertex. This is usually indicated by an arrowhead located somewhere along the edge that points to the terminal vertex.

Definition 2.3 A graph is *simple* if it is an undirected graph that contains neither loops nor multiple edges connecting the same vertices.

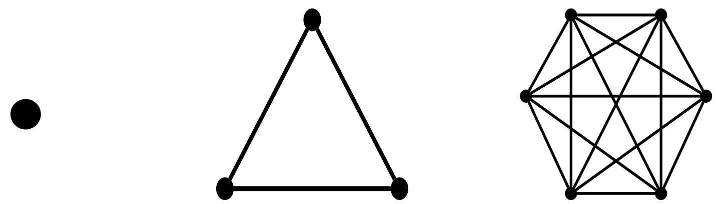
Definition 2.4 A *pseudograph* is a graph that may contain loops (i.e. an edge connecting a vertex to itself) or multiple edges connecting the same pair of vertices.

Definition 2.5 Two vertices are *adjacent* if there is an edge connecting them.

Example 2.1 (cont.) To classify the graphs in Figure 2, consider whether each graph has multiple edges connecting a pair of vertices or a loop, if the edges are directed, and whether the graph is connected. Since graph C has a loop, three edges connecting the same vertices, and no directions indicated along its edges, we can classify graph C as an undirected pseudograph that is not connected. Similarly, the edges of graph B are directed, yet there is still a path from any vertex of B to every other vertex of B. As there are two edges connecting the top and bottom vertices, graph B is a connected, directed pseudograph. Finally, graph A contains no loops or multiple edges and each vertex of A is adjacent to every other vertex of A. Therefore graph A is a simple, connected graph.

Based on these properties and others, graph theorists organize graphs into "families" of graphs that share some predetermined traits. These families often have special names, such as *tree graphs*, *wheel graphs*, and *spraying pipe graphs*. Example 2.2 defines and explains another family of graphs, called *complete graphs*, which we will use later.

Example 2.2 Consider the graphs in Figure 3.



A B C

Figure 3

Graphs A, B, and C are all members of the complete graph family. To be a member of this family, every pair of vertices in the graph must be connected by an edge. These graphs are referred to as K_n where n is the number of vertices in the graph. Since graph A has only one vertex, there is no pair of vertices that can be connected. Hence the lone vertex with no edges is the complete graph K_1. Graphs B and C are the complete graphs K_3 and K_6 respectively.

However, it should be noted that a graph can belong to more than one family. For example, graph B is also the cycle on 3 vertices. So graph B is both the complete graph K_3 and the cycle graph C_3.

3. Graph Labelings

Along with its inherent structure, a graph can be assigned additional properties. This is often accomplished by assigning labels to the vertices and edges of the graph.

Example 3.1 Suppose Figure 4 is modeling the population of frogs in a certain pond.

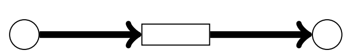


Figure 4

In its current state, the graph provides very little information about the size of the frog population and how it changes over time. If we were to label the directed edges with the birth rate and death rate of the population and the central vertex with the current frog population, Figure 4 would become a more useful model of the frog population.

Definition 3.1 A *graph labeling* is an assignment of integers to the vertices, edges, or both of a graph that is subject to certain conditions.

The suggested changes to Figure 4 are an example of a graph labeling. Specifically, the edges and vertices would be assigned integers that are subject to the conditions "the directed edges are labeled with the birth rate and death rate of the population" and "the central vertex is labeled with the current frog population," respectively.

Example 3.2

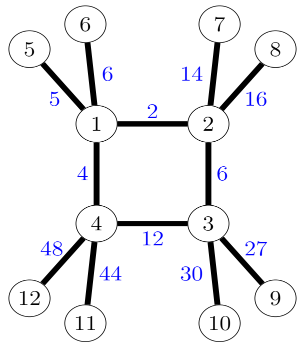


Figure 5

Figure 5 depicts another graph labeling where edges are labeled with the product of the labels of the vertices they connect. Note that two edges are assigned the same label. This is acceptable so long as the conditions chosen for the labeling do not require labels to be unique.

Example 3.3 Consider the graph in Figure 6. The 6 vertices and 6 edges have been labeled according to the following rules:

1. The edges must be labeled using each of the integers 1 through 6 once.
2. The vertices must be labeled using each of the integers 0 through 6 no more than once.
3. Every edge must be the positive difference of the vertex labels it connects.

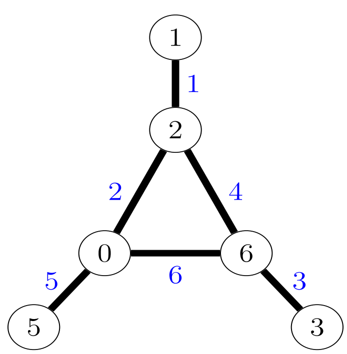


Figure 6

The graph labeling shown in Example 3.3 is called a *graceful labeling*.

4. Graceful Labelings

In 1967, Alexander Rosa [5] introduced a new type of graph labeling which he named a β-*labeling*. As Rosa reflects in [6], it was believed these labelings would help solve Ringel's conjecture, which involves decomposing a complete graph into isomorphic subgraphs. This type of labeling has since been renamed a *graceful labeling* and has been used to model situations in many fields, including radio astronomy, circuit design, coding theory, and X-ray crystallography. It is formally stated in Definition 4.1.

Definition 4.1 A simple, connected graph with q edges is said to be *graceful* if its vertices and edges are labeled such that:

(a) each vertex is assigned a distinct integer from the set $\{0, \ldots, q\}$,

(b) each edge is assigned a distinct integer from the set $\{1, \ldots, q\}$ such that the label is equal to the absolute value of the difference of the two vertices the edge connects.

By returning to the labeling rules used in Example 3.3, it can be seen that the vertices are assigned distinct integers from the set $\{0, \ldots, 6\}$ (by rule 2) and the edges are assigned distinct integers from the set $\{1, \ldots, 6\}$ (by rule 1) so that each edge is the positive difference of the vertex labels it connects (by rule 3).

The graceful labeling shown in Figure 6 is by no means the only graceful labeling for the graph shown. In fact, this particular graph has 6 distinct graceful labelings. Other graphs have more than 10,000 distinct graceful labelings. Yet many graphs cannot be gracefully labeled.

Example 4.1 Reconsider the graphs from Figure 2 (shown below).

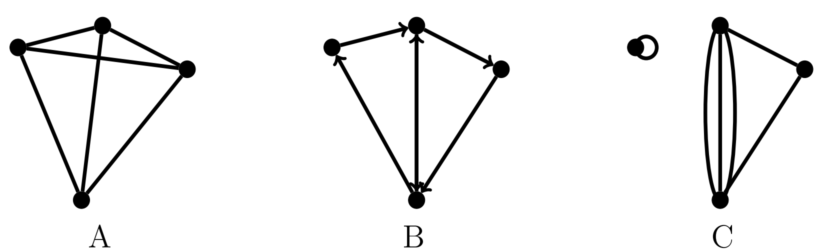


Figure 2

The definition of a graceful labeling requires a simple, connected graph. Thus, graphs B and C do not have graceful labelings since neither is a simple, connected graph. Graph A, however, might have a graceful labeling. In fact, an example of such a labeling is shown in Figure 7.

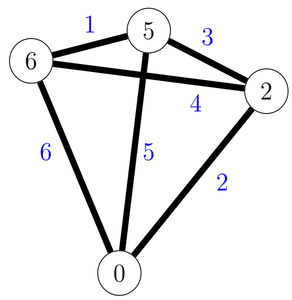


Figure 7

Finding a graceful labeling for a graph is usually accomplished through trial and error. Investigators assign the vertices of the graph eligible labels and constantly check the differences between the labels of adjacent vertices to ensure that the induced edge labels are distinct. This process is tedious and does not efficiently find every possible graceful labeling of a graph. Also, graph theorists tend to study families of graphs rather than individual graphs. These families are often infinite since one can often add vertices and edges to a member of a family to create a larger graph that still shares the same defining characteristics. Thus, since there are usually infinitely many members of a family of graphs, proving that every graph of a certain family has a graceful labeling cannot be accomplished by finding examples of such a labeling for only finitely many family members.

Instead, researchers develop functions that, when given an input of q, return an ordered set of integers. These integers, when used as vertex labels for the q vertices of a graph in the specified order, produce a graceful labeling for the graph. By proving that the function's output will always induce a graceful labeling for a graph with specified characteristics (such as belonging to a particular family of graphs), it then follows that all graphs that satisfy those characteristics have a graceful labeling.

Example 4.2 Hebbare's formula [4] for graceful labeling cycles C_n where $n \equiv 0, 3 \pmod 4$ is the sequence: $0, n, 1, n-1, 2, n-2, \ldots$, where the integer $\lfloor \frac{n+1}{4} \rfloor$ is omitted. Figure 8 shows the graceful labeling of C_{12} that follows this formula.

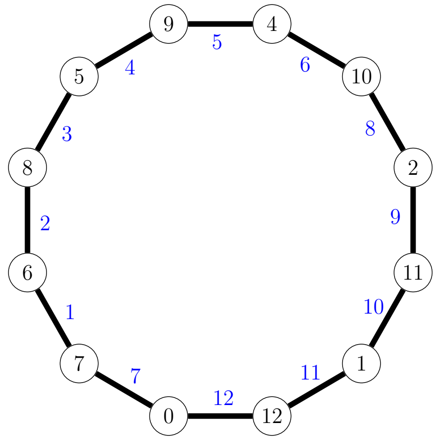


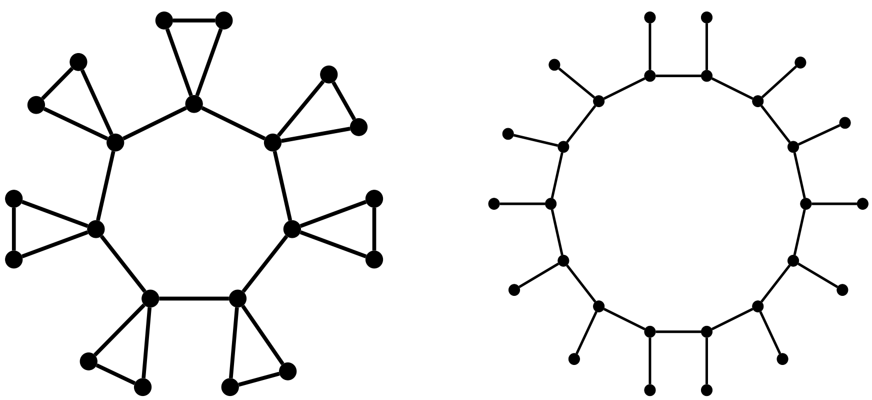
Figure 8

Formulas like the one shown in Example 4.2 often rely on properties specific to the family of graphs as well as the properties of graceful labelings. For instance, an n-cycle (or C_n) has exactly n vertices and n edges. Since the vertices are labeled with integers from the set $\{0, \ldots, n\}$, which has $n+1$ elements, exactly one of the possible vertex labels will not appear in the graceful labeling. Also, a graceful labeling of the n-cycle requires the edges to be labeled with distinct integers from the set $\{1, \ldots, n\}$. This forces the vertex labels 0 and n to be used in the labeling and to be assigned to adjacent vertices (since $n - 0$ is the only eligible difference that induces an edge label of n). These and other properties help investigators develop new formulas for families of graphs that share these traits.

5. Pendant Graphs and Their Graceful Labelings

Definition 5.1 The corona $G_1 \odot G_2$ of graphs G_1 and G_2 is the graph obtained by taking one copy of G_1, which has p_1 vertices, and p_1 copies of G_2, and then joining the ith vertex of G_1 by an edge to every vertex in the ith copy of G_2.

Example 5.1: The following are two examples of the coronas of two graphs.



$C_7 \odot K_2$ $C_{14} \odot K_1$

Figure 9

Definition 5.2 A *pendant graph* is a corona of the form $C_n \odot K_1$ where $n \geq 3$.

Example 5.2 Figure 6 and the second graph in Figure 9 are examples of pendant graphs.

Since the cycle C_n has n vertices and the complete graph K_1 has 1 vertex, a pendant graph consists of each vertex of the cycle being joined by an edge to a single vertex in the corresponding copy of K_1. The vertices of the copies of K_1 are then referred to as *pendant vertices*. Thus, a pendant graph has $2n$ vertices (n from the cycle and n from the copies of K_1) and $2n$ edges (n from the cycle and n edges joining the cycle to the copies of K_1).

As evidenced by the graceful labeling shown in Figure 6, some pendant graphs have graceful labelings. Graceful labelings of pendant graphs were investigated by Roberto Frucht [1] in 1979. He found that these labelings fell into one of three categories.

Definition 5.3 A graceful labeling of a pendant graph is of the:

1. *first kind* if the labels 0 and $2n$ are assigned to adjacent vertices of the n-gon.

2. *second kind* if $2n$ is assigned to a pendant vertex.

3. *third kind* if 0 is assigned to a pendant vertex.

Note that 0 and $2n$ cannot both be assigned to pendant vertices. This is because, like cycle graphs, in order to get an edge difference of $2n$, the vertex labels 0 and $2n$ must be assigned to adjacent vertices (since $2n - 0$ is the only eligible difference that induces an edge label of $2n$). Since no two pendant vertices are adjacent, 0 and $2n$ cannot both be assigned to pendant vertices.

Example 5.3: Figure 10 provides examples of graceful labelings of the first, second, and third kind.

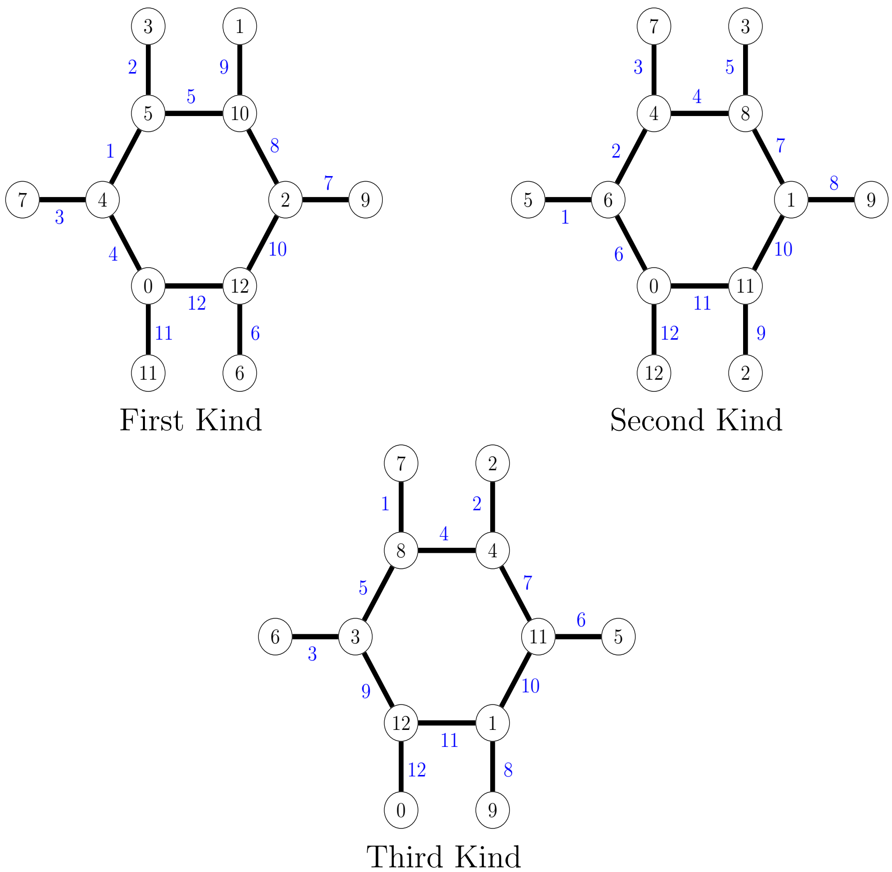


Figure 10

Frucht was able to prove that all pendant graphs have a graceful labeling using separate formulas for 4 different cases (one for each congruence class of $n \pmod 4$). However, his formulas only produced graceful labelings of the second kind. Frucht was able to find enough examples to conjecture that graceful labelings of the first kind exist for all pendant graphs as well, but he was unable to prove this claim.

In a 2013 paper [3], functions producing graceful labelings of the first kind for pendant graphs with $n \equiv 3, 4 \pmod 8$ were proved to exist (see Theorem 5.1 and Example 5.4).

Theorem 5.1 Let $\{v_1, v_2, \ldots, v_n\}$ be the set of cycle vertices and $\{u_1, u_2, \ldots, u_n\}$ be the set of pendant vertices for the pendant graph $C_n \odot K_1$ where v_i is adjacent to u_i, v_{i-1}, and v_{i+1} if $i \in \{2, 3, \ldots, n-1\}$. Note that v_1 is adjacent to u_1, v_n, as well as v_2 and that v_n is adjacent to u_n, v_{n-1}, and v_1. If $n \equiv 4 \pmod 8), n > 12$, the following function assigns a vertex label $f(v_i)$ to the cycle vertex v_i and assigns vertex label $f(u_i)$ to the pendant vertex u_i. The resulting labeling is a graceful labeling of the first kind for $C_n \odot K_1$.

$$f(v_i) = \begin{cases} 0 & \text{if } i = n \\ 2n - i + 1 & \text{if } i = 1, 3, 5, \ldots, n-1 \\ i + 2 & \text{if } i = \frac{n}{2}, \frac{n}{2} + 2, \frac{n}{2} + 4, \ldots, n-2 \\ i & \text{if } i = 2, 4, 6, \ldots, \frac{n}{2} - 2 \end{cases}$$

$$f(u_i) = \begin{cases} 2n - 1 & \text{if } i = n \\ 1 & \text{if } i = n - 1 \\ \frac{n}{2} + 1 & \text{if } i = n - 2 \\ 2n - i + 1 & \text{if } i = \frac{3n}{4} + 1, \frac{3n}{4} + 3, \frac{3n}{4} + 5, \ldots, n - 4 \\ i + 6 & \text{if } i = \frac{3n}{4}, \frac{3n}{4} + 2, \frac{3n}{4} + 4, \ldots, n - 3 \\ i + 5 & \text{if } i = \frac{3n}{4} - 1 \\ i + 4 & \text{if } i = \frac{n}{2} - 1, \frac{n}{2} + 1, \frac{n}{2} + 3, \ldots, \frac{3n}{4} - 2 \\ 2n - i - 1 & \text{if } i = 2, 4, 6, \ldots, \frac{3n}{4} - 3 \\ i + 2 & \text{if } i = 1, 3, 5, \ldots, \frac{n}{2} - 3 \end{cases}$$

Example 5.4: Figure 11 shows the graceful labeling of the first kind for a pendant graph with $n = 20$ produced by Theorem 5.1. Note that any cycle vertex can be chosen as v_1 and the labeling can proceed in either a clockwise or counterclockwise fashion without changing the labeling. In this example, v_1 (which receives the label 40) occurs at the bottom of the cycle and the labeling proceeds in a counterclockwise fashion.

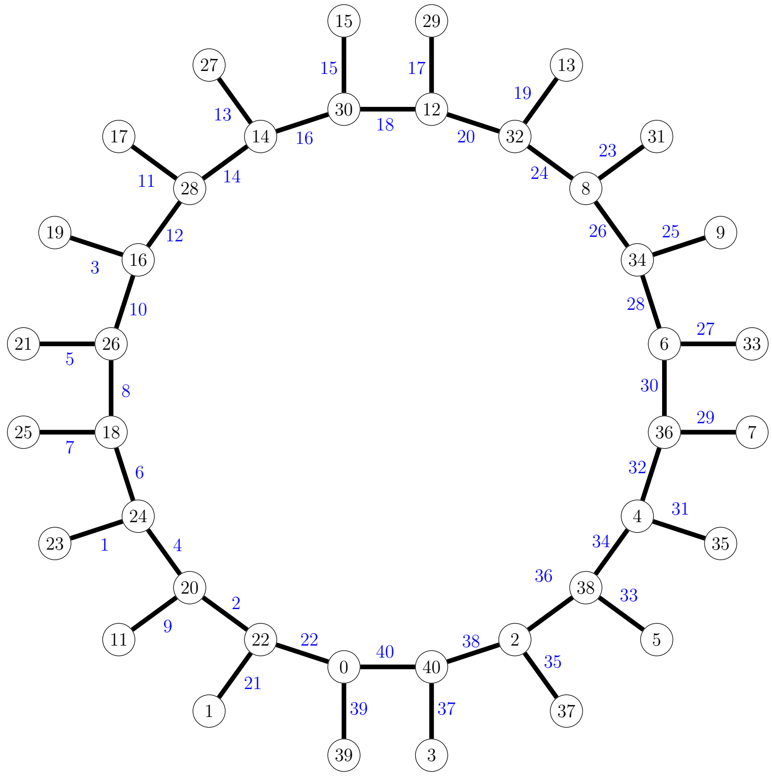


Figure 11

6. New Results

In my investigation, I have continued the efforts of proving Frucht's conjecture that every pendant graph has a graceful labeling of the first kind. Although difficult, the process became easier with the help of a Mathematica program I developed with the help of Ian Douglas. When given an integer n, this program was able to find and list all graceful labelings of a pendant graph of cycle size n. The program was then modified to accept a partial graceful labeling in addition to the size n so that the graceful labelings returned followed the provided labeling. This decreased the number of labelings the program returned from hundreds of thousands of possible labelings to in some cases less than 100 possibilities.

The program was used on values of n in the same congruence class (mod 8) to identify patterns present in some of the labelings for each of these values of n. [Note that congruence classes (mod 8) were chosen because the graceful labelings of pendant graphs with cycle sizes n and $n+8$ appeared to be very similar.] These patterns would then be added to the partial graceful labeling used by the Mathematica program and the program was run again for the same values of n. This process continued until the program returned exactly one graceful labeling for each tested n value. These labelings were then used to create a function that would produce a graceful labeling for any value n of the same congruence class (mod 8) (or in some cases (mod 16)). The function was then rigorously tested to ensure it was valid for all such n before a proof for the function was written. Theorem 6.1 is an example of one of the results my investigation produced.

Theorem 6.1 Let $\{v_1, v_2, \ldots, v_n\}$ be the set of cycle vertices and $\{u_1, u_2, \ldots, u_n\}$ be the set of pendant vertices for the pendant graph $C_n \odot K_1$ where v_i is adjacent to u_i, v_{i-1}, and v_{i+1} if $i \in \{2, 3, \ldots, n-1\}$. Note that v_1 is adjacent to u_1, v_n, as well as v_2 and that v_n is adjacent to u_n, v_{n-1}, and v_1. If $n \equiv 2 \pmod 8), n > 10$, the following function assigns vertex label $f(v_i)$ to the cycle vertex v_i and assigns vertex label $f(u_i)$ to the pendant vertex u_i. The resulting labeling is a graceful labeling of the first kind for $C_n \odot K_1$.

$$f(v_i) = \begin{cases} 2n - i + 1 & \text{if} \quad i = 1, 3, 5, \ldots, \frac{n}{2} \\ i & \text{if} \quad i = 2, 4, 6, \ldots, \frac{n}{2} - 1 \\ n + 2 & \text{if} \quad i = \frac{n}{2} + 1 \\ \frac{3n}{2} - 1 - i & \text{if} \quad i = \frac{n}{2} + 2, \frac{n}{2} + 4, \ldots, n - 3 \\ \frac{n}{2} + 1 + i & \text{if} \quad i = \frac{n}{2} + 3, \frac{n}{2} + 5, \ldots, n - 2 \\ \frac{n}{2} + 1 & \text{if} \quad i = n - 1 \\ 0 & \text{if} \quad i = n \end{cases}$$

$$f(u_i) = \begin{cases} \frac{3n}{2} & \text{if} \quad i = n - 1 \\ n & \text{if} \quad i = n \\ \frac{n}{2} + 2 + i & \text{if} \quad i = \frac{3n+2}{4} - 2 \\ \frac{n}{2} - 2 + i & \text{if} \quad i = \frac{3n+2}{4} \\ i & \text{if} \quad i = 1, 3, 5, \ldots, \frac{n}{2} \\ 2n - i + 1 & \text{if} \quad i = 2, 4, 6, \ldots, \frac{n}{2} - 1 \\ \frac{3n}{2} - 1 - i & \text{if} \quad i = \frac{n}{2} + 1, \frac{n}{2} + 3, \ldots, \frac{3n+2}{4} - 4 \\ \frac{n}{2} - 3 + i & \text{if} \quad i = \frac{n}{2} + 2, \frac{n}{2} + 4, \ldots, \frac{3n+2}{4} - 1 \\ \frac{n}{2} + 1 + i & \text{if} \quad i = \frac{3n+2}{4} + 1, \frac{3n+2}{4} + 3, \ldots, n - 3 \\ \frac{3n}{2} + 3 - i & \text{if} \quad i = \frac{3n+2}{4} + 2, \frac{3n+2}{4} + 4, \ldots, n - 2 \end{cases}$$

Example 6.1: Figure 12 shows the graceful labeling of the first kind for a pendant graph with $n = 18$ produced by Theorem 6.1. Note that any cycle vertex can be chosen as v_1 and the labeling can proceed in either a clockwise or counterclockwise fashion without changing the labeling. In this example, v_1 (which receives the label 36) occurs at the bottom of the cycle and the labeling proceeds in a counterclockwise fashion.

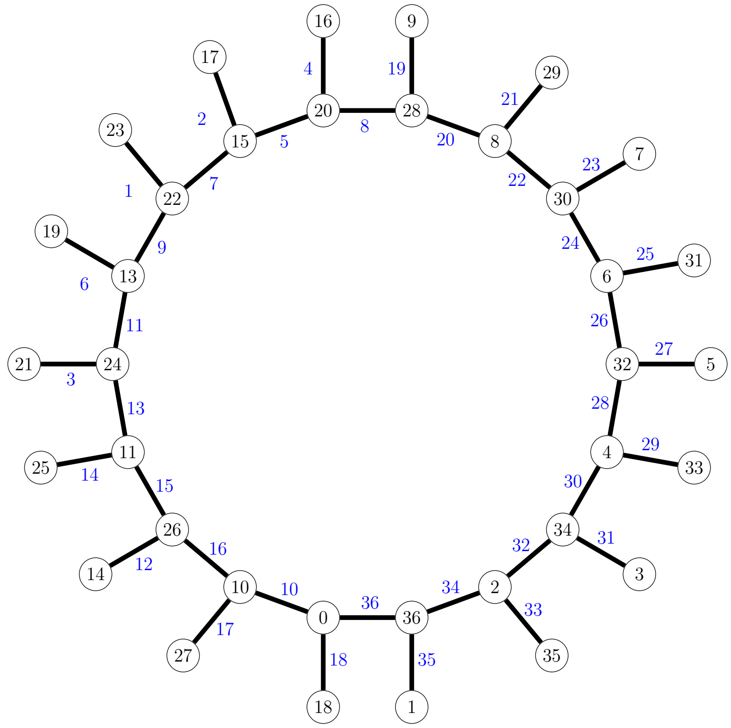


Figure 12

Proof. First, we shall verify that statement (a) of Definition 4.1 is satisfied. As no vertex belongs to more than one line of the function, each vertex is assigned precisely one label.

Through close examination of the above function f, it can be seen that each vertex is assigned a label from the set $\{0, \ldots, 2n\}$. Specifically, the first line of $f(u_i)$ assigns the odd integers from 1 to $\frac{n}{2}$ to unique pendant vertices of the graph. The second line of $f(v_i)$ assigns the even integers from 2 to $\frac{n}{2}+1$ to unique cycle vertices of the graph. Thus, all integers from 1 to $\frac{n}{2}+1$ are used in the graph labeling.

Similarly, the fourth line of $f(v_i)$ assigns the odd integers from $\frac{n}{2}+2$ to $n-3$ (since $\frac{3n}{2}-1-(\frac{n}{2}+2)=n-3$, $\frac{3n}{2}-1-(\frac{n}{2}+4)=n-5$, $\ldots$, $\frac{3n}{2}-1-(n-3)=\frac{n}{2}+2$) and the eighth line of $f(u_i)$ assigns the even integers from $\frac{n}{2}+5$ to $\frac{3n+2}{4}$ (as $\frac{3n}{2}+3-(\frac{3n+2}{4}+2)=\frac{3n+2}{4}$, $\frac{3n}{2}+3-(\frac{3n+2}{4}+4)=\frac{3n+2}{4}-2$, $\ldots$, $\frac{3n}{2}+3-(n-2)=\frac{n}{2}+5$). The odd integers from $n-1$ to $\frac{5n+2}{4}-4$ are assigned by the fourth line of $f(u_i)$ ($\frac{n}{2}-3+(\frac{n}{2}+2)=n-1$, $\frac{n}{2}-3+(\frac{n}{2}+4)=n+1$, $\ldots$, $\frac{n}{2}-3+(\frac{3n+2}{4}-1)=\frac{5n+2}{4}-4$) and the even integers from $\frac{3n-2}{4}+3$ to $n-2$ are assigned by the the third line of $f(u_i)$ ($\frac{3n}{2}-1-(\frac{n}{2}+1)=n-2$, $\frac{3n}{2}-1-(\frac{n}{2}+3)=n-4$, $\ldots$, $\frac{3n}{2}-1-(\frac{3n+2}{4}-4)=\frac{3n-2}{4}+3$).

Clearly, the fifth and sixth lines of $f(u_i)$ assign the odd labels $\frac{5n+2}{4}-2$ and $\frac{5n+2}{4}$ and the third line of $f(v_i)$ and last line of $f(u_i)$ assign the even labels $n+2$ and n. The seventh line of $f(u_i)$ assigns the odd labels from $\frac{5n+2}{4}+2$ to $\frac{3n}{2}-2$ (since $\frac{n}{2}+1+(\frac{3n+2}{4}+1)=\frac{5n+2}{4}+2$, $\frac{n}{2}+1+(\frac{3n+2}{4}+3)=\frac{5n+2}{4}+4$, $\ldots$, $\frac{n}{2}+1+(n-3)=\frac{3n}{2}-2$) and the fifth line of $f(v_i)$ assigns the even integers from $n+4$ to $\frac{3n}{2}-1$ ($\frac{n}{2}+1+(\frac{n}{2}+3)=n+4$, $\frac{n}{2}+1+(\frac{n}{2}+5)=n+6$, $\ldots$, $\frac{n}{2}+1+(n-2)=\frac{3n}{2}-1$). Finally, the ninth and second line of $f(u_i)$ assign the odd labels from $\frac{3n}{2}$ to $2n-1$ (since $2n-2+1=2n-1$, $2n-4+1=2n-3$, $\ldots$, $2n-(\frac{n}{2}-1)+1=\frac{3n}{2}+2$)

and the first line of $f(v_i)$ assigns the even integers from $\frac{3n}{2}+1$ to $2n$ (as $2n-1+1=2n$, $2n-3+1=2n-2$, ..., $2n-(\frac{n}{2})+1=\frac{3n}{2}+1$). Thus, all of the integers from 0 to $2n$ are assigned to a unique vertex except the even label $\frac{n}{2}+3$ which is omitted from the labeling. Therefore each vertex must be assigned a unique label, which shows statement (a) is verified.

To verify statement (b) of Definition 4.1, we must calculate the differences of adjacent vertex labels to determine the induced edge labels. We shall begin by checking the edge labels induced by the cycle vertex labels of i and $i+1$.

When $1 \le i < \frac{n}{2}$ and i is odd, the edge labels are $f(v_i)-f(v_{i+1})=(2n-i+1)-(i+1)=2n-2i$, which produces values that start at $2n-2$ and decrease by 4 as i increases by 2 until $i=\frac{n}{2}-2$ (which always produces the label n). When $1 \le i < \frac{n}{2}$ and i is even, the edge labels are $f(v_{i+1})-f(v_i)=(2n-(i+1)+1)-(i)=2n-2i$, which produces values that start at $2n-4$ and decrease by 4 as i increases by 2 until $i=\frac{n}{2}-1$ (which always produces the label $n+2$). Therefore the sequentially increasing path of cycle edges from vertex 1 to vertex $\frac{n}{2}$ have even edge labels $2n-2$, $2n-4$, ..., $n+2$.

The edge label between $i=\frac{n}{2}$ and $i+1$ is $f(v_i)-f(v_{i+1})=(2n-(\frac{n}{2})+1)-(n+2)=\frac{3n}{2}-1-n-2=\frac{n}{2}-3$. When $\frac{n}{2}<i\le n-3$ and i is odd, the edge labels are $f(v_{i+1})-f(v_i)=(\frac{n}{2}+1+(i+1))-(\frac{3n}{2}-1-(i))=2i+3-n$, which produces values that start at 7 and increase by 4 as i increases by 2 until $i=n-3$ (which always produces the label $n-3$). When i is even and $\frac{n}{2}+1\le i<n-2$, the edge labels are $f(v_i)-f(v_{i+1})=(\frac{n}{2}+1+(i))-(\frac{3n}{2}-1-(i+1))=2i+3-n$, which produces values that start at 5 and increase by 4 as i increases by 2 until $i=n-4$ (which always produces the label $n-5$).Therefore the sequentially increasing path of cycle edges from vertex $\frac{n}{2}+1$ to vertex $n-2$ have odd edge labels 5, 7, ..., $n-3$.

The edge label from $n-2$ to $n-1$ is $f(v_{n-2})-f(v_{n-1})=(\frac{n}{2}+1+(n-2))-(\frac{n}{2}+1)=n-2$, from $n-1$ to n is $f(v_{n-1})-f(v_n)=(\frac{n}{2}+1)-(0)=\frac{n}{2}+1$, and from n to 1 is $f(v_1)-f(v_n)=(2n-(1)+1)-(0)=2n$. Thus, the cycle edge labels are $5,7,9,\dots,n-3,n-2,\frac{n}{2}-3,\frac{n}{2}+1,n+2,n+4,\dots,2n$.

Next, we shall check the edge labels induced by the cycle and pendant vertex labels for each i. When $1\le i\le\frac{n}{2}$ and i is odd, the edge labels are $f(v_i)-f(u_i)=(2n-i+1)-(i)=2n-2i+1$, which produces values that start at $2n-1$ and decrease by 4 as i increases by 2 until $i=\frac{n}{2}$ (which always produces the label $n+1$). When i is even and $1\le i\le\frac{n}{2}-1$, the edge labels are $f(u_i)-f(v_i)=(2n-(i)+1)-(i)=2n-2i+1$, which produces values that start at $2n-3$ and decrease by 4 as i increases by 2 until $i=\frac{n}{2}-1$ (which always produces the label $n+3$). Therefore the pendant-cycle edges for $1\le i\le\frac{n}{2}$ have odd edge labels $2n-1$, $2n-3$, ..., $n+1$.

The pendant-cycle edge when $i=\frac{n}{2}+1$ is $f(v_i)-f(u_i)=(n+2)-(\frac{3n}{2}-1-(\frac{n}{2}+1))=n+2-(n-2)=4$. When $\frac{n}{2}+2\le i\le\frac{3n+2}{4}-1$ and i is odd, the edge labels are $f(u_i)-f(v_i)=(\frac{n}{2}-3+(i))-(\frac{3n}{2}-1-(i))=2i-n-2$, which produces values that start at 2 and increase by 4 as i increases by 2 until $i=\frac{3n+2}{4}-1$ (which always produces the label $\frac{n}{2}-3$). When i is even and $\frac{n}{2}+3\le i\le\frac{3n+2}{4}-4$, the edge labels are $f(v_i)-f(u_i)=(\frac{n}{2}+1+(i))-(\frac{3n}{2}-1-(i))=2i-n+2$, which produces values that start at 8 and increase by 4 as i increases by 2 until $i=\frac{3n+2}{4}-4$ (which always produces the label $\frac{n}{2}-5$).Therefore the pendant-cycle edges for even i where $\frac{n}{2}+1\le i\le\frac{3n+2}{4}-4$ have edge labels $4, 8, 12 \ldots, \frac{n}{2} - 5$ and for odd i where $\frac{n}{2} + 2 \leq i \leq \frac{3n+2}{4} - 1$ have edge labels $2, 6, 10, \ldots, \frac{n}{2} - 3$. Thus the even labels from 2 to $\frac{n}{2} - 3$ are induced.

When $i = \frac{3n+2}{4} - 2$, the pendant-cycle edge is assigned $f(u_i) - f(v_i) = (\frac{n}{2} + 2 + (\frac{3n+2}{4} - 2)) - (\frac{n}{2} + 1 + (\frac{3n+2}{4} - 2) = 1$ and when $i = \frac{3n+2}{4}$, the pendant-cycle edge is assigned $f(v_i) - f(u_i) = (\frac{n}{2} + 1 + (\frac{3n+2}{4})) - (\frac{n}{2} - 2 + (\frac{3n+2}{4})) = 3$. The edge labels assigned when i is odd and $\frac{3n+2}{4} + 1 \leq i \leq n - 3$ are $f(u_i) - f(v_i) = (\frac{n}{2} + 1 + (i)) - (\frac{3n}{2} - 1 - (i)) = 2i + 2 - n$, which produces values that start at $\frac{n}{2} + 5$ and increase by 4 as i increases by 2 until $i = n - 3$ (which always produces the label $n - 4$). Similarly, when i is even and $\frac{3n+2}{4} + 2 \leq i \leq n - 2$ are $f(v_i) - f(u_i) = (\frac{n}{2} + 1 + (i)) - (\frac{3n}{2} + 3 - (i)) = 2i - 2 - n$, which produces values that start at $\frac{n}{2} + 3$ and increase by 4 as i increases by 2 until $i = n - 2$ (which always produces the label $n - 6$). Thus the even labels from $\frac{n}{2} + 3$ to $n - 4$ are induced.

Finally, when $i = n - 1$, the pendant-cycle edge is labeled $f(u_i) - f(v_i) = (\frac{3n}{2}) - (\frac{n}{2} + 1) = n - 1$ and when $i = n$, the pendant-cycle edge is labeled $f(u_i) - f(v_i) = (n) - (0) = n$. Thus, the edge labels are unique and are from the set $\{1, \ldots, 2n\}$, so statement (b) is satisfied. Therefore the pendant graph is gracefully labeled. □

Remark 6.1 Graceful labelings of the first kind exist for those pendant graphs with values of n smaller than the restriction given in Theorem 6.1. For these small values of n, some vertices belong to multiple portions of the piecewise function. A graceful labeling can be produced using the function from Theorem 6.1 by letting the first label a vertex is assigned by the formula to be the label of that vertex (i.e. vertex labels are then assigned in the order they are listed in the piecewise function). For example, in the pendant graph $n = 10$, the vertex u_6 corresponds to both $i = \frac{3n+2}{4} - 2$ and $i = \frac{n}{2} + 1$. By following the rule stated above, this vertex is assigned the label $f(u_6) = \frac{10}{2} + 2 + (\frac{3(10)+2}{4} - 2) = 5 + 8 = 13$.

7. Conclusion and Open Questions

When Alexander Rosa [5] introduced his new β-labelings in 1967, he believed they would help solve Ringel's conjecture. Little did he know that graceful labelings would be used in a much broader scope. Over the past 45 years, graceful labelings have been used to model a variety of situations in several fields.

Finding graceful labelings for a specific graph or model, however, is often difficult. So, researchers have created formulas that produce a graceful labeling for specific types of models, which saves time and energy. Recently, new formulas have been found to provide graceful labelings of the first kind for pendant graphs of cycle size $n \equiv 2, 3, 4 \pmod 8$. Other formulas (which are still being verified) seem to produce graceful labelings of the first kind for pendant graphs with cycle sizes $n \equiv 5, 6 \pmod 8$. No such formulas have yet been identified for those of cycle size $n \equiv 0, 1, 7 \pmod 8$.

Other open questions involve adapting these or other formulas to find functions that produce graceful labelings for other similar families of graphs. For example, are unicyclic graphs, which are graphs that contain exactly one cycle, graceful so long as the cycle C_n is of size $n \equiv 0, 3 \pmod 4$? Could the graceful labeling formulas for pendant graphs (which are a type of unicyclic graph) be adapted to help prove this conjecture? Are there other formulas that produce graceful labelings for pendant graphs that would be more beneficial in certain applications?

There are many more unanswered questions concerning graceful labelings of graphs. The interested reader can find more conjectures and results for graceful and other types of graph labelings in a survey by Gallian [2].

8. References

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